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EXISTENCE UNIQUENESS AND CONTINUABILITY OF SOLUTIONS OF IMPULSIVE DIFFERETIAL-DIFFERENCE EQUATIONS

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The initial value problem for impulsive differential-difference equations is considered. There are obtained sufficient conditions on existence, uniqueness and continuability of solutions of such problems.

Key Words : Existence Uniqueness; Differetial-Difference Equations

1. INTRODUCTION

The necessity of studying of impulsive differential-difference equations due to the fact that these equations are a useful mathematical machinery in modeling of many processes and phenomena studied in chemie, biology, biotechnology, medicine, mechanics, theory of optimal control etc.

Processes, which adequate mathematical models are the impulsive differential-difference equations, are characterized with a per saltum changing of their state as well as with the fact that the processes under consideration depend on their prehistory at each moment of time.

Impulsive differential-difference equations are a natural generalizations to the impulsive differential equations. Their theory is more interesting than the theory of impulsive ordinary differential equations. However, this theory is developed slowly due to the difficulties of technical and theoretical character.

In the present work we consider the problems of existence, uniqueness and continuability of solutions of nonlinear impulsive differential-difference equations. Analogous results, concerning linear impulsive differential-difference equations, are obtained.

2. STATEMENT OF THE PROBLEM PRELIMINARY NOTES

Let $I\!R^n$ be the n-Euclidean space with elements $x = col\,(x_1, \ldots, x_n)$ and a norm $|\cdot| : \Omega \subset I\!R^n$, $\Omega \neq \phi$ and Ω is a domain; h = const ≥ 0; $(t_0, x_0) \in I\!R \times \Omega$.

We consider the next initial problem

$$\dot{x}(t) = f(t, x(t), x(t-h)),\ t \neq \tau_k(x(t)),\ t > t_0,$$

$$x(t) = \varphi(t),\ t \in [t_0 - h, t_0],$$

$$x(t_0 + 0) = x_0. \qquad \dots (1)$$

$$\Delta X(t)\,|_{t = \tau_k(x(t))} = I_k(x(t)),\ t > t_0,\ k = 1, 2, \dots$$

where $f : [t_0, \infty) \times \Omega \times \Omega \to I\!R^n$; $\tau_k : \Omega \to [t_0, \infty)$; $I_k : \Omega \to I\!R^n$, $k = 1, 2, \dots$; $\Delta x(t) = x(t+0) - x(t-0)$; $\varphi : [t_0 - h, t_0] \to I\!R^n$

Let us introduce the following notations :
$\sigma_k = \{(t, x) : t = \tau_k(x), x \in \Omega\}$, i.e. σ_k, $k = 1, 2, \dots$ are hypersurfaces with equations $t = \tau_k(x)$. We denote by $PC(t_0)$ the space of all functions $\varphi : [t_0 - h, t_0] \to I\!R^n$ having points of discontinuity $\theta_1, \theta_2, \dots, \theta_s \in (t_0 - h, t_0)$ of first kind and that are continuous from the left at these points. We shall denote by $x(t) = x(t; t_0, x_0, \varphi)$ solution of the problem (1), and by $J^+(t_0, x_0, \varphi)$ the maximal interval of the type $[t_0, \beta)$ where the solution $x(t; t_0, x_0, \varphi)$ is defined.

Let $\tau_0(x) = t_0$ for $x \in \Omega$ and the function $\varphi \in PC(t_0)$. We will give a precise description of the solution $x(t)$ of problem (1):

1. If $t_0 - h \leq t \leq t_0$ then the solution $x(t)$ of problem (1) coincides with the function $\varphi(t)$.

2. Assume $t_1, t_2, \dots$ $(t_0 < t_1 < t_2 < \dots < t_k < t_{k+1} < \dots)$ to be the moments at which the integral curve $(t, x(t))$ reaches the hypersurfaces $\{\sigma_k\}_{k=1}^{\infty}$, i.e. $t = t_k$, $k = 1, 2, \dots$ solves some of the equations $t = \tau_k(x(t))$. Let $t_l^h = t_l + h$, $l = 0, 1, 2, \dots$ and $\theta_z^h = \theta_z + h$, z = 1, 2, ... S. We define the sequence as follows:

a) $\{\tau_i\}_{i=0}^{\infty} = \{t_k\}_{k=0}^{\infty} \cup \{t_l^h\}_{l=0}^{\infty} \cup \{\theta_z^h\}_{z=1}^{s}$;

b) $\tau_0 = t_0$;

c) the sequence $\{\tau_i\}_{i=0}^{\infty}$ is monotonically increasing.

