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LEAST-SQUARES MIXED FINITE ELEMENT METHOD FOR SOBOLEV EQUATIONS*

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A least-squares mixed finite element (LSM) method is formulated for a class of Sobolev equations in two dimensional domains. Optimal L^2-error estimates are derived under the standard regularity assumption on the finite element partition (the LBB-condition is not required). Optimal L^2 estimates are derived under the conventional Raviart-Thomas spaces.

Key Words : Least Squares; Mixed Finite Element Method; Error Estimates

1. INTRODUCTION

In this paper, we consider the following Sobolev equations :

$$c(x)\frac{\partial u}{\partial t} - \nabla\cdot(a(x)\nabla u) - \nabla\cdot\left(b(x)\nabla\frac{\partial u}{\partial t}\right) = f(x, t),\ (x, (t) \in \Omega\times J = [0, T] \qquad \text{... (1.1a)}$$

$$u\,|_{\partial\Omega} = 0,\ (x, t) \in \Omega\times(0, T) \qquad \text{... (1.1b)}$$

$$u(x, 0) = u_0(x),\ x\in\Omega. \qquad \text{... (1.1c)}$$

When $\Omega\subset R^2$ is a bounded domain, with boundary Γ. A general theory of the least-squares method has been developed by Aziz *et al.*[1]. The most important advantage leads to a symmetric positive definite problem. Bramble and Nitshe presented a least-squares method for Dirichlet problems[2]. The method generates a positive definite linear system with condition number $O(h^{-2})$ but requires C^1 smoothness of the finite element spaces. So we can find numerical solution by using

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various rapid and effective methods. On the other hand, the mixed finite element method has been found very useful in some practical problems, because the solution u and the flux σ can be obtained directly. This method has been developed previously (see [3]). But this method must be subjected to the LBB condition.

The LBB condition is also the compatibility condition of finite element spaces. Let Q_h and V_h be two finite-dimensional spaces, such that

$$Q_h \subset H(\text{div}; \Omega); \; V_h \subset L^2(\Omega).$$

Lemma — (see [3]) Assume that

$$\left.\begin{array}{l} q_h \in Q_h, \\ \forall \qquad\qquad\qquad\qquad \Rightarrow \text{div}\, q_h = 0 \\ \quad v_h \in V_h, \int v_h \,\text{div}\, q_h \, dx = 0 \end{array}\right\}$$

and there exists a constant $\alpha > 0$ such that

$$\forall\, v_h \in V_h \quad \sup_{q_h \in Q_h} \frac{\int_\Omega v_h \,\text{div}\, q_h \, dx}{\| q_h \|_{H(\text{div};\, \Omega)}} \geq \alpha \| v_h \|_{0,\Omega}. \qquad \text{... (1.2)}$$

Then the mixed variational formulation has a unique solution.

This indicates that if finite-dimensional spaces subject to (1.2), the equation has a unique mixed finite element solution. (1.2) can also be regarded to be the LBB condition. It restricts the choice of approximation subspaces.

In the least-squares mixed (LSM) methods approach a least-squares residual minimization is introduced for the mixed system in u and σ. This method has some advantages as follows: Firstly, the LSM method is not subject to the LBB condition. It has been proved in [4]. We can choose various approximation spaces, in particular, the case of differing polynomial degrees for u_h and σ_h is analyzed. Secondly, when approximation spaces similar to, but not necessarily the same as, the Raviart Thomas spaces are used (the LBB condition is not required), we can get optimal error estimates. In this paper, we shall prove this. Thirdly, because the discrete system is symmetric positive definite, some rapid methods for systems are suggested.

Recently, there have been several related studies of LSM method (for example, see Pehlivanov etc.[4], Pehlivanov and Carey[5], Cai etc.[6]). All of these studies deal with the case of different degree polynomials for u_h and σ_h.

Problems of the form (1.1) arise in the flow of fluids through fissured rock[7], thermodynamics[8], and other applications. For a discussion of existence and uniqueness results (see [9]). In this paper, our goal is to extend the theory of LSM methods to the sobolev equations. This paper is organized as follows: In §2, we formulate the problem and the LSM method for Sobolev equations. In §3, Optimal error estimates in L^2 and H^1-norm are obtained under the standard regularity assumption on the finite element partition. In §4, Optimal error estimates in L^2-norm for the time-dependent LSM approximation is obtained under the conventional RT[3] spaces. We define C to be a generic positive constant, ε be a generic small positive constant, which be independent of t and all mesh parameters.

