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DUALITY IN THE OPTIMAL CONTROL OF DISTRIBUTED PARAMETER SYSTEMS GOVERNED BY HYPERBOLIC EQUATIONS

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In this paper, we study the duality theory of hyperbolic equations. The main objective is to prove the duality theorem under general conditions within an infinite-dimensional framework. An example of its application is also given.

Key Words : Duality; Optimal Control; Distributed Control Systems; Hyperbolic Equations

1 INTRODUCTION

To distributed parameter systems of some class we associate dual control systems and discuss relationships between their optimal values. Here, we consider the optimal control problems and the dual optimal control problems for hyperbolic systems. For optimal control of non-well-posed distributed systems, direction results have been given in [3, 4, 5]. More detailed accounts can be found in [6, 7]. Chan[1] and Tanimoto[8] studied duality theory for the corresponding non-well-posed parabolic equations with positive control. However, Tanimoto[8] dealt with for more general distributed control systems than those in Chan[1]. In this paper, we study duality theory for hyperbolic equations with time varing control constraints and convex cost functional. The main objective is to prove a duality theorem under general conditions within an infinite-dimensional framework. Our approach is based on an infinite dimensional version of the duality theorem in [8] for the optimality systems which characterize the optimal control. An example of its application is also given.

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2 PRELIMINARIES

Let X and H be two separable Hilbert spaces. The norm on X will be denoted by $\|\cdot\|$. The norm on H will be denoted by $|\cdot|$ and the corresponding inner product by $(\cdot,\cdot)$. We assume that $X \subseteq H$ and the injection of X into H is continuous, and that X is dense in H. We denote the dual space of X by X^* and the norm on it by $\|\cdot\|_*$. Identifying H with its dual space, we have $X \subseteq H \subseteq X^*$, where X is dense in H and H is dense in X^* and the corresponding injections are continuous. The usual bilinear form on the dual pair (X, X^*) is denoted by $\langle \cdot, \cdot \rangle$. When $x \in X \subseteq H$ and $h \in H \subseteq X^*$, the equality $\langle x, h \rangle = (x, h)$ will be frequently used. Let T be a fixed positive number.

We define a function space $W(0, T)$ by

$$W(0, T) = \{x \mid x \in L^2(0, T; X), \dot{x} \in L^2(0, T; H), \ddot{x} \in L^2(0, T; X^*)\},$$

with an inner product

$$(x_1, x_2)_{w(0,T)} = \int_0^T \{(x_1(t), x_2(t))_x + (\dot{x}_1(t), \dot{x}_2(t))_H + (\ddot{x}_1(t), \ddot{x}_2(t))_{X^*}\} dt$$

This becomes a Hilbert space with norm

$$\| x \|_{w(0,T)} = \left(\| x \|^2_{L^2(0;T;X)} + \| \dot{x} \|^2_{L^2(0,T;H)} + \| \ddot{x} \|^2_{L^2(0,T;X^*)} \right)^{\frac{1}{2}},$$

where $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$.

We consider the following optimal control problem :

$$(P) \begin{cases} \int_0^T (F(t, x(t)) + G(t, u(t)))\, dt \to \inf \\ \text{subject to} \\ \begin{cases} \ddot{x} = A(t)x(t) + B(t, u(t)) \text{ a. e} \\ x(0) \in C, \dot{x}(0) \in H, u(t) \in U(t) \text{ a. e..} \end{cases} \end{cases} \qquad \text{... (2.1)}$$

We assume that the control variable u takes its values in another separable Banach space Y, the norm of whose element is denoted by $\| \cdot \|_Y$. We denote by $S(Y)$ the set of nonempty closed, convex subsets of Y. The set of admissible control is given by

$$\mathcal{U}_{ad} = \{u(\cdot) \mid u(\cdot) \in L^2(0, T; Y), u(t) \in U(t) \text{ a.e.}\}.$$

We will describe the assumptions which are imposed on the optimal control problem (***P***).