It is possible

$$\{t_k\}_{k=1}^{\infty} \cap \left(\{t_l^h\}_{l=0}^{\infty} \cup \{\theta_z^h\}_{z=1}^{s} \right) \neq \phi$$

in general.

2.1. If $\tau_0 \leq t \leq \tau_1$ then the $x(t)$ of problem (1) coincides with the solution of the problem :

$$\dot{x}(t) = f(t, x(t), x(t-h)),$$

$$x(t) = \varphi(t),\ t \in [t_0 - h, t_0],$$

$$x(t_0 + 0) = x_0. \qquad \text{... (2)}$$

2.2 If $\tau_1 < t \leq \tau_{i+1}$, $i = 1, 2, \ldots$ then only one of the next three cases is possible :

a) If $\tau_i \in \{t_k\}_{k=1}^{\infty} \left(\{t_l^h\}_{l=0}^{\infty} \setminus \bigcup \{\theta_z^h\}_{z=1}^{s} \right) \tau_i = t_k$ and i_k is the number of the hypersurface, that the integral curve $(t, x(t))$ reaches at the moment t_k, then the solution $x(t)$ of problem (1) coincides with the solution of the problem :

$$\dot{y}(t) = f(t, y(t), x(t-h)),$$

$$y(t_k) = x(t_k) + I_{i_k}(x(t_k));$$

b) If $\tau_i \in \left(\{t_l^h\}_{l=0}^{\infty} \bigcup \{\theta_z^h\}_{z=1}^{s} \right) \setminus \{t_k\}_{k=1}^{\infty}$ then the solution $x(t)$ coincides with the solution of the problem :

$$\dot{y}(t) = f(t, y(t), x(t-h+0)),$$

$$y(\tau_i) = x(\tau_i);$$

c) If $\tau_i \in \{t_k\}_{k=1}^{\infty} \cap \left(\{t_l^h\}_{l=0}^{\infty} \bigcup \{\theta_z^h\}_{z=1}^{s} \right)$ and $\tau_i = t_k$ then the solution $x(t)$ coincides with the solution of the problem :

$$\dot{y}(t) = f(t, y(t), x(t-h+0)),$$

$$y(t_k) = x(t_k) + I_{i_k}(x(t_k)).$$

3. If the point $x(t_k) + I_{i_k}(x(t_k)) \,\bar{\varepsilon}\, \Omega$ then the solution $x(t)$ is not defined as $t > t_k$.

4. The function $x(t)$ is partially continuous on its definite domain $J^+(t_0, x_0, \varphi)$, continuous from the left at the points $t_k \in J^+(t_0, x_0, \varphi)$ and $x(t_k + 0) = x(t_k) + I_{i_k}(x(t_k))$, $k = 1, 2, \ldots$

5. The function $x(t)$ is partially differentiable with respect to t and

$$\dot{x}(t) = f(t, x(t), x(t-h)),\ t \in J^+(t_0, x_0, \varphi),\ t \neq \tau_i,\ i = 0, 1, \ldots$$

Let us define the following sets :

$$G_k = \{(t, x) \in [t_0, \infty) \times \Omega : \tau_{k-1}(x) < t < \tau_k(x)\},\ k = 1, 2, \ldots$$

$$F_k = \{(t, x) \in [t_0, \infty) \times \Omega : \tau_{k-1}(x) \leqq t < \tau_k(x)\} \quad k = 1, 2, \ldots$$

We shall say that the conditions (H) hold if the following conditions are valid :

H_1 : The function f is continuous with respect to its first argument on $[t, \infty) \times \Omega \times \Omega$.

H_2 : The function f is locally Lipschitz continuous with respect to its second and third arguments on $[t, \infty) \times \Omega \times \Omega$.

H_3 : There exists a constant $M > 0$ such that $|f(t, x, y)| \leq M$ for $(t, x, y) \in [t_0, \infty) \times \Omega \times \Omega$.

