Indian J. pure appl. Math., **31**(5) : 551-562, May 2000

EXISTENCE AND UNIQUENESS OF POSITIVE FIXED POINTS FOR NONCOMPACT DECREASING OPERATORS‡

DAJUN GUO

Department of Mathematics, Shandong University, Jinan, Shandong 250100, *People's Republic of China*

(*Received* 17 *March* 1997; *Accepted* 25 *October* 1999)

This paper obtains three new theorems on the existence and uniqueness of positive fixed points for noncompact decreasing operators. As application, an existence and uniqueness theorem of positive solutions for second order nonlinear integro-differential equation on infinite interval is given.

Key Words and Phrases : Fixed Point, Decreasing Operator; Cone Theory; Integro-differential Equation

1. INTRODUCTION

Let the real Banach space E be partially ordered by a cone P of E, i.e. $x \leq y$ if and only if $y - x \in P$. Let $D \subset E$. Operator $A : D \to E$ is said to be decreasing if $x_1 \leq r_2\,(x_1, x_2 \in D)$ implies $Ax_1 \geq Ax_2$. Cone P is said to be normal if there exists a positive constant N such that $0 \leq x \leq y$ implies $\| x \| \leq N \|y\|$, where 0 denotes the zero element of E, and the smallest N is called the normal constant of $P \cdot P$ is said to be strongly minihedral if the least upper bound sup S exists for any bounded above in order nonempty set $S \subset E$. For details on cone theory, see [2].

In [1, 2], an existence and uniqueness theorem of positive fixed points was established for decreasing operator $A : P \to P$, which is completely continuous (i.e. continuous and compact) (see [1] Theorem 2) or condensing (see [2] Theorem 2.1.5). This result was applied to a nonlinear integral equation on finite interval which is of interest in nuclear physics (see [1] Example 2).

Now, in this paper, we shall use a different method to obtain three theorems on the existence and uniqueness of positive fixed points for decreasing operator $A : P \to P$, which is noncompact and noncontinuous (see Theorems 1 to 3 in Section 2). As application, we give an existence and uniqueness theorem of positive solutions for a second order nonlinear integro-differential equation on infinite interval in which the corresponding operator is noncompact (see Theorem 4 in Section 3).

MAIN THEOREMS

Let us list some conditions for later use.

(H_1) Operator $A : P \to P$, satisfies $A^2\theta \geq \varepsilon A\theta$, where $0 < \varepsilon < 1$ and, for any $\varepsilon A\theta \leq x \leq A\theta$ and

‡The research was supported by the National Natural Science Foundation of China.

$\varepsilon \le t < 1$, there exists a $\eta = \eta(x, t) > 0$ such that

$$A(tx) \le [t(1+\eta)]^{-1} Ax.$$

(H_2) Operator $A : P \to P$ satisfies $A^2\theta \ge \varepsilon A\theta$, where $0 < \varepsilon < 1$, and for any $\varepsilon \le t^* < 1$, there exist a $\eta = \eta(t^*) > 0$ and a $\delta = \delta(t^*) > 0$ such that

$$A(tx) \le [t(1+\eta)]^{-1} Ax, \ \forall\ \varepsilon A\theta \le x \le A\theta, t^* - \delta \le t \le t^*.$$

(H_3) Operator $A : P \to P$ satisfies $A^2\theta \ge \varepsilon A\theta$, where $0 < \varepsilon < 1$, and there exists a $0 < \alpha < 1$ such that

$$A(tx) \le t^{-\alpha} Ax, \ \forall\ \varepsilon A\theta \le x \le A\theta, \varepsilon \le t < 1.$$

It is clear that (H_2) implies (H_1) and (H_3) implies (H_1).

Lemma 1 — Let $A : P \to P$ be a decreasing operator satisfying condition (H_1). If $u, \upsilon \in P$ satisfy $Au = \upsilon$ and $A\upsilon = u$, then $u = \upsilon$.

PROOF : Since $u \ge \theta, \upsilon \ge \theta$ and $A : P \to P$ is decreasing, we have

$$\theta \le \upsilon = Au \le A\theta,\ \theta \le u = A\upsilon \le A\theta,$$

and so, by (H_1),

$$\varepsilon A\theta \le A^2\theta \le A\upsilon = u \le A\theta,\ \varepsilon A\theta \le A^2\theta \le Au = \upsilon \le A\theta.$$

Consequently, $u \ge \varepsilon\upsilon$ and $\upsilon \ge \varepsilon u$. Let $t_0 = \sup\{t > 0 : u \ge t\upsilon\}$. We have $\varepsilon \le t_0 \le \infty$. If $t_0 < 1$, then, by virtue of (H_1), there exists a $\eta > 0$ such that

$$A(t_0\upsilon) \le [t_0(1+\eta)]^{-1} A\upsilon = [t_0(1+\eta)]^{-1} u. \quad \text{... (1)}$$

On the other and, from $u \ge t_0\upsilon$, we find

$$\upsilon = Au \le A(t_0\upsilon). \quad \text{... (2)}$$

It follows from (1) and (2) that $u \ge t_0(1+\eta)\upsilon$, which contradicts the definition of t_0. Hence $t_0 \ge 1$, and therefore $u \ge \upsilon$. Similarly, letting $t_1 = \sup\{t > 0 : \upsilon \ge tu\}$, we can show that $t_1 \ge 1$, and so $\upsilon \ge u$. Thus $u = \upsilon$. □

Theorem 1 — *Let $A : P \to P$ be a decreasing operator satisfying condition (H_1). If P is strongly minihedral, then A has a unique fixed point x^* in P.*

PROOF : Let $S = \{x \in P : x \ge A^2x\}$. Since $A^2\theta \ge \theta$, we have $A^3\theta \le A\theta$, and so $A\theta \in S$ and $S \ne \theta$. By the strong minihedrality of P, the greatest lower bound inf $S = u^*$ exists a $u^* \ge \theta$. For $x \in S$, we have $x \ge u^*$, so $Ax \le Au^*$, $x \ge A^2x \ge A^2u^*$. This shows that $u^* \ge A^2u^*$. On the other hand, from $u^* \ge A^2u^*$ we know that $Au^* \le A^3u^*$, so $A^2u^* \ge A^4u^*$, which implies that $A^2u^* \in S$, and therefore

$A^2u^* \geq u^*$. Hence, we have $A^2u^* = u^*$. Let $u_* = Au^*$, then $Au_* = A^2u^* = u^*$, so, by Lemma 1, we conclude that $u_* = u^*$, and $Au^* = u^*$.

Let $F = \{x \in P : Ax = x\}$. Since $u^* \in F$, we see that $F \neq \phi$. It is easy to see that $\theta \leq x \leq A\theta$ for $x \in F$, so, the strong minihedrality of P implies that $x_* = \inf F$ and $x^* = \sup F$ exist, and

$$\theta \leq x_* \leq x^* \leq A\theta. \qquad \text{... (3)}$$

Since $x_* \leq x \leq x^*$ for $x \in F$, we have

$$Ax^* \leq Ax = x \leq Ax_*, \quad \forall\, x \in F,$$

which implies that

$$Ax^* \leq x_* \leq x^* \leq Ax_*. \qquad \text{... (4)}$$

It follows from (3), (4) and (H_1) that

$$x_* \geq Ax^* \geq A^2\theta \geq \varepsilon A\theta \geq \varepsilon x^*.$$

Let $t^* = \sup\,\{t > 0 : x_* \geq tx^*\}$. We have $\varepsilon \leq t^* \leq \infty$. If $t^* < 1$, then (H_1) implies that there exists a $\eta^* > 0$ such that

$$A(t^*x^*) \leq [t^*(1+\eta^*)]^{-1}Ax^*. \qquad \text{... (5)}$$

From (4), (5) and the fact $x_* \geq t^*x^*$, we find

$$x^* \leq Ax_* \leq A(t^*x^*) \leq [t^*(1+\eta^*)]^{-1}\,Ax^* \leq [t^*\,(1+\eta^*)]^{-1}\,x_*,$$

so $x_* \geq t^*\,(1+\eta^*)x^*$, which contradicts the definition of t^*. Hence $t^* \geq 1$, and

$$x_* \geq x^*. \qquad \text{... (6)}$$

Finally, (4) and (6) imply that $x_* = x^*$ and $Ax^* = x^*$. It is clear, x^* is the unique fixed point of A in P.

***Theorem* 2** — *Let $A : P \to P$ be a decreasing operator satisfying condition (H_2). If P is normal, then A has a unique fixed point x^* in P, and, for any $x_0 \in P$, we have*

$$\| x_n - x^* \| \to 0 \;\; (n \to \infty), \qquad \text{... (7)}$$

where

$$x_n = Ax_{n-1} \;\; (n = 1, 2, 3, \ldots). \qquad \text{... (8)}$$

PROOF : If $A\theta=\theta$, then $x\geq\theta$ implies $\theta\leq Ax\leq A\theta=\theta$, hence $Ax = \theta$ for all $x\in P$. In this case, A has a unique fixed point $x^*=\theta$ in P, and (7) holds obviously. Thus, may assume that

$$A\theta>\theta. \quad \text{... (9)}$$

Let $u_0=\theta$, $u_n=Au_{n-1}$ ($n=1, 2, 3, \ldots$). Since $A:P\to P$ is decreasing, it is easy to see that

$$\theta=u_0\leq u_2\leq\ldots\leq u_{2n}\leq\ldots\leq u_{2n+1}\leq\ldots\leq u_3\leq u_1=A\theta. \quad \text{... (10)}$$

By (H_2) and (9), we have

$$u_2=A^2\theta\geq\varepsilon A\theta=\varepsilon u_1>\theta. \quad \text{... (11)}$$

It follows from (9), (10) and (11) that

$$u_{2n+1}\geq u_{2n}>\theta,\ u_{2n}\geq\varepsilon u_{2n+1}\ (n = 1, 2, 3, \ldots). \quad \text{... (12)}$$

Let $t_n = \sup\{t>0:u_{2n}\geq tu_{2n+1}\}$ ($n = 1, 2, 3, \ldots$). From (10) and (12), we find

$$\varepsilon\leq t_1\leq t_2\leq\ldots\leq t_n\leq\ldots\leq 1, \quad \text{... (13)}$$

and

$$u_{2n}\geq t_nu_{2n+1}\ (n = 1, 2, 3, \ldots). \quad \text{... (14)}$$

Hence

$$\varepsilon\leq\lim_{n\to\infty} t_n=t^*\leq 1.$$

We now prove

$$t^*=1. \quad \text{... (15)}$$

In fact, if $t^*<1$, then (H_2) implies that there exists a $\eta>0$ and a $\delta>0$ such that

$$A(tx)\leq[t(1+\eta)]^{-1}Ax,\ \forall\ \varepsilon A\theta\leq x\leq A\theta,\ t^*-\delta\leq t\leq t^*. \quad \text{... (16)}$$

Choosing positive integer m such that $t^*-\delta\leq t_n\leq t^*$ for $n\geq m$, we have by (16), (10) and (11),

$$A(t_nu_{2n+1})\leq[t_n(1+\eta)]^{-1}Au_{2n+1},\ \forall\ n\geq m. \quad \text{... (17)}$$

It follows from (14) and (17) that

$$u_{2n+3}\leq u_{2n+1}=Au_{2n}\leq A(t_nu_{2n+1})\leq[t_n(1+\eta)]^{-1}u_{2n+2},\ \forall\ n\geq m,$$

that is,

$$u_{2n+2} \geq t_n(1+\eta)u_{2n+3}, \;\; \forall\, n \geq m,$$

which implies that

$$t_{n+1} \geq t_n(1+\eta), \;\; \forall\, n \geq m.$$

Hence

$$t_{m+n} \geq t_m\,(1+\eta)^n \geq \varepsilon(1+\eta)^n \to \infty \;\; (n \to \infty),$$

a contradiction to (13). Thus (15) holds.

By virtue of (10) and (14), we have

$$\theta \leq u_{2n+2p} - u_{2n} \leq u_{2n+1} - u_{2n} \leq (1-t_n)\,u_{2n+1} \leq (1-t_n)u_1,$$

$$(n,\; p = 1, 2, 3, \ldots). \qquad \ldots (18)$$

It follows from (18), (15) and the normality of P that $\{u_{2n}\}$ is a fundamental sequence in E, so $u_{2n} \to u_*$ for some $u_* \in$ E. Similarly, we can show that $u_{2n+1} \to u^* \in E$. It is clear,

$$u_{2n} \leq u_* \leq u^* \leq u_{2n+1} \;\; (n = 1, 2, 3, \ldots), \qquad \ldots (19)$$

which implies that

$$u_{2n+2} = Au_{2n+1} \leq Au^* \leq Au_* \leq Au_{2n} = u_{2n+1}, \; (n = 1, 2, 3, \ldots). \qquad \ldots (20)$$

Taking limit as $n \to \infty$ in (20), we find

$$u_* \leq Au^* \leq Au_* \leq u^*. \qquad \ldots (21)$$

On the other hand, from (18) and (19) we have

$$\theta \leq u^* - u_* \leq u_{2n+1} - u_{2n} \leq (1-t_n)u_1, \; (n = 1, 2, 3, \ldots),$$

which implies by virtue of (15) and the normality of P that

$$u_* = u^*. \qquad \ldots (22)$$

It follows from (21) and (22) that $u_* = Au^* = Au_* = u^*$. Let $x^* = u_* = u^*$. Then $Ax^* = x^*$.

Let $x_0 \in P$ be given and $\{x_n\}$ be the sequence defined by (8). It is easy to see that $u_0 \leq x_1 \leq u_1$. Assume that $u_{2k} \leq x_{2k+1} \leq u_{2k+1}$. Then

$$u_{2k+1} \geq x_{2k+2} \geq u_{2k+2}, \; u_{2k+2} \leq x_{2k+3} \leq u_{2k+3}.$$

So, by induction, we have

$$u_{2n} \leq x_{2n+1} \leq u_{2n+1} \quad (n = 0, 1, 2, \ldots), \qquad \ldots (23)$$

and therefore

$$u_{2n+2} \leq x_{2n+2} \leq u_{2n+1} \quad (n = 0, 1, 2, \ldots), \qquad \ldots (24)$$

Since $u_{2n} \to u_* = x^*$ and $u_{2n+1} \to u^* = x^*$, it follows from (23), (24) and the normality of P that $x_{2n+1} \to x^*$ and $x_{2n+2} \to x^*$, that is, (7) holds. In particular, let $x_0 = \bar{x}$, where $\bar{x}$ is any fixed point of A in P, then $x_n = \bar{x}$ (n = 0, 1, 2, ...), and so, (7) implies that $\bar{x} = x^*$. This shows that A has only one fixed point x^* in P. □

***Theorem* 3** — *Let $A : P \to P$ be a decreasing operator satisfying condition (H3). If P is normal, then A has a unique fixed point x^* in P, and, for any $x_0 \in P$, (7) holds with the rate of convergence*

$$\| x_{2n+1} - x^* \| \leq 2N^2 \| A\theta \| (1 - \varepsilon^{\alpha^{n-1}}), \quad (n = 1, 2, 3, \ldots) \qquad \ldots (25)$$

and

$$\| x_{2n+2} - x^* \| \leq 2N^2 \| A\theta \| (1 - \varepsilon^{\alpha^{n-1}}), \quad (n = 1, 2, 3, \ldots) \qquad \ldots (26)$$

where sequence $\{x_n\}$ is defined by (8) and N denotes the normal constant of P.

PROOF : The proof is almost the same as that of Theorem 2. The only difference is the establishment of (15). We now prove

$$t_n \geq \varepsilon^{\alpha^{n-1}} \quad (n = 1, 2, 3, \ldots). \qquad \ldots (27)$$

In fact, from (13) we see that (27) is true for n = 1. Assume that (27) is true for $n = k$, i.e. $t_k \geq \varepsilon^{\alpha^{k-1}}$, then, by virtue of (H_3), (14) and the fact $\varepsilon \leq \varepsilon^{\alpha^{k-1}} < 1$, we have

$$u_{2k+1} = Au_{2k} \leq A(t_k u_{2k+1}) \leq A(\varepsilon^{\alpha^{k-1}} u_{2k+1}) \leq (\varepsilon^{\alpha^{k-1}})^{-\alpha} Au_{2k+1} = \varepsilon^{-\alpha^k} u_{2k+2},$$

and so

$$u_{2k+2} \geq \varepsilon^{\alpha^k} u_{2k+1} \geq \varepsilon^{\alpha^k} u_{2k+3},$$

which implies that $t_{k+1} \geq \varepsilon^{\alpha^k}$. Hence, by induction, (27) holds. Since $\varepsilon^{\alpha^{n-1}} \to 1$ as $n \to \infty$, from (13) and (27) we see that (15) holds.

Finally, we establish (25) and (26). By virtue of (19) and (23), we have

$$\| x_{2n+1} - x^* \| \leq \| x_{2n+1} - u_{2n} \| + \| u_{2n} - x^* \| \leq 2N \| u_{2n+1} - u_{2n} \|. \qquad \ldots (28)$$

On the other hand, (18) and (27) imply that

$$+ \;\theta \leq u_{2n=1} - u_{2n} \leq (1 - \varepsilon^{\alpha^{n-1}})\, A\theta,$$

so

$$\| u_{2n+1} - u_{2n} \| \leq N \| A\theta \| \,(1 - \varepsilon^{\alpha^{n-1}}). \qquad \ldots (29)$$

It follows from (28) and (29) that (25) holds. Similarly, using (24) instead of (23), we can show that (26) holds. □

Remark 1 : In Theorem 1 to 3, we do not require operator A to be compact or continuous.

3. APPLICATION

Consider the initial value problem for second order integro-differential equation on infinite interval:

$$\begin{cases} x''(t) = a(t) + \left\{ a_0(t) + \sum_{i=1}^{n} a_i(t)\, [x(t)]^{\alpha_i} + \int_0^t k(t, s) x(s) ds \right\}^{-1}, \\ \quad 0 \leq t < \infty; \\ x(0) = b_0,\, x'(0) = b_1, \end{cases} \qquad \ldots (30)$$

where $a(t), a_0(t), a_i(t)$ (i = 1, 2, ..., n) are non-negative continuous functions on $0 \leq t, \infty < k(t, s)$ is non-negative and continuous on $0 \leq s \leq t < \infty$, $0 < \alpha_1 < .... < \alpha_i < \ldots < \alpha_{n-1} < \alpha_n = 1$, $b_0 \geq 0$, $b_1 \geq 0$. Assume that

$$a(t) \leq (c_0 t^2 + c_1)^{-1},\; a_0(t) \geq c_2 t^m + c_3,$$

$$\sum_{i=1}^{n} a_i(t) \leq c_4 t^r + c_5, \;\; \forall\; 0 \leq t < \infty, \qquad \ldots (31)$$

and

$$k(t, s) \leq \beta_0 t^p + \beta_1 s^q + \beta_2, \;\; \forall\; 0 \leq s \leq t < \infty, \qquad \ldots (32)$$

where c_i (i = 0, 1, 2, 3, 4, 5), β_j (j = 0, 1, 2) are positive constants and m, r, p, q are non-negative integers such that

$$m \geq \max \{r + 1,\; p + 2,\; q + 2\}. \qquad \ldots (33)$$

Lemma 2 — $x(t)$ is a non-negative C^2 solution of initial value problem (30) if and only if $x(t)$ is a non-negative continuous solution of the following integral equation on infinite interval:

$$x(t) = b_0 + b_1 t + \int_0^t (t - s)\, a(s) ds$$

$$+\int_0^t (t-s)\left\{a_0(s)+\sum_{i=1}^n a_i(s)\,[x(s)]^{\alpha_i}+\int_0^s k(s,s_1)\,x(s_1)ds_1\right\}^{-1} ds, \qquad \dots (34)$$

$\forall\ 0 \le t < \infty.$

PROOF : If $x(t)$ is a C^2 function on $0 \le t < \infty$, then, substituting

$$x'(t) = x'(0) + \int_0^t x''(s)ds, \ \forall\ 0 \le t < \infty$$

into

$$x(t) = x(0) + \int_0^t x'(s)ds, \ \forall\ 0 \le t < \infty,$$

we get a simple formula:

$$x(t) = x(0) + tx'(0) + \int_0^t (t-s)x''(s)ds, \ \forall\ 0 \le t < \infty. \qquad \dots (35)$$

Now, if $x(t)$ is a non-negative C^2 solution of initial value problem (30), then, substituting (30) into (35), we find that $x(t)$ satisfies (34).

Conversely, if $x(t)$ is a non-negative continuous solution of Eq. (34), then, by direct differentiation of (34), we find

$$x'(t) = b_1 + \int_0^t a(s)ds + \int_0^t \left\{a_0(s)+\sum_{i=1}^n a_i(s)\,[x(s)]^{\alpha_i}+\int_0^s k(s,s_1)x(s_1)ds_1\right\}^{-1} ds,$$

$\forall\ 0 \le t < \infty.$

and

$$x''(t) = a(t) + \left\{a_0(t)+\sum_{i=1}^n a_i(t)\,[x(t)]^{\alpha_i}+\int_0^t k(t,s)x(s)ds\right\}^{-1}, \ \forall\ 0 \le t < \infty,$$

so $x(t)$ belongs to C^2 and satisfies (30). □

From (33) we know that $m \ge 2$, and therefore,

$$a^* = \int_0^\infty \frac{dt}{c_0t^2+c_1} < \infty,\ a_0^* = \int_0^\infty \frac{dt}{c_2t^m+c_3} < \infty,\ b_2 = b_1 + a^* + a_0^* > 0. \qquad \dots (36)$$

Lemma 3 — There exists a small constant $\varepsilon_0 > 0$, which satisfies the following condition: if

$x(t)$ is a non-negative continuous function on $0 \le t < \infty$ satisfying $x(t) \le b_0 + b_2 t$ for $0 \le t < \infty$, where b_2 is defined by (36), then

$$a_0(t) \ge \varepsilon_0 \left(\sum_{i=1}^{n} a_i(t)[x(t)]^{\alpha_i} + \int_0^t k(t,s)x(s)ds \right), \quad \forall\ 0 \le t < \infty, \qquad \text{... (37)}$$

PROOF : Let $x(t)$ be a non-negative continuous function on $0 \le t < \infty$, which satisfies $x(t) \le b_0 + b_2 t$ for $0 \le t < \infty$. By (31) and (32), we have

$$\sum_{i=1}^{n} a_i(t)\,[x(t)]^{\alpha_i} \le \left(\sum_{i=1}^{n} a_i(t) \right)(1 + b_0 + b_2 t) \le (c_4 t^r + c_5)\,(1 + b_0 + b_2 t)$$

$$\le b_3 t^{r+1} + b_4, \quad \forall\ 0 \le t < \infty \qquad \text{... (38)}$$

and

$$\int_0^t k(t,s)x(s)ds \le \int_0^t (\beta_0 t^p + \beta_1 s^q + \beta_2)\,(b_0 + b_2 s)ds$$

$$\le b_5 t^{p+2} + b_6\, t^{q+2} + b_7, \quad \forall\ 0 \le t < \infty, \qquad \text{... (39)}$$

where b_3, b_4, b_5, b_6, b_7 are positive constants. From (33) we see that there exists a small constant $\varepsilon_0 > 0$ such that

$$c_2 t^m + c_3 \ge \varepsilon_0\,(b_3 t^{r+1} + b_4 + b_5\, t^{p+2} + b_6\, t^{q+2} + b_7), \quad \forall\ 0 \le t < \infty. \qquad \text{... (40)}$$

It follows from (31), (38), (39) and (40) that (37) holds. □

***Theorem* 4** — *Initial value problem (30) has a unique non-negative C^2 solution $x^*(t)$ on $0 \le t < \infty$. Moreover, for any non-negative continuous on $0 \le t < \infty$ function $x_0(t)$, the sequence of functions $\{x_j(t)\}$ defined by*

$$x_j(t) = b_0 + b_1 t + \int_0^t (t-s)a(s)ds$$

$$+ \int_0^t (t-s) \left\{ a_0(s) + \sum_{i=1}^{n} a_i(s)\,[x_{j-1}(s)]^{\alpha_1} + \int_0^s k(s,s_1)x_{j-1}(s_1)ds_1 \right\}^{-1} ds, \qquad \text{... (41)}$$

$\forall\ 0 \le t < \infty$ *(j = 1, 2, 3, ...)*

converges to $x^*(t)$ uniformly on any finite interval [0, R], where R is any positive number.

PROOF : Let $J=[0,\infty)$, $C(J) = \{x : x(t)$ is a continuous function on $J\}$ and $E = \left\{ x \in C(J) : \sup_{t \in J} \frac{|x(t)|}{t+1} < \infty \right\}$. It is easy to see that E is a Banach space with norm

$$\|x\| = \sup_{t \in J} \frac{|x(t)|}{t+1}.$$

Let $C_+(J) = \{x \in C(J) : x(t) \geq 0 \text{ for } t \in J\}$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in J\}$. Obviously, $P \subset C_+(J)$ and P is a normal cone in E. Consider operator A defined by

$$(Ax)(t) = b_0 + b_1 t + \int_0^t (t-s)a(s)ds$$

$$+ \int_0^t (t-s) \left\{ a_0(s) + \sum_{i=1}^n a_i(s)\,[x(s)]^{\alpha_i} + \int_0^s k(s, s_1)\, x(s_1) ds_1 \right\}^{-1} ds, \quad \ldots (42)$$

$\forall\ 0 \leq t < \infty.$

For $x \in C_+(J)$, we have by virtue of (31),

$$0 \leq (Ax)(t) \leq b_0 + b_1 t + t \int_0^t a(s)ds + t \int_0^t \frac{ds}{a_0(s)}$$

$$\leq b_0 + b_1 t + t \left(\int_0^t \frac{ds}{c_0 s^2 + c_1} + \int_0^t \frac{ds}{c_2 s^m + c_3} \right)$$

$$\leq b_0 + b_1 t + t(a^* + a_0^*) = b_0 + b_2 t, \ \forall\ 0 \leq t < \infty, \quad \ldots (43)$$

where a^*, a_0^* and b_2 are defined by (36). Hence

$$A(C_+(J)) \subset P \text{ and } \|Ax\| \leq b_0 + b_2, \ \forall\ x \in C_+(J). \quad \ldots (44)$$

Consequently, $A : P \to P$ and it is clear from (42) that A is decreasing. Now, we verify that A satisfies condition (H_2). Letting $x = \theta$ in (43), we find

$$0 \leq (A\theta)(t) = b_0 + b_1 t + \int_0^t (t-s)a(s)ds + \int_0^t (t-s)\,[a_0(s)]^{-1} ds$$

$$\leq b_0 + b_2 t, \ \forall\ 0 \leq t < \infty. \quad \ldots (45)$$

so, Lemma 3 implies that

$$a_0(t) \geq \varepsilon_0 \left(\sum_{i=1}^{n} a_i(t) [(A\theta)(t)]^{\alpha_i} + \int_0^t k(t,s) [(A\theta)(s)] ds \right), \quad \forall\ 0 \leq t < \infty. \qquad \text{... (46)}$$

where $\varepsilon_0 > 0$ is the constant given in Lemma 3. It follows from (45) and (46) that

$$(A^2\theta)(t) \geq b_0 + b_1 t + \int_0^t (t-s)a(s)ds + \int_0^t (t-s) \{a_0(s) + \varepsilon_0^{-1} a_0(s)\}^{-1} ds$$

$$\geq \frac{\varepsilon_0}{1+\varepsilon_0} \left\{ b_0 + b_1 t + \int_0^t (t-s)a(s)ds + \int_0^t (t-s) [a_0(s)]^{-1} ds \right\}$$

$$= \varepsilon (A\theta)(t), \quad \forall\ 0 \leq t < \infty,$$

where $\varepsilon = \dfrac{\varepsilon_0}{1+\varepsilon_0}$, $0 < \varepsilon < 1$. Hence $A^2\theta \geq \varepsilon A\theta$.

Let $\varepsilon \leq t^* < 1$ be given. We prove the following inequality:

$$a_0(t) + \sum_{i=1}^{n} a_i(t) [x(t)]^{\alpha_i} + \int_0^t k(t,s)x(s)ds$$

$$\leq \tau \left\{ (t^*)^{-1} a_0(t) + \sum_{i=1}^{n} a_i(t) [x(t)]^{\alpha_i} + \int_0^t k(t,s)x(s)ds \right\}, \qquad \text{... (47)}$$

$\forall\ \theta \leq x \leq A\theta,\ 0 \leq t < \infty,$

where

$$\tau = t^* + \frac{(1-t^*)t^*}{t^* + \varepsilon_0},\ t^* < \tau < 1. \qquad \text{... (48)}$$

It is easy to see that (47) is equivalent to

$$(\tau - t^*)\, a_0(t) \geq (1-\tau)\, t^* \left\{ \sum_{i=1}^{n} a_i(t) [x(t)]^{\alpha_i} + \int_0^t k(t,s)x(s)ds \right\}, \qquad \text{... (49)}$$

$\forall\ \theta \leq x \leq A\theta,\ 0 \leq t < \infty.$

When $\theta \leq x \leq A\theta$, we have by (45) that $0 \leq x(t) \leq b_0 + b_2 t$ for $0 \leq t < \infty$, so, Lemma 3 implies that (37) holds. It follows from (48) and (37) that

$$(\tau - t^*)a_0(t) = \frac{(1-t^*)t^*}{t^* + \varepsilon_0} a_0(t) = (1-\tau)\, \varepsilon_0^{-1}\, t^*\, a_0(t)$$

$$\geq (1-\tau)t^* \left\{ \sum_{i=1}^{n} a_i(t)\,[x(t)]^{\alpha_i} + \int_0^t k(t,s)x(s)ds \right\},$$

that is, (49) holds, and therefore, (47) is true. Let $\eta = \tau^{-1} - 1$. Then $\eta > 0$.

For any $\theta \leq x \leq A\theta$ and $0 < \gamma \leq t^*$, we have by (47) and (48),

$$A(\gamma x)(t) = b_0 + b_1 t + \int_0^t (t-s)a(s)ds$$

$$+ \int_0^t (t-s) \left\{ a_0(s) + \sum_{i=1}^{n} a_i(s)\gamma^{\alpha_i}\,[x(s)]^{\alpha_i} + \gamma \int_0^s k(s,s_1)x(s_1)ds_1 \right\}^{-1} ds,$$

$$\leq \frac{\tau}{\gamma}\left[b_0 + b_1 t + \int_0^t (t-s)\,a(s)ds \right]$$

$$+ \frac{1}{\gamma} \int_0^t (t-s) \left\{ \frac{a_0(s)}{t^*} + \sum_{i=1}^{n} a_i(s)[x(s)]^{\alpha_i} + \int_0^s k(s,s_1)x(s_1)ds_1 \right\}^{-1} ds,$$

$$\leq \frac{\tau}{\gamma}\left[b_0 + b_1 t + \int_0^t (t-s)\,a(s)ds \right.$$

$$\left. + \int_0^t (t-s) \left\{ a_0(s) + \sum_{i=1}^{n} a_i(s)\,[x(s)]^{\alpha_i} + \int_0^s k(s,s_1)x(s_1)ds_1 \right\}^{-1} ds \right]$$

$$= [\gamma(1+\eta)]^{-1}\,(Ax)\,(t), \;\; \forall\; 0 \leq t < \infty.$$

So, (H_2) is satisfied. Hence, by Theorem 2, A has a unique fixed point x^* in P and (7) holds. Since $A(C_+(J)) \subset P$ on account of (44), x^* is the unique fixed point of A in $C_+(J)$, and therefore, Lemma 2 implies that $x^*(t)$ is the unique non-negative C^2 solution of (30). Finally, (7) implies that the sequence $\{x_j(t)\}$ defined by (41) converges to $x^*(t)$ uniformly on any finite interval $[0, R]$ $(R, > 0)$. □

Remark 2 : In general, operator A defined by (42) is noncompact in P.

REFERENCES

1. Dajun Guo, *Nonlinear Anal. TMA*, **10** (1986), 1293-1302.

2. Dajun Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston/New York, 1988.