$(A1)$ $A(t) : X \to X^*$ is a linear operator for each $t \in [0, T]$ such that for every $x, y \in X$, the function

$$a(t; x, y) = \langle A(t)x, y \rangle$$

of t is measurable and $| a(t; x, y) | \le M \| x \| \| y \|$ for some $M > 0$ and there exist a real number λ and a positive number α satisfying

$$a(t; x, x) + \lambda | x |^2 \ge \alpha \| x \|^2, \quad x \in X, t \in [0, T].$$

$(A2)$ $B : [0, T] \times Y \to X^*$ is a measurable mapping such that $B(t, \cdot)$ is continuous for each t and $B(t) : Y \to X^*$ is a continuous linear operator for each $t \in [0, T]$ such that $B(\cdot)u$ is a continuous mapping of t and $\| B(t)u \|_* \leq a(t) + b \| u \|_Y$ a.e. with $a \in L^2(0, T)$.

$(A3)$ $F : [0, T] \times H \to R$ is a function such that $F(\cdot, x)$ is a measurable function of t for each $x \in H$ and $F(t, \cdot)$ is a convex and continuously Gateaux differentiable function of c for each t.

$(A4)$ $G : [0, T] \times Y \to R$ is a measurable function such that $G(t, \cdot)$ is continuous for each t.

$(A5)$ $U(\cdot)$ is a mapping $[0, T] \to S(Y)$ such that the function

$t \to \sup\{\| u \|_Y \mid u \in U(t)\}$ belongs to $L^2(0, T)$ and such that its graph is a measurable subset of the product measurable space generated by the Borel σ-field of $[0, T]$ and the Borel σ-field of Y.

$(A6)$ $C \subseteq H$ is a closed and convex set with nonempty interior.

We associate another optimal control problem to **(P)**, which is called the dual problem of **(P)**. In order to describe it, we need a function $K : [0, T] \times H \to R$ defined by

$$K(t, p) = \underset{u \in U(t)}{\infty} \{G(t, u) + \langle p, B(t, u) \rangle\}, \quad p \in X$$

We make the following assumption on the function K.

$$(A7) \quad K(\cdot, p(\cdot)) \in L^1(0, T) \quad \text{for all} \quad p(\cdot) \in W(0, T).$$

We call the following optimal control problem of the dual problem :

$$(D) \begin{cases} \int_0^T (K(t, p(t)) - \langle p(t), \ddot{x} - A(t)x(t) \rangle + F(t, x(t)))\, dt \to \sup \\ \text{subject to} \\ \begin{cases} \ddot{p} = A^*(t)p(t) + F_x(t, x(t)) \text{ a. e} \\ p(T) = 0, \dot{p}(T) = 0, x \in W(0, T), \\ (p(0), h - \dot{x}(0)) \geq 0, (\dot{p}(0), x(0) - c) \geq 0 \text{ for all } h \in H, c \in C. \end{cases} \end{cases} \qquad \ldots (2.2)$$

Here $A^*(t)$ is the adjoint operator for $A(t)$ and $F_x(t, x)$ the derivative of $F(t, x)$ with respect to x. It should be noted that the control variable of this problem is x. By making use of Lemma 5.5.1 of [9] and (2.2) we have

$$\int_0^T \langle p(t), \ddot{x}(t) \rangle\, dt = (\dot{x}(t), p(t)) \Big|_0^T - (x(t), \dot{p}(t)) \Big|_0^T + \int_0^T \langle \ddot{p}(t), x(t) \rangle\, dt$$

$$= -\langle \dot{x}(0), p(0) \rangle + \langle x(0), \dot{p}(0) \rangle + \int_0^T \langle p(t), A(t)x(t) \rangle\, dt$$

$$+\int_0^T \langle x(t), F_x(t, x(t)) \rangle \, dt$$

since, as will be stated explicitly below, $p, x \in W(0, T)$. Thus the dual problem can be equivalently rewritten as follows (D):

$$\langle \dot{x}(0), p(0) \rangle - \langle x(0), \dot{p}(0) \rangle + \int_0^T \{K(t, p(t)) - \langle x(t), F_x(t, x(t)) \rangle + F(t, x)(t))\} \, dt \rightarrow \sup$$

subject to (2.2).

As for the differential equation in (2.2), we restrict ourselves to only solutions which belong to $W(0, T)$. Hence, if there exists no solution $(p, x) \in W(0, T) \times W(0, T)$ in (2.2), we define the supremum of problem (D) to be $-\infty$.

3 DUALITY

We first prove a weak duality theorem saying that the infimum of (P) is equal to or greater than the supremum of (D).

***Theorem* 3.1** — *Under the assumptions (A1)-(A7), if* $F(\cdot, x(\cdot)) \in L^1(0, T)$ *and* $F_x(\cdot, x(\cdot)) \in L^2(0, T; X^*)$ *for all* $x \in W(0, T)$ *and* $G(\cdot, u(\cdot)) \in L^1(0, T)$ *and* $B(\cdot, u(\cdot)) \in L^2(0, T; X^*)$ *for all* $u \in \mathcal{U}_{ad}$, *then the infimum of (P) is equal to or greater than the supremum of (D).*

PROOF : Let $u(\cdot) \in \mathcal{U}_{ad}$ be any admissible control and fix it for a moment. Due to J.L. Lions[2], there exists a solution $x \in W(0, T)$ of (2.1) with any initial condition $x(0) = x_0 \in C$ by the assumption $B(\cdot, u(\cdot)) \in L^2(0, T; X^*)$. Let $\bar{x}$ be any one of such solutions corresponding to u. That is,

$$\ddot{\bar{x}}(t) = A(t)\bar{x}(t) + B(t, u(t)) \qquad \text{... (3.1)}$$

$$\bar{x}(0) \in C, \dot{\bar{x}}(0) \in H.$$

To this u we associate the following problem (D_u):

$$\int_0^T \{F(t, x(t)) + G(t, u(t)) - \langle p(t), \ddot{x}(t) - A(t)x(t) - B(t, u(t)) \rangle\} dt \rightarrow \sup$$

subject to (2.2).

If there is no solution $(p, x) \in W(0, T) \times W(0, T)$ satisfying (2.2), we define the supremum of (D_u) to be $-\infty$. If $(p^\circ(\cdot), x^\circ(\cdot)) \in W(0, T) \times W(0, T)$ is an arbitrary solution of (2.2), then we can show that the value

$$\int_0^T \{F(t, \bar{x}(t)) + G(t, u(t))\}\, dt$$

$$- \int_0^T \{F(t, \overset{\circ}{x}(t)) + G(t, u(t)) - \langle \overset{\circ}{p}(t), \dot{\overset{\circ}{x}}(t) - A(t)\overset{\circ}{x}(t) - B(t, u(t)) \rangle\}dt$$

is nonnegative. To do this, note that

$$B(t, u(t)) = \dot{\bar{x}}(t) - A(t)\bar{x}(t) \text{ a.e.}$$

and observe that

$$F(t, \bar{x}(t)) - F(t, \overset{\circ}{x}(t)) \geq \langle \bar{x}(t) - \overset{\circ}{x}(t), F_x(t, \overset{\circ}{x}(t)) \rangle,$$

since F is a convex and differentiable function with respect to x. Substituting these into (3.2), we see that the value (3.2) is equal to or greater than

$$\int_0^T \langle \bar{x}(t) - \overset{\circ}{x}(t), F_x(t, \overset{\circ}{x}(t)) + A^*(t)\, \overset{\circ}{p}(t) \rangle\, dt + \int_0^T \langle \overset{\circ}{p}(t), \dot{\overset{\circ}{x}}(t) - \ddot{\bar{x}}(t)\ dt.$$