H_4 : The functions $I + I_k : \Omega \to \Omega$, k = 1, 2, ..., where I is the identity mapping on $\mathbb{R}^n$.

H_5 : $(t, x + I_k(x)) \in F_k$ for $(t, x) \in \sigma_{k-1}$ $k = 1, 2, \ldots$

H_6 : The functions τ_k are Lipschitz' functions with respect to x in Ω with a constants L_k, $0 \leq L_k < \frac{1}{M}$, $k = 1, 2, \ldots$

H_7 : The inequalities

$$t_0 = \tau_0(x) < \tau_1(x) < \ldots, \tau_k(x) < \tau_{k+1}(x) < \ldots, x \in \Omega$$

hold.

H_8 : $\tau_k(x) \to \infty$ as $k \to \infty$, uniformly on $x \in \Omega$.

$H9$: For each $(t_0, x_0, \varphi) \in \mathbb{R} \times \Omega \times PC(t_0)$ the solution $\phi(t; t_0, x_0, \varphi)$ of the initial problem (2) without impulses does not leave the domain Ω for $t \in J$ where

$$J = \begin{cases} (t_0, \infty) \text{ if the } t_i \text{ are a finite number,} \\ U(t_{i-1}, t_i] \text{ if the } t_i \text{ are infinitely many.} \end{cases}$$

Let us note that in the case of impulsive systems of differential-difference equations is possible to arise the so-called "beating" of solutions, i.e. a phenomena when the integral curve $(t, x(t))$ reaches finitely (or infinitely) many times one and the same hypersurface. In the present work we consider the case of absence of "beating" only. The phenomena "beating" was considered in details by Bainov and Dushliev[1] for impulsive functional-differential (as well as for impulsive differential-difference) equations. Effective sufficient conditions are proved for absence of "beating" of solutions. In the sequel we shall use the next lemma.

Lemma 1 — (Bainov and Dishliev[1]) Assume the following conditions hold :

1. Conditions $H_1, H_3 - H_9$ are valid.

2. $(t_0, x_0) \in F_k$, $k = 1, 2, \ldots$

Then the integral curves of problem (1) reach successively each of the hypersurfaces $\sigma_k, \sigma_{k+1}, \ldots$ only once.

The already formuled Lemma guarantees absence of "beating" of solutions of problem (1) as well as the validity of the following properties :

1. For each $(t_0, x_0, \varphi) \in \sigma_k \times PC(t_0)$, $k = 1, 2, \ldots$ there exists a constant $\beta > t_0$ such that $(t, \phi(t; t_0, x_0, \varphi)) \in G_{k+1}$, $t \in (t_0, \beta)$ where $\phi(t; t_0, x_0, \varphi)$ is a solution of problem (2).

2. If $(t_0, x_0, \varphi) \in G_{k+1} \times PC(t_0)$, $k = 1, 2, \ldots$ and $x(t; t_0, x_0, \varphi)$ is a solution of problem (1) then

$$\{(t, x(t; t_0, x_0, \varphi)); t \geq t_0\} \cap \{\sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_k\} = \phi.$$

3. Main Results

***Theorem* 1** — *Assume the conditions $H_1, H_3 - H_9$ hold.*

Then :

1. For each point $(t_0, x_0) \in \mathbb{R} \times \Omega$ and for each fnction $\varphi \in PC(t_0)$ there exists solution $x(t) = x(t; t_0, x_0, \varphi)$ of the initial value problem (1) defined on $J^+(t_0, x_0, \varphi)$.

2. $J^+(t_0, x_0, \varphi) = [t_0, \infty)$.

3. If, moreover, condition H_2 is fulfilled then the solution $x(t; t_0, x_0, \varphi)$ is unique.

Proof of Assertion 1 : The validity of H1, H9 as well as the existence theorem applied to the problem (2) (cf. Hale[2]) imply that for each point $(t_0, x_0) \in \mathbb{R} \times \Omega$ and for each function $\varphi \in PC(t_0)$ there exists a solution $\phi_i(t)$ of problem (2) as $t \geq t_0$. Moreover, $\phi_i(t) = \varphi(t)$ as $t \in [t_0 - h, t_0]$ $\phi_i(t_0) = x_0$ and this solution does not leave the domain Ω. