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CONSTRUCTION OF *r*-REGULAR SINGULAR GRAPHS

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A graph G is singular if the adjacency matrix of G is a singular matrix. For each integer $r \geq 2$, we determine all values of n= for which there exist r-regular connected singular graphs of order n. Our constructions make use of 1-factors in even complete graphs.

Key Words : Singular graphs; Set; Integer

1. INTRODUCTION

A graph G is a pair $\langle V(G), E(G) \rangle$ where $V(G)$ is a non-empty set of vertices and $E(G)$ is a set of unordered pairs xy called edges, where $x, y \in V(G)$. If $xy \in E(G)$, then x and y are adjacent. If $x \in V(G)$, the number of vertices adjacent to x is called the degree of x in G, denoted by $deg_G(x)$. The cardinality of $V(G)$ is the order of G, and the number of edges in G is the size of G. If the vertices of G are denoted by $x_1, x_2, \ldots, x_n$, then the adjacency matrix of G is the $n \times n$ matrix $A(G) = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } x_i x_j \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

If $A(G)$ is singular, then the graph G is a singular graph; otherwise, G is non-singular.

If r is a nonnegative integer, then G is r-regular if for each $x \in V(G)$, we have $\deg_G(x) = r$. Thus, the cycle C_n is 2-regular, while the complete graph K_n of order n is $(n-1)$-regular.

Note that $A(G)$ depends on a given ordering $x_1, x_2, ..., x_n$ of the vertices of G. If M_1 and M_2 are two adjacency matrices based on two orderings of the vertices of G, it is easy to see that M_1 and M_2 are either both singular or both nonsingular. Indeed, the determinants of M_1 and M_2 are either equal or opposite in sign.

In this paper, we determine for what orders n connected r-regular singular graphs exist.

2. PRELIMINARY RESULTS

If G is a graph and $x \in V(G)$, define the neighbour set of x by

$$N(x) = \{y \in V(G) : xy \in E(G)\}$$

The following result gives a sufficient condition for a graph G to be singular, and forms the basis for the constructions that will be used in this paper:

***Theorem* 1** — *([1]) If a graph G has two distinct vertices x and y such that $N(x) = N(y)$, then G is singular.*

PROOF : Let x and y be two vertices of G such that $N(x) = N(y)$. Then the rows of $A(G)$ corresponding to these two vertices have exactly the same entries and so $A(G)$ is singular. Consequently, G is singular. □

***Theorem* 2** — *If G has connected components $G_1, G_2, ..., G_p$, then G is singular if and only if at least one G_i is singular.*

PROOF : This follows directly from the fact that the adjacency matrix $A(G)$ can be represented as follows :

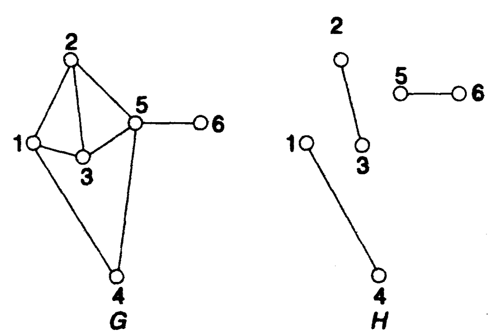
$$A(G) = \begin{bmatrix} A(G_1) & \dots & 0 \\ \dots & A(G_2) & 0 \\ 0 & \dots & \dots \\ 0 & \dots & A(G_p) \end{bmatrix}$$

Thus, G is singular if and only if at least one of its components is singular. □

This showed that we may restrict our study of singular graphs to connected graphs.

2.1 Partition the Edges of an Even Complete Graph into 1-Factors

A 1-factor of G is a 1-regular spanning subgraph of G. For example, the following diagram illustrates a graph G and a 1-factor H.

