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GEOMETRIC RESULTS IN POLYNOMIAL APPROXIMATION VIA THE 1-1 MINIMAX TRANSFORM

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This paper deals on a theory of extrernal circles in polynomial approximation. The first paper in connection with the theory of extremal circles in polynomial approximation was[4]. In such paper was introduced the called γ—q minimax transform. In this paper we work with the 1-1 minimax transform linked with minimax series considered in other papers[2, 5-6] which is in connection with the norm of the Besov space $B^{\infty}_{1,1}$.

The main goal of this paper is to prove several results in polynomial approximation. The existence of an extremal universal circle denominated the great Besov's circle, the extremal circles of functions, the existence of an hollow called the Besov's hollow, the existence of extremal crowns and finally certain special results for polynomials, in particular the existence of a sequence of extremal maximal circles.

Key Words : Minimaxs; Polynomial Approximation; q-Minimax Series; γ—q Minimax Series; Besov Spaces; Extremal Circles; Extremal Crowns

1. INTRODUCTION

For $\gamma > 0$ and $q \geq 1$ the γ—q minimax series of $f \in C[a, b]$ is given by

$$S_{\gamma, q}(f) = (2^{\gamma q - 1} E_0^q(f) + 3^{\gamma q - 1} E_1^q(f) + \dots + (n+2)^{\gamma q - 1} E_n^q(f) + \dots)^{1/q},$$

where $E_k(f)$ is the kth minimax of $f \in C[a, b]$, i.e.

$$E_k(f) = \operatorname*{Inf}_{p \in \Pi_k} \| f - p \|_\infty$$

and Π_k is the set of all polynomials of degree k at most. See [1], [2] and [7] for the importance of these weighted minimax series.

The weights $(n+2)^{\gamma q - 1}$ appear in the norm of the Besov spaces,

$$\| f \|_{B_\infty}^{\gamma, q} = \left(\| f \|_\infty^q + S_{\gamma, q}^q(f) \right)^{1/q},$$

(see [2]). These weights was considered by Pietsch in [7] for the construction of certain spaces of approximation which are generalized Besov spaces, its origin is an inequality by Stechking. The Besov's space $B^{\infty}_{\gamma, q}$ is the set of continuous functions in $[a, b]$ such that $S_{\gamma, q}(f) < \infty$.

If $\gamma q = 1$ then result the q-minimax series

$$S_q(f) = (E_0^q(f) + E_1^q(f) + \ldots + E_n^q(f) + \ldots)^{1/q},$$

and if $\gamma = q = 1$ result the minimax series of f,

$$S(f) \equiv S_{1,1}(f) = E_0(f) + E_1(f) + .. + E_n(f) + \ldots.$$

Recently the author has proved the existence of certain extremal circles in connection with the polynomial minimaxs in [4]. Closely to this circles certain Banach algebras was considered.

In this paper we prove several results in polynomial approximation, firstly it is proved the existence of an universal circle denominated the great Besov's circle, secondly, we assign an extremal circle to each function $f \in B^{\infty}_{1,1}$ in connection with the functions of $B^{\infty}_{1,1}$ with zeros in $[a, b]$ denominated the fundamental circle of f.

It is also proved the existence of an hollow, denominated the Besov' hollow which establishes a barrier to the radius of the fundamental circles of all the functions of $B^{\infty}_{1,1}$. Moreover, we prove the existence of an extremal universal crown denominated the great Besov's crown, we assign a crown to each function of $B^{\infty}_{1,1}$ and we prove the existence of certain restrictions in the relation between the radius of the crowns of functions of $B^{\infty}_{1,1}$ with zeros in $[a, b]$.

Finally we prove certain results for the polynomials in particular the existence of a sequence of extremal circles.

2. Preliminary Results

The following lemma and theorem are well known. See [1, pp. 210-211].

Lemma 1 — Let C_r designate the closed circle $|z - z_0| \leq r$. Suppose that $f(z)$ is analytic in C_r. If

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

then

$$\iint_{C_r} |f(z)|^2 \, dz = \pi \sum_{n=0}^{\infty} |a_n|^2 \frac{r^{2n+2}}{n+1}.$$

Theorem 1 — *Let* $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. *Then* $S = \pi \sum_{n=0}^{\infty} |a_n| \frac{r^{2n+2}}{n+1} < \infty$ *if and only if* $f \in L^2(C_r)$. *If* $f \in L^2(C_r)$ *then* $S = \|f\|^2$.

In [4] the author defines the γ–q minimax transform of $f \in B^{\infty}_{\gamma, q}$ by

$$F(f; \gamma, q)(z) = \sum_{n=0}^{\infty} a_n z^n$$

where

$$a_n = [(n+1)(n+2)^{\gamma q - 1} E_n^q(f)]^{1/2} \ (n \geq 0).$$

Then from Lemma 1

$$\int\int_{C_r} | F(f; \gamma, q)(z) |^2 dz = \pi r^2 \sum_{n=0}^{\infty} (n+2)^{\gamma q - 1} E_n^q(f) r^{2n}. \quad \text{... (2.1)}$$

In this paper, we consider the case $\gamma = q = 1$. Then the 1-1 minimax transform is given by

$$F(f)(z) = \sum_{n=0}^{\infty} a_n z^n,$$

with

$$a_n = ((n+1) E_n(f))^{1/2}.$$

Then

$$\int\int_{C_r} | F(f)(z) |^2 dz = \pi r^2 \sum_{n=0}^{\infty} E_n(f) r^{2n}.$$

Hence, we denote by $S_r(f)$ the series $\sum_{n=0}^{\infty} E_n(f) r^{2n}$.

***Theorem* 2** — *The next inequalities hold*

(*i*) $S_r(fg) \leq 2[\|f\|_{\infty} S_r(g) + \|g\|_{\infty} S_r(f) + S_r(f) S_r(g)]$

(*ii*) $S_r(fg) \leq 3[\|f\|_{\infty} S_r(g) + \|g\|_{\infty} S_r(f)]$

for all f and $g \in C[a, b]$ and $0 \leq r \leq 1$.

PROOF : Let p_k and q_k be the best approximations of f and g in Π_k respectively. Then

$$E_{2k}(fg) \leq \|f\|_{\infty} E_k(g) + \|g\|_{\infty} E_k(f) + E_k(f) E_k(g),$$

see [5] or [6]. Then

$$E_{2k}(fg)r^{4k} \leq \|f\|_\infty E_k(g)r^{2k} + \|g\|_\infty E_k(f)\, r^{2k} + E_k(f)\, r^{2k}\, E_k(g)\, r^{2k};$$

on the other hand

$$E_{2k+1}(fg)\, r^{4k+2} \leq E_{2k}(fg)\, r^{4k}.$$

Thus

$$S_r(fg) = \sum_{k=0}^{\infty} E_k(fg)\, r^{2k} \leq 2 \sum_{k=0}^{\infty} E_{2k}(fg)\, r^{4k}$$

$$\leq 2\left[\|f\|_\infty \sum_{k=0}^{\infty} E_k(g)\, r^{2k} + \|g\|_\infty \sum_{k=0}^{\infty} E_k(f)\, r^{2k} + \sum_{k=0}^{\infty} E_k(f)\, r^{2k} \sum_{k=0}^{\infty} E_k(g)\, r^{2k}\right]$$

and (i) it is proved

$$= 2[\|f\|_g S_r(g) + \|g\|_\infty S_r(f) + S_r(f)\, S_r(g)]. \qquad \text{... (2.2)}$$

Also

$$E_{2k}(fg) \leq \|f\|_\infty E_k(g) + 2\|g\|_\infty E_k(f),$$

See [6, pp. 320]. Then

$$E_{2k}(fg)\, r^{4k} \leq \|f\|_\infty E_k(g)\, r^{2k} + 2\|g\|_\infty E_k(f) r^{2k},$$

$$\sum_{k=0}^{\infty} E_k(fg)\, r^{2k} \leq 2 \sum_{k=0}^{\infty} E_k(fg) r^{2k} \leq \|f\|_\infty S_r(g) + 2\|g\|_\infty S_r(f),$$

and

$$\sum_{k=0}^{\infty} E_k(fg)\, r^{2k} \leq 2\|f\|_\infty S_r(g) + \|g\|_\infty S_r(f).$$

Consequently

$$S_r(fg) \leq 3[\|f\|_\infty S_r(g) + \|g\|_\infty S_r(f)].$$

Lemma 2 — If f have a zero in $[a, b]$ then $\|f\|_\infty \leq 2 S_r(f)$, $\forall\ r > 0$.

PROOF : Note that $E_0(f) = \|f - c\|_\infty$, being c a certain constant. Let $z_0 \in [a, b]$ such that $f(z_0) = 0$. Then

$$|c| = |f(z_0) - c| \leq \|f - c\|_\infty$$

and

$$\|f\|_\infty \leq \|f - c\|_\infty + |c| \leq 2E_0(f) \leq 2S_r(f).$$

As a consequence of this lemma and the Theorem 2(i) we have that

$$S_r(fg) \leq 10 S_r(f) S_r(g),$$

for all f and $g \in C[a, b]$ with zeros in $[a, b]$ and $0 \leq r \leq 1$. In fact we have a more fine result

Theorem 3 — *If f and $g \in C[a, b]$ with zeros in $[a, b]$ then*

$$S_r(fg) \leq 6 S_r(f)\, S_r(g), \text{ for all } 0 \leq r \leq 1.$$

PROOF : The proof is analogous to one which appears in [6, pp. 321]. For functions f and g with zeros in $[a, b]$ we have

$$E_0(fg) \leq 3 E_0(f)\, E_0(g),$$

and

$$E_{2k}(fg) \leq 2[E_0(f) E_k(g) + E_0(g) E_k(f) + E_k(f)\, E_k(g)],$$

see [6, pp. 321].

Then

$$E_{2k}(fg) r^{4k} \leq 2[E_0(f)\, E_k(g) r^{2k} + E_0(g)\, E_k(f) r^{2k} + E_k(f)\, r^{2k}\, E_k(g) r^{2k}].$$

We denote by $S_r^m(f)$ the $2(m+1)$th partial sum of $S_r(f)$, namely

$$S_r^m(f) = \sum_{k=0}^{2m} E_k(f) r^{2k},$$

taking in mind that $E_{2k+1}(fg) r^{4k+2} \leq E_{2k}(fg) r^{4k}$ and $E_1(fg) r \leq E_0(fg)$, result

$$S_r^m(fg) \leq 6 E_0(f)\, E_0(g) + 4 \left[\sum_{k=1}^{m} E_0(f)\, E_k(g) r^{2k} + E_0(g)\, E_k(f) r^{2k} + E_k(f)\, r^{2k}\, E_k(g) r^{2k} \right]$$

$$\leq 6 S_r^m(f)\, S_r^m(g).$$

Then the proof follows taking limits form n tending to infinity.

3. The Circle's Generating Functional

The circles's generating functional is defined by

$$\psi(h, r) = \frac{1}{\pi r^3} \iint_{C_r} |F(h)(z)|^2\, dz,\ h \in B^{\infty}_{1,1},\ 0 < r \leq 1.$$

Let f and $g \in B^{\infty}_{1,1}$ with zeros in $[a, b]$. Then

$$\frac{1}{\pi r^2} \iint_{C_r} |F(fg)(z)|^2\, dz = S_r(fg) \le 3\,[\|f\|_\infty S_r(g) + \|g\|_\infty S_r(f)]$$

$$\le 3(2+2)\, S_r(f)\, S_r(g)$$

$$= \frac{12}{\pi r^2} \iint_{C_r} |F(f)(z)|^2\, dz \frac{1}{\pi r^2} \iint_{C_r} |F(g)(z)|^2\, dz.$$

Then it is clear that there exists an optimus constant $\mu > 0$ such that

$$\iint_{C_r} |F(fg)(z)|^2\, dz \le \frac{\mu}{\pi r^2} \iint_{C_r} |F(f)(z)|^2\, dz \iint_{C_r} |F(g)(z)|^2\, dz \qquad \dots (3.1)$$

for all f and $g \in B^{\infty}_{1,1}$ with zeros in $[a, b]$ and $0 < r \le 1$.

The constant μ optimus independent of r is

$$\mu = \operatorname*{Sup}_{0<r\le 1} \Big\{ \pi r^2 \iint_{C_r} |F(fg)(z)|^2\, dz \Big/ \iint_{C_r} |F(f)(z)|^2\, dz \iint_{C_r} |F(g)(z)|^2\, dz :$$

$$f,\ g \in B^{\infty}_{1,1} \text{ with zeros in } [a, b] \text{ and } f, g \ne 0\}.$$

Then we have that

$$\psi(fg, r) \le \mu r \psi(f, r)\, \psi(g, r) \text{ for all } f \text{ and } g \in B^{\infty}_{1,1} \text{ and } 0 \le r \le 1.$$

Definitions 1 — The great Besov's circle is the circle with centre in the origin and radius $\tau = \frac{1}{\mu}$.

On such circle we have

$$\psi(fg, r) \le \psi(f, r)\, \psi(g, r) \text{ for all } f \text{ and } g \in B^{\infty}_{1,1} \text{ and } r \le \tau.$$

Let $f \in B^{\infty}_{1,1}$ be a fixed function. Let α_1 be determined by the relation

$$\|f\|_\infty = \frac{\alpha_1}{\pi r^2} \iint_{C_r} |F(f)(z)|^2\, dz.$$

Then

$$\iint_{C_r} |F(fg)(z)|^2\, dz \le 3\,\frac{(2+\alpha_1)}{\pi r^2} \iint_{C_r} |F(f)(z)|^2\, dz \iint_{C_r} |F(g)(z)|^2 dz,$$

for all $g \in B^{\infty}_{1,1}$ with zeros in $[a, b]$. Thus there exists an optimum constant $\mu(f) > 0$ such that

$$\int\int_{C_r} |F(fg)(z)|^2 dz \leq \frac{\mu(f)}{\pi r^2} \int\int_{C_r} |F(f)(z)|^2 dz \int\int_{C_r} |F(g)(z)|^2 dz.$$

For all $g \in B^{\infty}_{1,1}$ with zeros in $[a, b]$ and $0 \leq r \leq 1$. Thus

$$\psi(fg, r) \leq \mu(f)\, r\psi(f, r)\, \psi(g, r) \text{ for all } g \in B^{\infty}_{1,1} \text{ with zeros in } [a, b] \text{ and } 0 < r \leq 1.$$

The radius of the fundamental circle of f is the biggest value of $r \leq 1$ such that $\mu(f)r \leq 1$. If $\mu(f) \leq 1$ then $r = 1$ and if $\mu(f) > 1$ then $r = \frac{1}{\mu(f)}$.

Analagously we can define the fundamental circle of a finite number of functions $\{f_1, f_2, \ldots, f_n\} \subseteq B^{\infty}_{1,1}$, let $\mu(f_1, f_2, \ldots, f_n)$ the optimum constant such that

$$\int\int_{C_r} |F(f_i g)(z)|^2 dz \leq \frac{\mu(f_1, f_2, \ldots, f_n)}{\pi r^2} \int\int_{C_r} |F(f_i)(z)|^2 dz \int\int_{C_r} |F(g)(z)|^2 dz,$$

for all $g \in B^{\infty}_{1,1}$ with zeros in $[a, b]$, $1 \leq i \leq n$ and $0 < r \leq 1$.

The radius of the fundamental circle of $f_1, f_2, \ldots, f_n$ is the biggest value of $r \leq 1$ such that $\mu(f_1, f_2, \ldots, f_n)\, r \leq 1$.

4. The Besov's Hollow for Functions with Zeros

In this section we prove that the radius of the fundamental circles of the functions of $B^{\infty}_{1,1}$ with zeros do not make full the unit disk, in fact we prove that there exists an hollow which we denominate the Besov's hollow.

Theorem 4 — *The radius of the fundamental circles of a function* $f \in B^{\infty}_{1,1}$ *(or a finite number) with zeros in [a, b] is* $\geq \frac{1}{6}$.

PROOF : Suppose that f have zeros in $[a, b]$, from theorem 3,

$$\frac{S_r(f)\, S_r(g)}{S_r(fg)} \geq \frac{1}{6},$$

for all $g \in B^{\infty}_{1,1}$ with zeros in $[a, b]$. Then

$$\frac{1}{\mu f} = \text{Sup}\left\{ \frac{S_r(f)\, S_r(g)}{S_r(fg)} : g \in B^{\infty}_{1,1} \text{ with zeros in } [a, b], f, g \neq 0 \text{ and } C < r \leq 1 \right\} \geq \frac{1}{6}.$$

The result also holds for a finite number of functions.

4. The Maximal Crowns

Let $0 \le r_1 < r_2 \le 1$. Then let us consider the crown

$$C(r_1, r_2) = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$$

and the integrals

$$\int\int_{C_{(r_1, r_2)}} |F(f)(z)|^2 \, dz, f \in B^{\infty}_{1,1}.$$

Note that

$$\int\int_{C(r_1, r_2)} |F(f)(z)|^2 \, dz = \int\int_{C_{r_2}} |F(f)(z)|^2 \, dz - \int\int_{C_{r_1}} |F(f)(z)|^2 \, dz$$

$$= \pi r_2^2 \sum_{n=0}^{\infty} E_n(f)\, r_2^{2n} - \pi r_1^2 \sum_{n=0}^{\infty} E_n(f)\, r_1^{2n}$$

$$= \pi(r_2^2 - r_1^2) \sum_{n=0}^{\infty} E_n(f)\,(\lambda r_2^{2n} - r_1^{2n}(\lambda - 1)),$$

where

$$\lambda = \frac{r_2^2}{r_2^2 - r_1^2}.$$

Note that $\lambda \ge 1$ and $\lambda\, r_2^{2n} + r_1^{2n}(1 - \lambda) = r_1^{2n} + \lambda\left(r_2^{2n} - r_1^{n} \right) > 0,\ \forall_n$.

Then let us consider $S_{(r_2, r_1)}(f) = \sum_{n=0}^{\infty} E_n(f)\,(\lambda r_2^{2n} + r_1^{2n}(1 - \lambda))$.

Hence

$$\int\int_{C(r_1, r_2)} |F(f)(z)|^2 \, dz = \pi(r_2^2 - r_1^2)\, S_{(r_2, r_1)}(f).$$

Note that

$$\frac{r_2^{4n}\lambda + r_1^{4n}(1 - \lambda)}{r_2^{2n}\lambda + r_1^{2n}(1 - \lambda)} \le 2.$$

Let us consider the function

$$\varphi(\lambda) = \frac{r_2^{4n}\lambda + r_1^{4n}(1-\lambda)}{r_2^{2n}\lambda + r_1^{2n}(1-\lambda)}.$$

Then

$$\varphi(1) = r_2^{2n} \le 1,\ \varphi(+\infty) = r_1^{2n} + r_2^{2n} \le 2 \text{ and } \varphi \text{ is an increasing function of } \lambda.$$

Similarly

$$\frac{r_2^{4n+2}\lambda + r_1^{4n+2}(1-\lambda)}{r_2^{2n}\lambda + r_1^{2n}(1-\lambda)} = \frac{r_2^{4n+4}\lambda - r_1^{4n+4}(1-\lambda)}{r_2^{2n+2} - r_1^{2n+2}} = r_2^{2n+2} + r_1^{2n+2} \le 2.$$

Then from (2.2)

$$E_{2n}(fg)\,(r_2^{4n}\lambda + r_1^{4n}(1-\lambda) \le 3\,(\|f\|_\infty E_n(g) + \|g\|_\infty E_n(f)$$

$$\times\left[r_2^{2n}\lambda + r_1^{2n}(1-\lambda)\right]. \qquad \dots (4.1)$$

and

$$E_{2n+1}(fg)\,(r_2^{4n+2}\lambda + r_1^{4n+2}(1-\lambda)) \le 3\,\|f\|_\infty E_n(g) + \|g\|_\infty E_n(f)$$

$$\times\left[r_2^{2n}\lambda + r_1^{2n}(1-\lambda)\right]. \qquad \dots (4.2)$$

Then analyzing the proof of theorem 2 (ii) it is clear that there exists a constant $L > 0$ such that

$$\int\int_{C(r_1, r_2)} |F(fg)(z)|^2\, dz$$

$$\le L\left[\lambda \|f\|_\infty \int\int_{C(r_1, r_2)} |F(g)(t)|^2\, dt + \|g\|_\infty \int\int_{C(r_1, r_2)} |F(f)(\lambda)|^2\, dt\right],$$

for all f and $g \in B^{\infty}_{1,1}$. Then there exists a constant $v > 0$ such that

$$\int\int_{C(r_1, r_2)} |F(fg)(z)|^2\, dz$$

$$\le \frac{v}{\Pi(r_2^2 - r_1^2)} \int\int_{C(r_1, r_2)} |F(g)(z)|^2\, dz \int\int_{C(r_1, r_2)} |F(g)(z)|^2\, dz],$$

for all f and $g \in B^{\infty}_{1,1}$ with zeros in $[a, b]$.

5. The Crown's Generating Functional

Let $s = (r_2^2 - r_1^2)^{1/2}$. Then such functional is defined by

$$\zeta(h, r_1, r_2) = \frac{1}{\pi s^3} \int\int_{C(r_1, r_2)} |F(h)(z)|^2 dz, h \in B_{1,1}^{\infty}, 0 \le r_1 < r_2 \le 1.$$

From (4.1)

$$\zeta(fg, r_1, r_2) \le vs\zeta(f, r_1, r_2)\, \zeta(g, r_1, r_2),$$

for all f and $g \in B_{1,1}^{\infty}$ with zeros in $[a, b]$.

Definition 3 — The great Besov's crown is the crown with centre in the origin and radius $r_2 = 1$ and $r_1 = (r_2^2 - s^2)^{1/2}$ with $s = \frac{1}{v}$ if $v \ge 1$ and $r_1 = 1$ if $v \le 1$.

Similarly to the definitions in section 3, we can assign an extremal crown to each function $f \in B_{1,1}^{\infty}$ (or a finite number of functions $\{f_1, f_2, \ldots, f_n\} \subseteq B_{1,1}^{\infty}$) in connection with the functions with zeros in $[a, b]$. Such crown is denominated the fundamental crown of f (of $\{f_1, f_2, \ldots, f_n\}$).

6. Acosta's Rings

The F. Pérez Acosta rings are small crowns continued in the unit disk associated to the functions of $B_{1,1}^{\infty}$ with zeros in $[a, b]$. We have defined

$$S_{(r_2, r_1)}(f) = \sum_{n=0}^{\infty} E_n(f)\,(\lambda r_2^{2n} + r_1^{2n}(1 - \lambda)),$$

from (4.1) and (4.2) result

$$S_{(r_2, r_1)}(fg) \le 6 \left(\|f\|_{\infty} S_{)r_2, r_1)}(g) + \|f\|_{\infty} S_{(r_2, r_1)}(f) \right),$$

$$\forall f, g \in B_{1,1}^{\infty}.$$

Moreover, if f and g have zeros in $[a, b]$

$$\|f\|_{\infty} \le \frac{2}{\Pi\left(r_2^2 - r_1^2\right)} \int\int_{C_r} |F(f)(t)|^2 dt$$

and similarly for g.