Using $p(T) = 0$ and $\dot{p}(T) = 0$ and $\langle p(0), h - \dot{x}(0) \rangle \geq 0$ and $\langle \dot{p}(0), x(0) - c \rangle \geq 0$, we have

$$\int_0^T \langle \overset{\circ}{p}(t), \ddot{\overset{\circ}{\bar{x}}}(t) - \ddot{\bar{x}}(t) \rangle\, dt$$

$$= (\overset{\circ}{p}(t), \dot{\overset{\circ}{x}}(t) - \dot{\bar{x}}(t)) \Big|_0^T - (\dot{\overset{\circ}{p}}(t), \overset{\circ}{x}(t) - \bar{x}(t)) \Big|_0^T + \int_0^T \langle \ddot{\overset{\circ}{p}}(t), \overset{\circ}{x}(t) - \bar{x}(t) \rangle\, dt$$

$$\geq \int_0^T \langle \ddot{\overset{\circ}{p}}(t), \overset{\circ}{x}(t) - \bar{x}(t) \rangle\, dt.$$

Therefore, the value (3.2) is equal to or greater than

$$\int_0^T \langle \bar{x}(t) - \overset{\circ}{x}(t), F_x(t, \overset{\circ}{x}(t)) + A^*(t)\, \overset{\circ}{p}(t) - \ddot{\overset{\circ}{p}}(t) \rangle\, dt = 0,$$

since $(\overset{\circ}{p}, \overset{\circ}{x})$ is a solution of (2.2). Let us denote by $p(\bar{x}, u)$ the value

$$\int_0^T \{F(t, \bar{x}(t)) + G(t, u(t))\}dt,$$

and by $d(u)$ the supremum of problem (D_u). Then the above argument show that

$$p(\bar{x}, u) \geq d(u) \qquad \text{... (3.3)}$$

holds for every $u \in \mathcal{U}_{ad}$ and every $\bar{x}$ satisfying (3.1) together with u. If we denote by $J(p, x, u)$ the objective function of (D_u), that is, if we put

$$J(p, x, u) = \int_0^T \{F(t, x(t)) + G(t, u(t)) + \langle p(t), B(t, u(t)) \rangle$$

$$- \langle p(t), \dot{x}(t) - A(t)x(t) \rangle\}dt$$

it follows from (3.3) and a well-known inequality of game theory that

$$\inf_{\bar{x}, u} p(\bar{x}, u) \geq \inf_u d(u) = \inf_u \sup_{p, \dot{x}} J(p, x, u) \qquad \text{... (3.4)}$$

$$\geq \sup_{p, x} \inf_u J(p, x, u).$$

For a given $u(\cdot) \in \mathcal{U}_{ad}$, we have

$$\int_0^T \{G(t, u(t)) + \langle p(t), B(t, u(t)) \rangle\} \ dt \geq \int_0^T K(t, p(t))dt,$$

since $G(t, u(t)) + \langle p(t), B(t), u(t)) \rangle \geq K(t, p(t))$ a.e. by definition of the function K. Hence, it follows that

$$\inf_u J(p, x, u) \geq \int_0^T \{K(t, p(t)) - \langle p(t), \dot{x}(t) - A(t)x(t) \rangle + F(t, x(t))\} \ dt$$

for every solution (p, x) of (2.2). If we denote by inf (P) the infimum of (P) and by sup (D) the supremum of (D), it is obvious that $\inf_{\bar{x}, u} p(\bar{x}, u) = \inf (P)$ and $\sup_{p, x} \inf_u J(p, x, u) \geq \sup (D)$ by the above inequality. Therefore, we conlude from (3.4) that inf $(P) \geq$ sup (D). This completes the proof.

We prove next the duality theorem that under certain conditions the infimum of (P) coincides with the supremum of (D). We make the following assumptions:

(A4′) $G : [0, T] \times Y \rightarrow R$ is a measurable function such that $G(t, \cdot)$ is convex and continuously Gateaux differentiable with respect to u for each $t \in [0, T]$.

