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RAYLEIGH-TAYLOR INSTABILITY OF RIVLIN-ERICKSEN ELASTICO-VISCOUS FLUIDS IN PRESENCE OF SUSPENDED PARTICLES THROUGH POROUS MEDIUM

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The stability of superposed Rivlin-Ericksen elastico-viscous fluids permeated with suspended particles in a porous medium is considered following the linearized perturbation theory and normal mode analysis. The cases of two uniform elastico-viscous fluids separated by a horizontal boundary and exponentially varying density are considered. In each case, the perturbations decay with time for potentially stable configuration/stable stratification and grow with time for potentially unstable configuration/unstable stratification. The growth rates both increase (for certain wave numbers) and decrease (for different wave numbers) with the increase in suspended particles number density and medium permeability.

Key Words : Rivlin-Ericksen; Elastico-Viscous; Fluid/Rayleigh-Taylor; Instability/Suspended; Particles/Porous Medium

1. INTRODUCTION

A comprehensive account of the instability of a plane interface between two Newtonian fluids in non-porous medium, under various assumptions of hydrodynamics and hydromagnetics has been given by Chandrasekhar[2]. Bhatia[1] has considered the Rayleigh-Taylor instability of two viscous superposed conducting fluids in the presence of a uniform horizontal magnetic field. The stability of superposed fluids in the presence of a variable magnetic field has been studied by Sharma and Thakur[8].

With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. Oldroyd[9] proposed a theoretical model for a class of viscoelastic fluids. Toms and Strawbridge[11] revealed that a dilute solution of methyl methacrylate in n-butyl acetate agrees well with a theoretical model of Oldroyd fluid. Sharma et al.[5] have studied the effect of suspended particles on the onset of Bénard convetion in hydromagnetics. There are many elastico-viscous fluids that cannot be characterized by Oldroyd's constitutive relations. The Rivlin-Ericksen[4] fluid is one such fluid. Many research workers have paid their attention towards the study of Rivlin-Ericksen fluid.

Sisodia and Gupta and Srivastava and Singh[10] have studied the unsteady flow of a dusty elastico-viscous Rivlin-Ericksen fluid through channel of different cross-sections in the presence of the time dependent pressure gradient. Sharma and Kumar[7] have studied the thermal instability of a layer of Rivlin-Ericksen elastico-viscous fluid acted on by a uniform rotation and found that rotation has a stabilising effect and introduce oscillatory modes in the system.

In all the above studies, the medium has been considered to be non-porous. When the fluid permeates a porous material, the gross effect is represented by the Darcy's law. As a result of this macroscopic law, the usual viscous term in the equations of Rivlin-Ericksen fluid motion is replaced by the resistance term $\left[-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\boldsymbol{v}\right]$, where μ and μ' are the viscosity and viscoelasticity of the Rivlin-Ericksen fluid, k_1 is the medium permeability and $\boldsymbol{v}$ is the Darcian (filter) velocity of the fluid. The thermal instability of fluids in a porous medium in the presence of suspended particles has been studied by Sharma and Sharma[6]. The suspended particles and the permeablity of the medium were found to destabilize the layer.

The present paper attempts to study the stability of two superposed Rivlin-Ericksen elastico-viscous fluids permeated with suspended particles in porous medium. The knowledge regarding viscoelastic fluid-particle mixtures is not commesurate with their scientific and industrial importance. The analysis would be relevant to the stability of some polymer solutions and the problem finds its usefulness in several Geophysical situations and in Chemical technology. These aspects form the motivation for the present study.

2. Perturbation Equations

Consider an incompressible viscoelastic (Rivlin-Ericksen) fluid-particle layer consisting of a viscoelastic fluid of density ρ, permeated with suspended particles of density mN, arranged in horizontal strata in porous medium. The system is acted on by gravity force $\boldsymbol{g}(0, 0, -g)$. Let p, ρ, μ, μ' and $\boldsymbol{v}(u, v, w)$ denote respectively the pressure, density, viscosity, viscoelasticity and velocity of the pure fluid; $\boldsymbol{u}\,(l, r, s)$, m and $N(\bar{x}, t)$ denote the velocity, mass and number density of the particles respectively, ε is the medium porosity, k_1 is the medium permeability and $\bar{x}=(x, y, z)$. The equations of motion and continuity for the fluid are

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \boldsymbol{v}}{\partial t}+\frac{1}{\varepsilon}(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p+\rho\boldsymbol{g}-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\boldsymbol{v}+\frac{KN}{\varepsilon}(\boldsymbol{u}-\boldsymbol{v}) \quad \dots (1)$$

and

$$\nabla\cdot\boldsymbol{v}=0, \quad \dots (2)$$

where $K=6\pi\mu\eta$, η being the particle radius, is the Stokes' drag coefficient.

Since the density of every fluid particle remains unchanged as we follow it with its motion, we have

$$\varepsilon\frac{\partial\rho}{\partial t}+(\boldsymbol{v}\cdot\nabla)\rho=0. \quad \dots (3)$$

Assuming uniform particle size, spherical shape and small relative velocities between the fluid and particles, the presence of particles adds an extra force term in the equations of motion (1), proportional to the velocity difference between particles and fluid. The force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid. Interparticle reactions are ignored for we assume that the distances between particles are quite large compared with their diameter. The effects of pressure, gravity and Darcian force on the suspended particles are negligibly small and therefore ignored. Under the above assumptions, the equations of motion and continuity for the particles are

$$mN\left[\frac{\partial\boldsymbol{u}}{\partial t}+\frac{1}{\varepsilon}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right]=KN(\boldsymbol{v}-\boldsymbol{u}) \quad \dots (4)$$

and

$$\varepsilon \frac{\partial N}{\partial t} + \nabla \cdot (N \boldsymbol{u}) = 0. \qquad \dots (5)$$

The initial state of the system is taken to be a quiescent layer (no settling) with a uniform particle distribution N_0 i.e. $\boldsymbol{v} = (0, 0, 0)$, $\boldsymbol{u} = (0, 0, 0)$ and $N = N_0$, is a constant. The character of the equilibrium of this initial static state can be determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\delta\rho$, δp, $\boldsymbol{v}(u, v, w)$ and $\boldsymbol{u}(l, r, s)$ denote respectively the perturbations in density ρ, pressure p, velocity of fluid and velocity of particles. Then the linearized perturbed forms of eqs. (1)-(5) become

$$\frac{\rho}{\varepsilon}\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla \delta p + \boldsymbol{g}\,\delta\rho - \frac{\rho}{k_1}\left(\nu + \nu' \frac{\partial}{\partial t}\right)\boldsymbol{v} + \frac{KN_0}{\varepsilon}(\boldsymbol{u} - \boldsymbol{v}), \qquad \dots (6)$$

$$\nabla \cdot \boldsymbol{v} = 0, \qquad \dots (7)$$

$$\varepsilon \frac{\partial}{\partial t}(\delta\rho) = -w(D\rho), \qquad \dots (8)$$

$$\left(\frac{m}{K}\frac{\partial}{\partial t} + 1\right)\boldsymbol{u} = \boldsymbol{v} \qquad \dots (9)$$

and

$$\frac{\partial M}{\partial t} + \nabla \cdot \boldsymbol{u} = 0, \qquad \dots (10)$$

where $M = \frac{\varepsilon N}{N_0}$ and $N_0, N, \nu(=\mu/\rho), \nu'(=\mu'/\rho)$ stand for initial uniform number density, perturbations in number density, kinematic viscosity, kinematic viscoelasticity respectively and $D = \frac{d}{dz}$.

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is given by

$$\exp(ik_x x + ik_y y + nt), \qquad \dots (11)$$

where k_x, k_y are horizontal wave numbers, $k^2 = k_x^2 + k_y^2$, and n is a complex constant.

For a perturbation of the form (11), eqs. (6)-(9) after eliminating **u**, give

$$\left[n' + \frac{\nu + \nu' n}{k_1}\right]\rho u = -ik_x \delta p, \qquad \dots (12)$$

$$\left[n' + \frac{\nu + \nu' n}{k}\right]\rho v = -ik_y \delta p, \qquad \dots (13)$$

$$\left[n' + \frac{\nu + \nu' n}{k_1}\right]\rho w = -D\delta p - g\delta\rho, \qquad \dots (14)$$

$$ik_x u + ik_y v + Dw = 0. \qquad ... (15)$$

and

$$\varepsilon n \delta\rho = -wD\rho, \qquad ... (16)$$

where

$$n' = \frac{n}{\varepsilon}\left(1 + \frac{mN_0K/\rho}{mn + K}\right).$$

Eliminating δp between eqns (12)-(14) and using (15) and (16), we obtain

$$n'[D(\rho Dw) - k^2\rho w] + \frac{1}{k_1}[D(\rho v Dw) - k^2\rho v w] + \frac{n}{k_1}[D(\rho v' Dw) - k^2 \rho v' w]$$

$$= -\frac{gk^2}{\varepsilon n}(D\rho)w. \qquad ... (17)$$

3. Two Superposed Viscoelastic (Rivlin-Ericksen) Fluids separated by a Horizontal Boundary

Consider the case when two superposed Rivlin-Ericksen fluids of uniform densities ρ_1 and ρ_2, uniform viscosities μ_1 and μ_2, uniform viscoelasticities μ_1' and μ_2' are separated by a horizontal boundary at $z = 0$. The subscripts 1 and 2 distinguish the upper and the lower fluids, respectively. Then in each region of constant ρ, constant μ and constant μ', eq. (17) reduces to

$$(D^2 - k^2)w = 0. \qquad ... (18)$$

The general solution of eqn. (18) is

$$w = Ae^{+kz} + B^{-kz}, \qquad ... (19)$$

where A and B are arbitrary constants.

The boundary conditions to be satisfied in the present problem are :

(i) The velocity should vanish when $z \to +\infty$ (for the upper fluid) and $z \to -\infty$ (for the lower fluid).

(ii) $w(z)$ is continuous at $z = 0$.

(iii) The jump condition at the interface.

Applying the boundary conditions (i) and (ii), we have

$$w_1 = Ae^{+kz}, (z < 0) \qquad ... (20)$$

$$w_2 = Ae^{-kz}, (z > 0) \qquad ... (21)$$

where the same constant A has been chosen to ensure the continuity of w at $z = 0$.

Equation (17) gives the jump condition at the interface $z = 0$ as

$$n' \Delta_0(\rho Dw) + \frac{1}{k_1}\Delta_0(\rho v Dw) + \frac{n}{k_1}\Delta_0(\rho v' Dw) + \frac{gk^2}{\varepsilon n}\nabla_0(\rho)w_0 = 0. \qquad ... (22)$$

Applying the condition (22) to the solutions (20) and (21), we obtain

$$\left[1+\frac{\varepsilon}{k_1}(\alpha_2 v_2' + \alpha_1 v_1')\right]n^3 + \left[\frac{k}{m}+\frac{2N_0K}{\rho_1+\rho_2}+\frac{\varepsilon}{k_1}(\alpha_2 v_2 + \alpha_1 v_1)\right.$$

$$\left. + \frac{K\varepsilon}{mk_1}(\alpha_2 v_2' + \alpha_1 v_1')\right]n^2 + \left[\frac{K\varepsilon}{mk_1}(\alpha_2 v_2 + \alpha_1 v_1) + gk(\alpha_1 - \alpha_2)\right]$$

$$n + \frac{Kgk}{m}(\alpha_1 - \alpha_2) = 0, \qquad \dots (23)$$

where $\alpha_{1,2} = \frac{\rho_{1,2}}{\rho_1+\rho_2}$, $v_{1,2} = \frac{\mu_{1,2}}{\rho_{1,2}}$, $v_{1,2}' = \frac{\mu_{1,2}'}{\rho_{1,2}}$.

(a) Stable Case $(\alpha_2 < \alpha_1)$

For the potentially stable case $\alpha_2 < \alpha_1$, eq. (23) does not involve any change of sign and so does not allow any positive root. The system is therefore stable.

(b) Unstable Case $(\alpha_2 > \alpha_1)$

For the potentially unstable configuration $\alpha_2 > \alpha_1$, the constant term in eq. (23) is negative. Equation (23) therefore allows one change of sign and so has one positive root and hence the system is unstable.