Then

$$S_{(r_2, r_1)}(fg) \le 12\, S_{(r_2, r_1)}(f)\, S_{(r_2, r_1)}(g)$$

and

$$\iint_{C(r_1, r_2)} |F(fg)(t)|^1 \, dt \leq \frac{12}{\Pi\left(r_2^2 - r_1^2\right)} \iint_{C(r_1, r_2)} |F(f)(z)|^2 \, dt \iint_{C(r_1, r_2)} |F(g)(t)|^2 \, dt$$

and in analogy with the theorem 4, for the s in the definition 3 we obtain the restriction $s \geq \frac{1}{12}$. Thus the great Besov's crown and similarly the fundamental crowns of functions of $B_{1,1}^{\infty}$ are with zeros in $[a, b]$ are rings (small crowns) with radius

$$r_1 \leq \left(\frac{143}{144}\right)^{1/2} \text{ and } r_2 = 1,$$

These rings of functions are denominated the Pérez Acosta rings. All them contain the great Besor's crown and with zeros cannot be very small because $r_2 - r_1 \geq 3.\ 478 \times 10^{-3}$.

7. A Sequence of Extremal Circles in Connection with the Polynomials

For each n let us consider the polynomials Π_n of degree n at most.

It is clear that for $f \in \Pi_n$, $S_r(f) = \sum_{k=0}^{n-1} E_k(f) r^k$ is a finite same. In this case we have similar inequalities to the obtained in theorem 2, for $r \leq M$, with $M \geq 1$.

Theorem 5 — The next inequalities holds

$$\text{(i) } S_r(fg) \leq (1 + M^2)\,[M^{2(n-1)}\,(\|f\|_\infty S_r(g) + \|g\|_\infty S_r(f)) + S_r(f)\,S_r(g)].$$

$$\text{(ii) } S_r(fg) \leq \frac{3}{2}(1 + M^2)\,M^{2(n-1)}\,[\|f\|_\infty S_r(g) + \|g\|_\infty S_r(f)].$$

for all f and $g \in \Pi_n$ and $0 \leq r \leq M$ and $M \geq 1$.

PROOF : Let f and $g \in \Pi_n$. Let p_k and q_k be the best approximations of f and g in Π_k respectively, then

$$E_{2k}(fg) \leq \|f\|_\infty E_k(g) + \|g\|_\infty E_k(f) + E_k(f)\,E_k(g),$$

and

$$E_{2k}(fg) r^{4k} \leq M^{2k}\,(\|f\|_\infty E_k(g) r^{2k} + \|g\|_\infty E_k(f)\, r^{2k}) + E_k(f)\, r^{2k}\, E_k(g) r^{2k};$$

on the other hand

$$E_{2k+1}(fg)\, r^{4k+2} \leq M^2\, E_{2k}(fg)\, r^{4k}.$$

Thus

$$S_r(fg) = \sum_{k=0}^{2n-1} E_k(fg)\, r^{2k} \le (1+M^2) \sum_{k=0}^{n-1} E_{2k}(fg)\, r^{4k}$$

$$\le (1+M^2)\left[M^{2(n-1)} \left(\| f \|_\infty \sum_{k=0}^{n-1} E_k(g)\, r^{2k} + \| g \|_\infty \sum_{k=0}^{n-1} E_k(f)\, r^{2k} \right) + \sum_{k=0}^{n-1} E_k(f)\, r^{2k} \sum_{k=0}^{n-1} E_k(g)\, r^{2k} \right]$$

$$= (1+M^2)\,[M^{2(n-1)}\,(\| f \|_\infty S_r(g) + \| g \|_\infty S_r(f)) + S_r(f)\, S_r(g)].$$