2. LSM Methods for Sobolev Equations

Let

$$W_p^k = \left\{ \phi : \frac{\partial^\alpha \phi}{\partial x^\alpha} \in L^p(\Omega), \text{ for } |\alpha| \le k \right\}.$$

If $p = 2$, we write $H^k = W_2^k$, with norm

$$\|\phi\|_k = \|\phi\|_{H^k}, \|\phi\| = \|\phi\|_0 = \|\phi\|_{L}^2.$$

Coefficients $c(x)$, $a(x)$, $b(x)$, $f(x, t)$ are catalogued as follows:

$$|f(x, t)| \le C, 0 < c_1 \le c(x) \le c_2,$$

$$0 < a_1 \le a(x) \le a_2, 0 < b_1 \le b(x) \le b_2.$$

Also set

$$\sigma(x, t) = -\left(a(x) \nabla u + b(x) \nabla \frac{\partial u}{\partial t} \right). \qquad \text{... (2.1)}$$

We have

$$\sigma_t = -\left(a(x) \nabla u_t + b(x) \nabla \frac{\partial^2 u}{\partial t^2} \right).$$

Note $u_t = w$, so that

$$\sigma_t = -\left(a(x) \nabla w + b(x) \nabla \frac{\partial^2 w}{\partial t} \right).$$

And then, the equivalent form of (1.1) is

$$c(x)\, w + div\, \sigma = f(x, t),$$

$$\sigma_t = -\left(a(x) \nabla w + b(x) \nabla \frac{\partial w}{\partial t} \right). \qquad \text{... (2.2)}$$

Let $\tau > 0, N = T/\tau \in Z$, and $t^n = n\tau$. Also, for integer n, let $u^n = u(x, t^n)$. We have

$$w^n = \frac{u^n - u^{n-1}}{\tau}.$$

Combining (2.1), we can choose a suitable initial valuc σ_0 and $w^0 = u_0$. Now, introducing the spaces

$$V = \{v \in H^1(\Omega); v|_\Gamma = 0\},$$

$$H = \{w \in (L^2(\Omega))^2; \text{div}\, w \in L^2(\Omega)\},$$

we obtain the discrete form respect to the time. Find $(w^n, \sigma^n) \in V \times H$, such that

$$\left\{\begin{array}{l} c(x)w^n + \operatorname{div} \sigma^n = f^n, \\ \dfrac{\sigma^n - \sigma^{n-1}}{\tau} + a(x)\, \nabla w^n + b(x) \dfrac{\nabla w^n - \nabla w^{n-1}}{\tau} = 0, \\ \sigma^0 = \sigma_0, \\ w\,|_\Gamma = u\,|_\Gamma = 0. \end{array}\right\}. \quad \text{... (2.3)}$$

So the least-squares functional $J(v^n, q^n)$ for the first order systems (2.3) is

$$J(v^n, q^n) = \left(\frac{1}{c}(cw^n + \operatorname{div} \sigma^n - f^n),\, cw^n + \operatorname{div} \sigma^n - f^n \right)$$
$$+ \left(\frac{1}{a}\left(\frac{\sigma^n - \sigma^{n-1}}{\tau} + a\nabla w^n + b\frac{\nabla w^n - \nabla w^{n-1}}{\tau} \right),\right.$$
$$\left. \frac{\sigma^n - \sigma^{n-1}}{\tau} + a\nabla w^n + b\frac{\nabla w^n - \nabla w^{n-1}}{\tau} \right).$$

Then, the least-squares minimization problem: Find $(w^n, \sigma^n) \in V \times H$, such that

$$J(w^n, \sigma^n) = \inf_{v \in V,\, w \in H} J(v, q). \quad \text{... (2.4)}$$

Let $J^n = [t^{n-1}, t^n]$. The corresponding variational statement: Find $(w^n, \sigma^n) : J^n \to V \times H$ such that

$$a(w^n, \sigma^n; v, q) = \left(\frac{1}{c}\tau f^n, cv + \operatorname{div} q \right), \quad \forall\ (v, q) \in \times H,$$

where

$$a(w^n, \sigma^n; v, q) = \tau\left(\frac{1}{c}(cw^n + \operatorname{div} \sigma^n), cv + \operatorname{div} q \right)$$
$$+ \left(\frac{1}{a}(\sigma^n - \sigma^{n-1} + a\tau\nabla w^n + b(\nabla w^n - \nabla w^{n-1})), q + a\tau\nabla v + b\nabla v \right). \quad \text{... (2.5)}$$

Eq. (2.4) contains the boundary condition $u\,|_\Gamma = 0$. Let $\mathcal{J}_{h_u}, \mathcal{J}_{h_\sigma}$ be two classes quasi-uniform regular partitions of Ω, with diameter h_u, h_σ. h_u and h_σ are allowed to be different. Let V_{h_u} and H_{h_σ} be finite-dimensional subspaces of V and H. $m \geq 0, k \geq 1$ are integers. A standard choice for such spaces is the span of piecewise polynomial basis of degree k and m, respectively, which have the following approximation properties:

$$\inf_{v_I \in V_{h_u}} \{\| v - v_I \|_0 + h_u \| \nabla(v - v_I) \|_0\} \le C h_u^{m+1} \| v \|_{m+1} \quad \text{... (2.6)}$$

$$\inf_{\sigma_I \in H_{h_\sigma}} \{\| \sigma - \sigma_I \|_0 + h_\sigma \| \operatorname{div} (\sigma - \sigma_I) \|_0\} \le C h_\sigma^{k+1} \| \sigma \|_{k+1}. \quad \text{... (2.7)}$$

Then, the time-dependent LSM finite element approximation for (2.4) is: Find $(w_h^n, \sigma_h^n) : J^n \to V_{h_u} \times H_{h_\sigma}$, such that

$$\left.\begin{aligned} &\left(\frac{1}{a}\left(\sigma_h^n - \sigma_h^{n-1} + a\tau \nabla w_h^n + b(\nabla w_h^n - \nabla w_h^{n-1}\right)\right), q_h + a\tau \nabla v_h + b \nabla v_h) \\ &+ \tau\left(\frac{1}{c}(c w_h^n + \operatorname{div} \sigma_h^n, c v_h + \operatorname{div} q_h)\right) = \tau\left(\frac{1}{c} f^n, c v_h + \operatorname{div} q_h\right), v_h \in V_{h_u}, q_h \in H_{h_\sigma}, \\ &\sigma_h^0 = \sigma_{0h} \end{aligned}\right\} \quad \text{... (2.8)}$$

where $\sigma_{0h} \in V_{h_u}$ is a approximation of σ_0. Obviously (2.8) has a unique solution.

3. Error Estimates under the Standard Spaces

In the analysis to follow we shall employ a projection $\tilde{w} : J^n \to V_{h_u}$ respectively defined by:

$$(b(x) \nabla (w - \tilde{w}), \nabla v_h) = 0, \ \forall \ v_h \in V_{h_u}. \quad \text{... (3.1)}$$

Let $\eta = w - \tilde{w}$, $\pi = \tilde{w} - w_h$. From[10], the following results are standard:

$$\| \eta \|_0 = h_u \| \eta \|_1 \le C \| u \|_{m+1} h_u^{m+1}$$

$$\left\| \frac{\partial \eta}{\partial t} \right\|_0 + h_u \left\| \frac{\partial \eta}{\partial t} \right\| \le C\left(\| u \|_{m+1} + \left\| \frac{\partial u}{\partial t} \right\|_{m+1} \right) h_u^{m+1}. \quad \text{... (3.2)}$$

From (3), we define projection $\Pi_{h_\sigma} : H \to H_{h_\sigma}$ by

$$(\operatorname{div} \Pi_{h_\sigma} \chi, v_h) = (\operatorname{div} \chi, v_h), \ \forall \ v_h \in V_{h_u}, \chi \in H. \quad \text{... (3.3)}$$

Then, the following results are standard:

$$\| \Pi_{h_\sigma} \chi - \chi \|_0 \le C h_\sigma^{k+1} \| \chi \|_{k+1},$$

$$\| \operatorname{div} (\Pi_{h_\sigma} \chi - \chi) \|_0 \, C h_\sigma^{k+1} \{\| \chi \|_{k+1} + \| \operatorname{div} \chi \|_{k+1}\}. \quad \text{... (3.4)}$$

And let $\rho = \sigma - \Pi_{h_\sigma}\sigma, \xi = \Pi_{h_\sigma}\sigma - \sigma_h$. From (2.4), (2.8), the error equation is

$$\left(\frac{1}{a}(\rho^n - \rho^{n-1} + \xi^n - \xi^{n-1} + a\tau(\nabla\eta^n + \nabla\pi^n) + b(\nabla\eta^n + \nabla\pi^n - \nabla\eta^{n-1} - \nabla\pi^{n-1})\right),$$

$$q_h + a\tau(\nabla v_h + b\nabla v_h) + \tau\left(\frac{1}{c}(c(\eta^n + \pi^n) + \operatorname{div}(\rho^n + \xi^n)\right),$$

$$c v_h + \operatorname{div} q_h) = 0, \ \forall\ (v_h, q_h) \in V_{h_u} \times H_{h_\sigma}. \qquad \text{... (3.5)}$$

Choosing $q_h = \xi^n, v_h = \pi^n$ in (3.5), we have

$$\left(\frac{1}{a}(\xi^n - \xi^{n-1} + a\tau\nabla\pi^n + b(\nabla\pi^n - \nabla\pi^{n-1})), \xi^n + a\tau\nabla\pi^n + b\nabla\pi^n\right)$$

$$+ \tau\left(\frac{1}{c}(c\pi^n + \operatorname{div}\xi^n), c\pi^n + \operatorname{div}\xi^n\right) = -\tau\left(\frac{1}{c}(c\eta^n + \operatorname{div}\rho^n), c\pi^n + \operatorname{div}\xi^n\right)$$

$$-\left(\frac{1}{a}(\rho^n - \rho^{n-1} + a\tau\nabla\eta^n + b(\nabla\eta^n - \nabla\eta^{n-1})), \xi^n + a\tau\nabla\pi^n + b\nabla\pi^n\right) \qquad \text{... (3.6)}$$

Then, note the left-hand side of (3.6) to be $I = I_1 + I_2$, we have

$$I_1 = \left(\frac{1}{a}(\xi^n + a\tau\nabla\pi^n + b\nabla\pi^n), \xi^n + a\tau\nabla\pi^n + b\nabla\pi^n\right)$$

$$-\left(\frac{1}{a}(\xi^{n-1} + b\nabla\pi^{n-1}), \xi^n + a\tau\nabla\pi^n + b\nabla\pi^n\right)$$

$$\geq \frac{1}{2}\left(\frac{1}{a}(\xi^n + a\tau\nabla\pi^n + b\nabla\pi^n), \xi^n + a\tau\nabla\pi^n + b\nabla\pi^n\right)$$

$$-\frac{1}{2}\left(\frac{1}{a}(\xi^{n-1} + b\nabla\pi^{n-1}), \xi^{n-1} + b\nabla\pi^{n-1}\right).$$

We obtain

$$I \geq \frac{1}{2}\left(\frac{1}{a}(\xi^n + a\tau\nabla\pi^n + b\nabla\pi^n), \xi^n + a\tau\nabla\pi^n + b\nabla\pi^n\right) - \frac{1}{2}\left(\frac{1}{a}(\xi^{n-1} + b\nabla\pi^{n-1}),\right.$$

$$\left.\xi^{n-1} + b\nabla\pi^{n-1}\right) + \frac{1}{2}\tau\left(\frac{1}{c}(c\pi^n + \operatorname{div}\xi^n), c\pi^n + \operatorname{div}\xi^n\right)$$

$$\geq \frac{1}{2}\min\left\{\frac{1}{a_2}, 2b_1, a_1, c_1, \frac{1}{c_2}\right\}\ \{(\| \xi^n + b\nabla\pi^n \|_0^2$$

$$-\| \xi^{n-1} + b\nabla\pi^{n-1} \|_0^2) + 2\tau \| \nabla\pi^n \|_0^2 + \tau^2 \| \nabla\pi^n \|_0^2$$

$$+ \tau \| \pi^n \|_0^2 + \tau \| \operatorname{div}\xi^n \|_0^2\} \qquad \text{... (3.7)}$$

where, the following Green's formulas are applied

$$(\xi^n, \nabla\pi^n) + (\pi^n, \operatorname{div} \xi^n) = 0$$

and the right-hand side is,

$$- \tau(\eta^n, c\pi^n) - \tau(\eta^n, \operatorname{div} \xi^n) - \tau(\pi^n, \operatorname{div} \rho^n) - \tau\left(\frac{1}{c}\operatorname{div} \rho^n, \operatorname{div} \xi^n\right)$$

$$- \left(\frac{1}{a}(\rho^n - \rho^{n-1}), \xi^n\right) - \tau(\rho^n - \rho^{n-1}, \nabla\pi^n) - \left(\frac{1}{a}(\rho^n - \rho^{n-1}), b\nabla\pi^n\right)$$

$$- \tau(\nabla\eta^n, \xi^n) - \tau^2(\nabla\eta^n, a\nabla\pi^n) - \tau(\nabla\eta^n, b\nabla\pi^n)$$

$$- \left(\frac{b}{a}(\nabla\eta^n - \eta^{n-1}), \xi^n\right) - \tau(b(\nabla\eta^n - \nabla\eta^{n-1}), \nabla\pi^n)$$

$$- \left(\frac{b}{a}(\nabla\eta^n - \nabla\eta^{n-1}), b\nabla\pi^n\right). \qquad \text{... (3.8)}$$

By (3.1) and (3.3), we have

$$(b\nabla\eta^n, \nabla\pi^n) = 0, (\operatorname{div} \rho^n, \pi^n) = 0. \qquad \text{... (3.9)}$$

From (3.8) and (3.9), we sum for $n = 0, 1, ..., N$, and apply

$$\| \xi^N \|_0^2 \le \| \xi^N + b\nabla\pi^N \|_0^2,$$

we obtain

$$\| \xi^N \|_0^2 + \sum_{n=0}^{N} [\| \nabla\pi^n \|_0^2 + \tau \| \nabla\pi^n \|_0^2 \| \pi^n \|_0^2 + \| \operatorname{div} \xi^n \|_0^2] \tau \le \sum_{i=1}^{10} | T_i |, \qquad \text{... (3.11)}$$

where

$$T_1 = \sum_{n=0}^{N} (\eta^n, c\pi^n)\tau, \; T_2 = \sum_{n=0}^{N} (\eta^n, \operatorname{div} \xi^n) \tau,$$

$$T_3 = \sum_{n=0}^{N} \left(\frac{1}{c}\operatorname{div} \rho^n, \operatorname{div} \xi^n\right)\tau, \; T_4 = \sum_{n=0}^{N} \left(\frac{1}{a}\rho^n - \rho^{n-1}), \xi^n\right),$$

$$T_5 = \sum_{n=0}^{N} (\rho^n - \rho^{n-1}, \nabla\pi^n)\tau, \; T_6 = \sum_{n=0}^{N} \left(\frac{1}{a}(\rho^n - \rho^{n-1}), b\nabla\pi^n\right),$$

$$T_7=\sum_{n=0}^{N}(\nabla\eta^n,\xi^n)\tau,\ T_8=\sum_{n=0}^{N}\left(\frac{b}{a}(\nabla\eta^n-\nabla\eta^{n-1}),\xi^n\right),$$

$$T_9=\sum_{n=0}^{N}\left(b(\nabla\eta^n-\nabla\eta^{n-1}),\nabla\pi^n\right)\tau,\ T_{10}=\sum_{n=0}^{N}\left(\frac{b}{a}(\nabla\eta^n-\nabla\eta^{n-1}),b\nabla\pi^n\right)$$

and

$$T_{11}=\sum_{n=0}^{N}(a\nabla\eta^n,\nabla\pi^n)\tau^2.$$

Applying Green's formula, we have

$$T_2+T_7=-\tau(\eta^n,\operatorname{div}\xi^n)-\tau(\nabla\eta^n,\xi^n)=0.$$

The estimates of T_i are

$$|T_1|\le\varepsilon\sum_{n=0}^{N}\|\pi^n\|_0^2\tau+Ch_u^{2m+2} \quad \text{... (3.11)}$$

and

$$|T_3|\le\varepsilon\sum_{n=0}^{N}\|div\,\xi^n\|_0^2\tau+Ch_\sigma^{2k+2}. \quad \text{... (3.12)}$$

Since $$\left\|\frac{\rho^n-\rho^{n-1}}{\tau}\right\|_0^2\le C_\tau\int_\Omega\int_{t^{n-1}}^{t^n}+\left|\frac{\partial\rho}{\partial t}\right|^2dt\,dx,$$

$$|T_4|\le\varepsilon\sum_{n=0}^{N}\|\xi^n\|_0^2\tau+C\tau^2\left\|\frac{\partial\rho}{\partial t}\right\|_0^2, \quad \text{... (3.13)}$$

$$|T_5|\le\varepsilon\sum_{n=0}^{N}\|\nabla\pi^n\|_0^2\tau+C\tau^2\left\|\frac{\partial\rho}{\partial t}\right\|_0^2, \quad \text{... (3.14)}$$

$$|T_6|\le\varepsilon\sum_{n=0}^{N}\|\nabla\pi^n\|_0^2\tau+C\tau^2\left\|\frac{\partial\rho}{\partial t}\right\|_0^2, \quad \text{... (3.15)}$$

$$|T_8|\le\left|\frac{b_2}{b_1}\sum_{n=0}^{N}(\eta^n-\eta^{n-1},\operatorname{div}\xi^n)\right|\le\varepsilon\sum_{n=0}^{N}\|\operatorname{div}\xi^n\|_0^2\tau+C\tau^2\left\|\frac{\partial\eta}{\partial t}\right\|_0^2 \quad \text{... (3.16)}$$

$$|T_9| \le \varepsilon \sum_{n=0}^{N} \|\nabla\pi^n\|_0^2 \tau^2 + C\tau^2 \left\| \frac{\partial^2\eta}{\partial t^2} \right\|_0^2 \quad \ldots (3.17)$$

and

$$|T_{10}| \le \varepsilon \sum_{n=0}^{N} \|\nabla\pi^n\|_0^2 \tau + C\tau^2 \left\| \frac{\partial^2\eta}{\partial t^2} \right\|_0^2 \quad \ldots (3.18)$$

At last, let us estimate T_{11}. Then

$$\begin{aligned}(a\nabla\eta^n, \nabla\pi^n) &= \left(b\nabla\eta^n, \frac{a}{b}\nabla\pi^n \right) \\ &= \left(b\nabla\eta^n, \nabla\left(\frac{a}{b}\pi^n - w_h \right) \right) - \left(b\nabla\eta^n, \left(\nabla\frac{a}{b} \right) \cdot \pi^n \right) + (b\nabla\eta^n, \nabla w_h) \quad \ldots (3.19) \\ &= M_1 + M_2 + M_3.\end{aligned}$$

Where $w_h \in V_{h_u}$ is an approximation of $\frac{a}{b}\pi^n$. Therefore, By (3.9), we obtain $M_3 = 0$, and by the results of finite element spaces, we have

$$\begin{aligned}|M_1| &= \left| \left(b\nabla\eta^n, \nabla\left(\frac{a}{b}\pi^n - w_h \right) \right) \right| \\ &\le C\|\nabla\eta^n\| \cdot \inf_{w_h \in V_{h_u}} \left\| \nabla\left(\frac{a}{b}\pi^n - w_h \right) \right\| \\ &\le Ch_u^k \|\nabla\eta^n\| \cdot \left\| \frac{a}{b}\pi^n \right\|_{H^{k+1}(\Omega)} \\ &= Ch_u^k \|\nabla\eta^n\| \cdot \sum_{e \in \mathcal{J}_{h_u}} \left\| \frac{a}{b}\pi^n \right\|_{H^{k+1}(e)} \\ &= Ch_u^k \|\nabla\eta^n\| \cdot \sum_{e \in \mathcal{J}_{h_u}} \left(\sum_{j=0}^{k+1} \left\| D^j\left(\frac{a}{b}\pi^n \right) \right\|^2 \right)^{\frac{1}{2}} \\ &= Ch_u^k \|\nabla\eta^n\| \cdot \sum_{e \in \mathcal{J}_{h_u}} \left(\sum_{j=0}^{k+1} \left\| \sum_{i=0}^{j} D^i\left(\frac{a}{b} \right) D^{j-i}(\pi^n) \right\|^2 \right)^{\frac{1}{2}}\end{aligned}$$