4. The Case of Exponentially Varying Density

Here we consider the density stratification in the fluid of depth d as

$$\rho(z) = \rho_0 e^{\beta z}, \qquad \dots (24)$$

where ρ_0 and β are constants. Assume that $\beta d << 1$, i.e, the variation of density at two neighbouring points in the velocity field, which is much less than the average density, has a negligible effect on the inertia of the fluid.

The boundary conditions for the case of two free surfaces are

$$w = 0,\ D^2 w = 0 \text{ at } z = 0 \text{ and } z = d. \qquad \dots (25)$$

The proper solution of eqn. (17) satisfying (25) is

$$w = w_0 \sin\frac{s\pi z}{d}, \qquad \dots (26)$$

where w_0 is a constant and s is an integer.

Substituting (26) in eqn. (17) and neglecting the effect of heterogeneity on the inertia, we get

$$\left[1+\frac{\varepsilon V}{k_1}\right]n^3+\left[\frac{K}{m}\left(1+\frac{mN_0}{\rho}\right)+\frac{\varepsilon v}{k_1}+\frac{\varepsilon V K}{mk_1}\right]n^2$$

$$+\left[\frac{\varepsilon v K}{mk_1}-\frac{g\beta k^2}{\left(\frac{s\pi}{d}\right)^2+k^2}\right]n-\frac{g\beta k^2\,K/m}{\left(\frac{s\pi}{d}\right)^2+k^2}=0. \qquad \text{... (27)}$$

For the stable stratifications ($\beta<0$), eqn. (27) does not have any positive root implying thereby that the system is stable.

For the unstable stratifications ($\beta>0$), the constant term in eqn. (27) is negative. Equation (27), therefore, allows one change of sign and so has one positive root and hence the system is unstable.

Let n_0 denote the positive root of eqn. (27). Then

$$\left[1+\frac{\varepsilon V}{k_1}\right]n_0^3+\left[\frac{K}{m}\left(1+\frac{mN_0}{\rho}\right)+\frac{\varepsilon v}{k_1}+\frac{\varepsilon V K}{mk_1}\right]n_0^2$$

$$+\left[\frac{\varepsilon v K}{mk_1}-\frac{g\beta k^2}{\left(\frac{s\pi}{d}\right)^2+k^2}\right]n_0-\frac{g\beta k^2\,K/m}{\left(\frac{s\pi}{d}\right)^2+k^2}=0. \qquad \text{... (28)}$$

To find the role of particle number density and medium permeability concerning the growth rate of unstable modes, we examine the natures of $\frac{dn_0}{dN_0}$ and $\frac{dn_0}{dk_1}$ analytically.

Equation (28) yields

$$\frac{dn_0}{dN_0}=\frac{(K/\rho)\,n_0^2}{\frac{g\beta k^2}{\left(\frac{s\pi}{d}\right)^2+k^2}-\left[3n_0^2\left(1+\frac{\varepsilon V}{k_1}\right)+2n_0\left\{\frac{K}{m}\left(1+\frac{mN_0}{\rho}\right)+\frac{\varepsilon v}{k_1}+\frac{\varepsilon V K}{mk_1}\right\}+\frac{\varepsilon v K}{mk_1}\right]} \qquad \text{... (29)}$$

and

$$\frac{dn_0}{dk_1}=\frac{\left[\frac{\varepsilon V}{k^2}\right]n_0^3+\left[\frac{\varepsilon v}{k_1}+\frac{\varepsilon V K}{mk_1^2}\right]n_0^2+\left[\frac{\varepsilon v K}{mk_1^2}\right]n_0}{3n_0^2\left[1+\frac{\varepsilon V}{k_1}\right]+2n_0\left[\frac{K}{m}\left(1+\frac{mN_0}{\rho}\right)+\frac{\varepsilon v}{k_1}+\frac{\varepsilon V K}{mk_1}\right]}+\left[\frac{\varepsilon v K}{mk_1}-\frac{g\beta k^2}{\left(\frac{s\pi}{d}\right)^2+k^2}\right]. \qquad \text{... (30)}$$

It is clear from (29) and (30) that $\frac{dn_0}{dN_0}$ and $\frac{dn_0}{dk_1}$ may be positive or negative. The growth rates, thus, both decrease (for certain wave numbers) and increase (for different wave numbers) with the increase in particle number density and medium permeability for the unstable stratifications.

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