Also from the inequality

$$E_{2k}(fg) \le \| f \|_\infty E_k(g) + 2\,\| g \|_\infty E_k(f),$$

we have that

$$E_{2k}(fg) r^{4k} \le M^{2k}\,(\| f \|_\infty\, E_k(g) r^{2k} + 2\,\| g \|_\infty E_k(f) r^{2k}),$$

and

$$\sum_{k=0}^{2n-1} E_k(fg)\, r^{2k} \le (1+M^2) \sum_{k=0}^{n-1} E_k(fg) r^{2k}$$

$$\le (1+M^2)\, M^{2(n-1)}\,[\| f \|_\infty S_r(g) + 2\,\| g \|_\infty S_r(f)],$$

consequently

$$S_r(fg) \le \frac{3}{2}(1+M^2)\, M^{2(n-1)}\,[\| f \|_\infty S_r(g) + \| g \|_\infty S_r(f)].$$

Corollary 1 — (*i*) $S_r(fg) \le (1+M^2)\,[4M^{2(n-1)}+1]\, S_r(f)\, S_r(g)$

(*ii*) $S_r(fg) \le 6(1+M^2)\, M^{2(n-1)}\, S_r(f)\, S_r(g)$

for all f and $g \in \Pi_n$ with zeros in $[a, b]$ $0 \le r \le M$ and $M \ge 1$.

Corollary 2 — For each $n \in I\!N$ and $M > 0$ there exists a $K \equiv K(M, n) > 0$ minimal such that

(*i*) $\iint_{C_r} | F(fg)(z) |^2\, dz \le \frac{K(M, n)}{\pi r^2} \iint_{C_r} | F(f)(z) |^2\, dz \iint_{C_r} | F(g)(z) |^2\, dz$

(*ii*) $\psi(f\, g\, r) \le K(M, n) r \psi(f, r) \psi(g, r)$

for all f and $g \in \Pi_n$ with zeros in $[a, b]$ and $0 \leq r \leq M$.

PROOF :

Case 1 — $0 < M \leq 1$. It follows from the inequality (3.1). We can take $K(M, n) = \mu$.

Case 2 — $M \geq 1$. It follows from corollary 1.

Definition 4 — The dual circle of C_M induced by Π_n is C_τ where τ is the biggest value of $r = \tau \leq M$ such that

$$K(M, n)\,\tau \leq 1.$$

On such circle we have

$$\psi(fg, r) \leq \psi(f, r)\,\psi(g, r), \text{ for all } f \text{ and } g \in \Pi_n \text{ and } 0 \leq r \leq \tau,$$

and it is the biggest circle contained in C_M with this property.

Theorem 6 — *(A fixed circle's theorem) For each $n \in I\!N$ there exists an $M > 0$ maximal such that the dual circle of C_M is the same C_M.*

PROOF : Note that

$$K(M, n) \leq \mu \text{ for } 0 \leq M \leq 1,$$

taking $M = r \equiv s$ we have, $K(s, n)s$ tends to zero when s tends to zero.

On the other hand it is clear that

$$K(M, n) = \text{Sup}\left\{ \pi r^2 \int\int_{C_r} |F(fg)(z)|^2\,dz \Big/ \int\int_{C_r} |F(f)(z)|^2\,dz \int\int_{C_r} |F(g)(z)|^2\,dz \right\},$$

$$f, g \in \Pi_n\; f,\, g \text{ with zeros in } [a, b]\; f,\, g \neq 0\} \text{ and } 0 < r \leq M).$$

Then $K(s, n)\,s$ tends to infinity when s tends to infinity. Moreover, $K(s, n)$ is an increasing function of s. Then there exists an s maximal such that

$$K(s, n)s \leq 1.$$

REFERENCES

1. Z. Ditzian and V. Totik, *Proc. Amer. Math. Soc.* **104** (1998), 1059-1066.
2. F. Pérez Acosta, *J Comput. Appl. Math.* **85** (1997), 315-23.
3. F. Pérez Acosta, *J Comput. Appl. Math.* **81** (1997) 101-106.
4. F. Pérez Acosta, The Besov and F. Pérez Acosta's circles. Preprint submitted.
5. F. Pérez Acosta, *J Comput. Appl. Math.* **83** (1997), 55-69.
6. F. Pérez Acosta, J. M. Almira Picazo, *J Comput. Appl. Math.* **99** (1998), 319-25.
7. A. Pietsch, *J Approx. Theory* **32** (1985), 115-134.