$$\leq Ch_u^k \| \nabla\eta^n \| \cdot \sum_{e \in \mathcal{J}_{h_u}} \left(\sum_{i=0}^{k+1} \left| \left| D^i \left(\frac{b}{a} \right) D^{k+1-i} (\pi^n) \right| \right|^2 \right)^{\frac{1}{2}}$$

$$\leq Ch_u^k \| \nabla\eta^n \| \cdot \sum_{e \in \mathcal{J}_{h_u}} \left(\sum_{i=1}^{k+1} \| D^{k+1-i} (\pi^n) \|^2 \right)^{\frac{1}{2}}$$

$$= Ch_u^k \| \nabla\eta^n \| \cdot \sum_{e \in \mathcal{J}_{h_u}} \| \pi^n \|_{H^k(e)}.$$

Applying the inverse inequalities of the finite dimensional spaces, we obtain

$$| M_1 | \leq C_{h_u}^k \cdot h_u^{-(k-1)} \| \nabla\eta^n \| \cdot \sum_{e \in \mathcal{J}_{h_u}} \| \pi^n \|_{H^1(e)}$$

$$= Ch_u \| \nabla\eta^n \| \cdot \| \pi^n \|_{H^1(\Omega)}.$$

By the Young inequalities, we obtain :

$$| M_1 | \leq Ch_u^2 \| \nabla\eta^n \|^2 + \varepsilon \| \pi^n \|_1^2. \quad \text{... (3.20)}$$

For M_2, we have

$$| M_2 | \leq \| \nabla\eta^n \|_{-1} \cdot \| \pi^n \|_1 \leq C \| \eta^n \|_0^2 + \varepsilon \| \pi^n \|_1^2 \quad \text{... (3.21)}$$

So, we have

$$| T_{11} | \leq \varepsilon \sum_{n=0}^{N} \| \pi^n \|_1^2 \, \tau | \, C(\tau^2 + h_u^{2m+2}). \quad \text{... (3.22)}$$

Assume $\| \xi^{n*} \|_0$ to be the maximum of $\| \xi^n \|_0$, for the sake of simplicity, still note n^* to be N. We obtain

$$\| \xi^N \|_0^2 + \sum_{n=0}^{N} [\| \pi^n \|_0^2 + \| \nabla\pi^n \|_0^2 + \| \nabla\pi^n \|_0^2 \, \tau + \| \operatorname{div} \xi^n \|_0^2] \, \tau$$

$$\leq C(\tau^2 + h_u^{2m+2} + h_\sigma^{2k+2}). \quad \text{... (3.23)}$$

Notice that

$$\| u^N \|_0^2 = \sum_{n=1}^{N} \frac{\| u^n \|_0^2 - \| u^{n-1} \|_0^2}{\tau} \tau \le \varepsilon \sum_{n=1}^{N} \tau \| u^n \|_0^2 + C \| w^n \|_{L^2(J, L^2)}^2. \quad \text{... (3.24)}$$

where $w^n = \frac{u^n - u^{n-1}}{\tau}$. From (3.21) and (3.22), we have

$$\max_{0 \le n \le t/\tau} \left\{ \| u^n - u_h^n \|_L^2 + \| \sigma^n - \sigma_h^n \|_L^2 \right\}$$

$$+ \| w - w_h \|_{L^2(J, L^2)}^2 + \tau \| w - w_h \|_{L^2(J, H^1)}^2 \le C \left(\tau^2 + h_u^{2m+2} + h_\sigma^{2k+2} \right). \quad \text{... (3.25)}$$

Thus, we obtain

***Theorem* 3.1** — *For $\tau > 0$, the approximation from (2.8) has unique solution, and*

$$\max_{0 \le n \le t/\tau} \left\{ \| u^n - u_h^n \|_L^2 + \| \sigma^n - \sigma_h^n \|_L^2 \right\}$$

$$+ \| w - w_h \|_{L^2(J, L^2)}^2 + \tau \| w - w_h \|_{L^2(J, H^1)}^2 \le C \left(\tau^2 + h_u^{2m+2} + h_\sigma^{2k+2} \right). \quad \text{... (3.26)}$$

C is independent of τ, h.