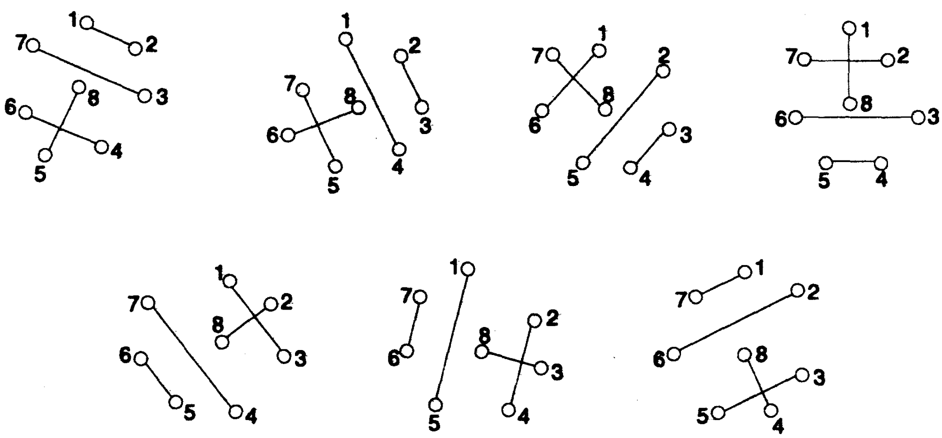


If n is even, then the edges of the complete graph K_n can be partitioned into $n - 1$ 1-factors. In order to do so, construct a regular polygon on $n - 1$ sides and consecutively label the vertices with 1, 2, 3, ..., $n - 1$. Place another vertex at the center of this polygon and label this n. Since the number of sides is odd, no two sides of the polygon are parallel. Starting at the side corresponding to the edge joining 1 and 2, consider all chords joining two vertices parallel to the given side. The vertex left without a pair is joined to the central vertex. The edges corresponding to the vertex pairs form a 1-factor for K_n. To obtain the other 1-factors, repeat the procedure with each of the remaining sides of the polygon.

For example, when $n = 8$, we have the following decomposition of K_n into 1-factors:

Let n be even and let $H_1, H_2, \dots, H_{n-1}$ form a partition of K_n into 1-factors. Then

$$G_1 = K_n \setminus H_1, G_2 = G_1 \setminus H_2, \dots, G_{n-1} = G_{n-2} \setminus H_{n-1}$$



are regular graphs with degrees $n - 2$, $n - 3$, ..., 0, respectively.

3. Main Results

Let G be a connected r-regular singular graph of order n, where $r \geq 2$. If $r = n - 1$, then $G = K_n$, which is known to be non-singular. Thus, we must have $r < n - 1$. Moreover, if r is odd, then n is even, and so the minimum possible order for a connected r-regular singular graph is $r + 3$.

If $r = 2$, then $G = C_n$, and we have the following known result:

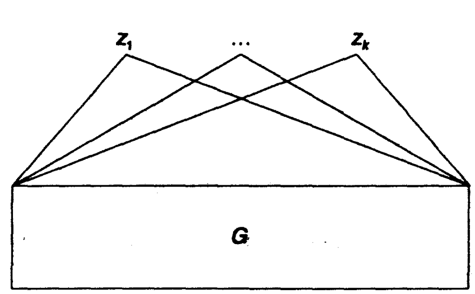
Theorem 3 — *([2]) C_n is singular if and only if $n \equiv 0$ mod 4.*

Thus, the remaining problem is to investigate the existence of connected r-regular singular graphs for $3 \leq r \leq n - 2$. The solution is given in the following results:

Theorem 4 — *Let $r \geq 4$ be an even integer. Then there exist connected r-regular singular graphs of order n for all $n \geq r + 2$.*

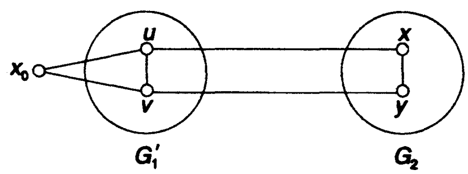
PROOF : The proof is by construction, and we consider the following cases:

Case 1 — If $n = r + k$, where $2 \leq k \leq r$, consider K_r, and delete $k - 1$ 1-factors from K_r to obtain a $(r - k)$-regular graph G of order r. Add k new vertices $z_1, z_2, \dots, z_k$ to G and join each of these vertices to all the vertices of G (see diagram below). This produces a graph G' which is connected, r-regular and of order $n = r + k$. If x, y are two of the k vertices added to G, then

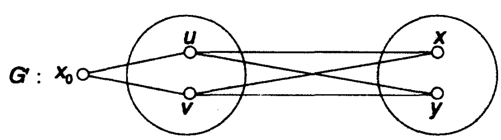


$N(x) = N(y)$, and hence G' is singular, by Theorem 1.

Case 2 — If $n = 2r + 1$, consider two disjoint copies of K_r, and call these G_1 and G_2. Delete one 1-factor from G_1 to obtain the graph G_1', and let $\phi : V(G_1') \rightarrow V(G_2)$ be a bijection. For each $v \in V(G_1')$, add the edge joining v and $\phi(v)$. Add a new vertex x_0 and join this to every vertex of G_1'. The resulting graph G is connected, and r-regular of order $2r + 1$.

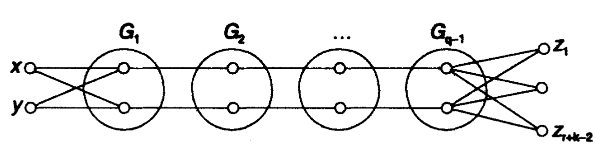


Let $u, v \in V(G_1')$ be such that $uv \in E(G)$. Let $\phi(u) = x$ and $\phi(v) = y$. Since $G_2 = K_r$, $xy \in E(G)$. Delete the edges uv and xy and replace them by uy and xv. Call the resulting graph G' (see the diagram below). Since $N(x) = N(y)$, G is connected r-regular graph of order $n = 2r + 1$.



Case 3 — If $n > 2r + 1, n = qr + k$, where $q \geq 2, 2 \leq k < r - 1$, consider q disjoint copies of K_r. Delete one 1-factor from each of copy 2 to copy $(q - 1)$ and k 1-factors from copy q. Call the resulting graphs $G_2, G_3, \ldots, G_q$, and call the first copy G_1. Let $\phi_i : V(G_i) \rightarrow V(G_{i+1})$, $i = 1, 2, \ldots, q - 1$ be a bijection. For each $v \in V(G_i)$, add the edge joining v and $\phi_i(v)$]. Add k vertices to G_q and join these to each of the vertices of G_q. Call the resulting graph G. If x and y are two of the k vertices added to G_q, then $N(x) = N(y)$, and G is a connected r-regular graph of order n.

Case 4 — If $n = qr + k, q > 2, k = 0$ or 1, consider $q - 1$ disjoint copies of K_r. Delete two 1-factors from copy 1 and $r + k - 2$ factors from copy $(q - 1)$. Delete one 1-factor from each of copy 2 to copy $(q - 2)$. Call the resulting graphs $G_1, G_2, \ldots, G_{q-1}$. Add two vertices x and y to G_1 and join these to each of the vertices of G_1. Add $r + k - 2$ vertices to G_{q-1} and join these to each of the vertices of G_{q-1}. For $i = 1, 2, \ldots, q - 2$, let $\phi_i : V(G_i) \rightarrow V(G_{i+1})$ be a bijection. For each $v \in V(G_i)$, add the edge joining v and $\phi_i(v)$. We have $N(x) = N(y)$, so the resulting graph is connected, r-regular and singular.



This completes the proof of Theorem 3. □

When r is odd, we shall use the following lemma :

Lemma 1 — Let $n \geq 3$ be any integer. and let r be an even integer r with $2 \leq r < n$. Then there exists a connected r-regular graph of order n.