$(A8)$ The interior of $\mathcal{U}_{ad}$ is nonempty.

Theorem 3.2 — *Assume (A1), (A2), (A3), (A4′), (A5)-(A8) and that* $F(\cdot, x(\cdot)) \in L^1(0, T)$ *and* $F_x(\cdot, x(\cdot)) \in L^2(0, T; H)$ *for all* $x \in W(0, T)$ *and* $G(\cdot, u(\cdot)) \in L^1(0, T)$ *for all* $u \in \mathcal{U}_{ad}$ *attains the infimum of (P), then there exists* $\overset{\circ}{p} \in W(0, T)$ *such that* $(\overset{\circ}{p}, \overset{\circ}{x})$ *attains the supremum of (D). Furthermore, the infimum of (P) is equal to supremum of (D).*

PROOF : If $(\overset{\circ}{x}(0), \overset{\circ}{x}, \overset{\circ}{u}) \in C \times W(0, T) \times \mathcal{U}_{ad}$ attains the infimum of (P). There exists $\overset{\circ}{p} \in W(0, T)$ satisfying

$$\ddot{\overset{\circ}{p}}(t) = A^*(t)\, \overset{\circ}{p}(t) + F_x(t, \overset{\circ}{x}(t)) \text{ a.e. } p(T) = 0, \dot{p}(T) = 0, \quad \ldots (3.5)$$

$$(G_u(t, \overset{\circ}{u}(t)) + B^*(t)\, \overset{\circ}{p}(t), v - \overset{\circ}{u}(t))_{Y, Y^*} \geq 0 \quad \ldots (3.6)$$

for all $v \in U(t)$ a.e., and

$$(p(0), h - \dot{x}(0)) \geq 0, (\dot{p}(0), x(0) - c) \geq 0, \text{ for all } h \in H, c \in C, \quad \ldots (3.7)$$

where $B^*(t)$ denotes the adjoint operator of $B(t)$, $(\cdot, \cdot)_{Y, Y^*}$ the bilinear form on the dual pair (Y, Y^*) and G_u the derivative of $G(t, u)$ with respect to u. By convexity of G, (3.6) implies $G(t, v) + \langle \overset{\circ}{p}(t), B(t)v \rangle \geq G(t, \overset{\circ}{u}(t)) + \langle \overset{\circ}{p}(t), B(t)\overset{\circ}{u}(t) \rangle$ for all $v \in U(t)$ a.e., from which we have

$$K(t, \overset{\circ}{p}(t)) = G(t, \overset{\circ}{u}(t)) + \langle \overset{\circ}{p}(t), B(t)\overset{\circ}{u}(t) \rangle \text{ a.e.} \quad \ldots (3.8)$$

From (3.5) and (3.7) we see that $(\overset{\circ}{p}, \overset{\circ}{x})$ is a solution of (2.2) and from (3.8) that we obtain

$$\begin{aligned} &K(t, \overset{\circ}{p}(t)) - \langle \overset{\circ}{p}, \ddot{\overset{\circ}{x}}(t) - A(t)\overset{\circ}{x}(t) \rangle + F(t, \overset{\circ}{x}(t)) \\ &\quad = G(t, \overset{\circ}{u}(t)) + \langle \overset{\circ}{p}(t), B(t)\overset{\circ}{u}(t) \rangle - \langle \overset{\circ}{p}(t), \ddot{\overset{\circ}{x}}(t) - A(t)\overset{\circ}{x}(t) \rangle + F(t, \overset{\circ}{x}(t)) \\ &\quad = G(t, \overset{\circ}{u}(t)) + \langle \overset{\circ}{p}(t), A(t)\overset{\circ}{x}(t) + B(t)\overset{\circ}{u}(t) - \ddot{\overset{\circ}{x}}(t) \rangle + F(t, \overset{\circ}{x}(t)) \\ &\quad = G(t, \overset{\circ}{u}(t)) + F(t, \overset{\circ}{x}(t)) \text{ a.e.,} \end{aligned}$$