4. Error Estimates Under the RT Spaces

We now introduce the RT spaces. We assume that $u \in H^{k+2}(\Omega)$ and $\Delta u \in H^{k+1}(\Omega)$ for some integer $k \ge 0$, $\mathcal{K}_h$ is a triangulation of $\overline{\Omega}$ made up with triangles K whose diameters are $\le h$. we have

$$Q_h = \{ q_h \in H(\text{div}; \Omega);\ \forall\, K \in \mathcal{K}_h, q_h|_K \in Q_K \}$$

where, for all $K \in \mathcal{K}_h$, the space Q_K is defined as

$$Q_K = \{ q \in H(\text{div}; K), \hat{q} \in \hat{Q} \}$$

where, the space $\hat{Q}$ is required to satisfy the five conditions (see [3], (3.1)-(3.5)).

On the other hand, for any $q_h \in Q_h$ and any $K \in \mathcal{K}_h$, we have $(\text{div}\ q_h)|_K \in P_k$ (P_k are polynomials of degree $\le k$. Hence, a natural choice for the space V_h is given by

$$V_h = \{ v_h \in L^2(\Omega);\ \forall\, K \in \mathcal{K}_h, v_h|_K \in P_k \},$$

Q_h, V_h are associated with a regular family of triangulations, and regarded as RT spaces with index k.

In this section, assume that the spaces V_h, H_h to be the RT spaces with $r \ge 0$. Rewrite (3.2) and (3.4), we obtain

$$\| \eta \|_0 + h_u \| \eta \|_1 \leq C \| u \|_{r+1} h^{r+1}$$

$$\left\| \frac{\partial \eta}{\partial t} \right\|_0 + h_u \left\| \frac{\partial \eta}{\partial t} \right\|_1 \leq C \left(\| u \|_{r+1} + \left\| \frac{\partial u}{\partial t} \right\|_{r+1} \right) h^{r+1}. \quad \dots (4.1)$$

$$\| \rho \|_0 \leq C h^{r+1} \| \chi \|_{r+1},$$

$$\| \operatorname{div} \rho \|_0 \leq C h^{r+1} \{ \| \chi \|_{r+1} + \| \operatorname{div} \chi \|_{r+1} \}. \quad \dots (4.2)$$

Since div $V_h = H_h$, we have

$$T_3 = (\operatorname{div} \rho^n, \operatorname{div} \xi^n) = 0.$$

The estimates of T_i is the same as §3. But the order of all mesh parameter h is $r + 1$. Therefore, we have

$$\max_{0 \leq n \leq t/\tau} \left\{ \| u^n - u_h^n \|_{L^2}^2 + \| \sigma^n - \sigma_h^n \|_{L^2}^2 \right\}$$

$$+ \| w - w_h \|_{L^2(J, L^2)}^2 + \tau \| w - w_h \|_{L^2(J, H^1)}^2 \leq C(\tau^2 + h^{2r+2}). \quad \dots (4.3)$$

We obtain the following results:

Theorem 4.1 — *The RT spaces are choosed, For* $\tau > 0$, *the approximation form (2.8) has unique solution, and*

$$\max_{0 \leq n \leq t/\tau} \left\{ \| u^n - u_h^n \|_{L^2}^2 + \| \sigma^n - \sigma_h^n \|_{L^2}^2 \right\}$$

$$+ \| w - w_h \|_{L^2(J, L^2)}^2 + \tau \| w - w_h \|_{L^2(J, H^1)}^2 \leq C(\tau^2 + h^{2r+2}). \quad \dots (4.4)$$

C is independent of τ, h. These results indicate that optimal error estimates is obtained, even if the standard RT spaces are chosen.

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