PROOF : If $r = 2$, we take the cycle C_n which is clearly a connected 2-regular graph of order n. If $r = 2t$, $t > 1$, let C_n^t be the graph obtained from the cycle C_n by adding an edge joining two vertices x and y in C_n whenever they are joined by a path in C_n whose length is at most t. It is easy to see that C_n^t is a connected r-regular graph of order n. □

Theorem 5 — *Let $r \geq 3$ be an odd integer. There exist connected, r-regular singular graphs of order n only for $n = r + 3, r + 5, \ldots$*

PROOF : We have noted earlier that when the degree r is odd, the order of the graph is necessarily even. We consider the following cases:

Case 1 — $n = r + k$, $k = 3, 5, \ldots, r - 2$, $r \geq 5$. Since r and k are both odd and $r > k$, $r - k = p$ is a positive even integer. Let G be a connected p-regular graph of order r. Take k new vertices $z_1, z_2, \ldots, z_k$ and join each q_i by an edge to each vertex of G. Then the resulting graph G' is of order $r + k = n$. Furthermore, each z_i has degree r and each vertex of G' in G has degree $p + k = r$. Thus G' is a connected r-regular graph. If x and y are any two distinct vertices in $\{z_1, z_2, \ldots, z_k\}$, then $N(x) = N(y) = V(G)$. Thus, G' is singular.

Case 2 — $n = qr + k, q \geq 2, 0 \leq k < r$. Since n is even and r is odd, q and k have the same parity, and so $q + k$ is even.

Subcase 2.1 — $0 \leq k \leq r - 2$. Let $G_1 = K_{r-1}$ and let G_q be obtained from another copy of K_{r-1} by removing k edge-disjoint 1-factors. For $2 \leq j \leq q - 1$, let G_j be obtained from a copy of K_{r-1} by deleting one 1-factor. Let $y_1, y_2, \ldots, y_q$ be new vertices and for each $i = 1, 2, \ldots, q$, join y_i by an edge to each vertex of G_i. For $i = 1, 2, \ldots, q - 1$, let $\phi_i : V(G_i) \to V(G_{i+1})$ be a bijection. For each i and for each vertex t of G_i, add the edge joining t and $\phi_i(t)$]. Let $z_1, z_2, \ldots, z_k$ be k new vertices and join each z_i by an edge to each vertex of G_q. The resulting graph is a connected graph of order $n = qr + k$. The vertices $y_1, y_2, \ldots, y_q, z_1, z_2, \ldots, z_k$ all have degree $r - 1$ while the other vertices have degree r. Since the number of vertices with degree $r - 1$ is $q + k$, which is even, and these vertices are mutually non-adjacent, we can pair them off and form an edge with each pair. This would make their degrees all equal to r. Now, let x and y be any two vertices in G_1. Since G_1 is complete, x and y are adjacent. Let $\phi_1(x) = u$ and $\phi_1(y) = v$. We may assume that u and v are adjacent in G_2. Remove the edges xy and uv and then add the edges xv and yu. The final graph G is still r-regular and $N(x) = N(y)$. Therefore, G is singular.

Subcase 2.2 — $k = r - 1$. Let G_1 be obtained from K_{r-1} by removing two edge-disjoint 1-factors and let G_q be obtained from another copy of K_{r-1} be deleting $k - 2$ edge-disjoint 1-factors. For $2 \leq j \leq q - 1$, let G_j be obtained from a copy of K_{r-1} by deleting one 1-factor. Let a and b be new vertices each joined by an edge to each vertex of G_1 ad to each other. Add the edge ab. For $i = 1, 2, \ldots, q - 1$, let $\phi_i : V(G_i) \to V(G_{i+1})$ be a bijection. Join the vertices t and $\phi_i(t)$ for each $t \in V(G_i)$, for each i. Let $y_1, y_2, \ldots, y_q$ be new vertices and join y_i by an edge to each vertex of G_i. Let $z_1, z_2, \ldots, z_{k-2}$ be new vertices and join each z_i by an edge to each vertex of G_q. In the resulting graph, all vertices have degree r except the y_i and the z_i which have degree $r - 1$. Since their number is $q + k - 2$, which is even, we can pair them off and form an edge out of each pair. This produces a connected r-regular graph of order n. Just as in Subcase 2.1, we let x and y be distinct vertices in G_1 such that $\phi_1(x) = u$ and $\phi_1(y) = v$ are adjacent vertices in G_2.

We delete the edges xy and uv and add the edges xv and yu. We then see that $N(x) = N(y)$ and hence we have a singular graph. □

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