since $(\overset{\circ}{x}, \overset{\circ}{p})$ satisfies $\ddot{\overset{\circ}{x}}(t) = A(t)\overset{\circ}{x}(t) + B(t)\overset{\circ}{u}(t)$ a.e.. Therefore, we conclude that

$$\begin{aligned} &\int_0^T \{F(t, \overset{\circ}{x}(t)) + G(t, \overset{\circ}{u}(t))\}\, dt \\ &\qquad = \int_0^T \{K(t, \overset{\circ}{p}(t)) - \langle \overset{\circ}{p}(t), \ddot{\overset{\circ}{x}}(t) - A(t)\overset{\circ}{x}(t) \rangle + F(t, \overset{\circ}{x}(t))\} dt. \end{aligned}$$

Thus Theorem 3.1 implies that the infimum of (P) is equal to the supremum of (D) and that $(\overset{\circ}{p}, \overset{\circ}{x})$ attains the supremum of (D). This completes the proof.

4 EXAMPLE

In this section we give an example satisfying Theorem 3.2 (duality theorem) to a hyperbolic distributed system.

Let $T > 0$ and Ω be a bounded domain in R^n with a smooth boundary $\partial\Omega = \Gamma$. On $[0, T] \times \Omega$ we consider the control problem :

$$(P')\begin{cases} \int_0^T \int_\Omega (f(t, z, x(t, z)) + g(t, z, u(t, z))\, dtdz \to \inf \\ \text{subject to} \\ \dfrac{\partial^2 x(t, z)}{\partial t^2} = A(t)x(t, z) + u(t, z) \text{ on } [0, T] \times \Omega \\ x(t, z) = 0 \text{ on } [0, T] \times \Gamma,\ x(0, \cdot) \in C \subseteq L^2(\Omega),\ \dot{x}(0, \cdot) \in H \\ \left(\int_\Omega |\, u(t, z)\,|^2 dz \right)^{\frac{1}{2}} \leq L. \end{cases}$$

Here $A(t)$ is the formal second order elliptic italic italic partial differential operator defined by

$$A(t)y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t, z) \frac{\partial y(z)}{\partial x_j} \right).$$

We assume that $a_{ij}(\cdot, \cdot) \in L^\infty([0, T] \times \Omega)$ and that they satisfy the strong ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(t, z)\xi_i \xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2$$

for all $(t, z) \in [0, T] \times \Omega$, $(\xi_i)_{i=1}^n \in R^n$ and for some $\alpha > 0$. Taking $X = H_0^1(\Omega)$, $H = L^2(\Omega)$, $X^* = H^{-1}(\Omega)$, the triple (X, H, X^*) enjoys the properties mentioned in Section 2. The bilinear form

$$a(t; \phi, \psi) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(t, z) \frac{\partial \phi(z)}{\partial z_i} \frac{\partial \psi(z)}{\partial z_j}\, dz$$

on $X \times X$ gives rise to a continuous linear operator $A(t) : V = H_0^1(\Omega) \to V^* = H^{-1}(\Omega)$ characterized by

$$a(t; y, z) = \langle A(t)y, z \rangle, y, z \in H_0^1(\Omega).$$

The strong ellipticity condition and the assumption imposed on $a_{ij}(\cdot,\cdot)$ ensure that $A(t)$ satisfies $(A1)$ of Section 2. The control space is $Y = L^2(\Omega)$ and $U(t)$ is defined by $U(t) = \{u \in L^2(\Omega) \mid \| u \|_2 \leq L\}$ for $t \in [0, T]$, where $\| u \|_2 = \left(\int_\Omega | u(z) |^2 dz \right)^{\frac{1}{2}}$ and $L \geq 0$. Then $(A8)$ is fulfilled. Let $C \subseteq L^2(\Omega)$ be a closed, convex subset with nonempty interior. Moreover, we assume that the functions f and g in (P') satisfy

(i) $f(\cdot,\cdot,\cdot) : [0, T] \times \Omega \times R \to R$ is a measurable function such that $f(t, z, \cdot)$ is convex and continuously differentiable with respect to x. $f \in L^2(0, T; L^2(\Omega))$ for every $x \in L^\infty(0, T; L^2(\Omega))$.

(ii) $g(\cdot,\cdot,\cdot) : [0, T] \times \Omega \times R \to R$ is a measurable function such that $g(t, z, \cdot)$ is convex and continuously differentiable with respect to u. $g \in L^2(0, T; L^2(\Omega))$ for every $u \in L^2([0, T] \times \Omega)$.

If $F : [0, T] \times L^2(\Omega) \to R$ is given by $F(t, x) = \int_\Omega f(t, z, x(z))\, dz$ and $G : [0, T] \times L^2(\Omega) \to R$ by $G(t, u) = \int_\Omega g(t, z, u(z))\, dz$, our optimal control problem (P') reduces to the abstract form

$$\int_0^T \{F(t, x(t)) + G(t, u(t))\}\; dt \;\to\; \inf$$

subject to

$$\ddot{x}(t) = A(t)x(t) + u(t) \;\text{ a.e.}, \;\; x(0) \in C, \dot{x}(0) \in H, u(t) \in U(t) \;\text{ a.e.}$$

The dual problem of (P') can be formulated via the dual problem of this abstract one. In this case it is calculared as

$$K(t, p) = \inf_{\| u \|_2 \leq L} \int_\Omega \{g(t, z, u(z)) + p(z)u(z)\}dz, \; p \in H_0^1(\Omega).$$

Hence the dual problem written in the alternative form becomes :

$$
(D')\left\{
\begin{aligned}
&\int_{\Omega} \dot{x}(0, z)p(0, z)\,dz - \int_{\Omega} x(0, z)\,\dot{p}(0, z)dz \\
&+ \int_0^T K(t, p(t))dt - \int_0^T \int_{\Omega} (f_x(t, z, x(t, z))x(t, z) \\
&- f(t, z, x(t, z)))\, dtdz \to \sup \\
&\text{subject to} \\
&\frac{\partial^2 p(t, z)}{\partial t^2} = A^*(t)p(t, z) + f_x(t, z, x(t, z)) \text{ on } [0, T] \times \Omega, \\
&p(t, z) = 0 \text{ on } [0, T] \times \Gamma, p(T, z) = 0, \dot{p}(T, z) = 0 \text{for } z \in \Omega, \\
&x \in W(0, T) = (x(\cdot) \mid x \in L^2(0, T; X), \dot{x} \in L^2(0, T; H), \ddot{x} \in L^2(0, T; X^*)), \\
&\int_{\Omega} (h(z) - \dot{x}(0, z))\, p(0, z)dz \geq 0 \text{ for all } h(\cdot) \in H, \\
&\int_{\Omega} (x(0, z) - c)\dot{p}(0, z)\, dz \geq 0 \text{ all } (c(\cdot) \in C,
\end{aligned}
\right.
$$

where $p(t)$ in $K(t, p(t))$ means $p(t) = p(t, \cdot) \in W(0, T), A^*(t)$ is the formal adjacent of $A(t)$. Thus the following duality theorem is immediate from Theorem 3.2.

***Theorem* 4.1** — *Assume that* $(t, z) \to f_x(t, z, x(t, z)) \in L^2([0, T] \times \Omega)$ *for every* $x \in W(0, T)$. *Under the above assumptions, if* $(x^\circ(0), x^\circ, u^\circ) \in C \times W(0, T) \times \mathcal{U}_{ad}$ *attains the infimum of* (P'), *then there exists* $p^\circ \in W(0, T)$ *such that* (p°, x°) *attains the supremum of* (D'). *Furthermore, the infimum of* (P') *is equal to the supremum of* (D').

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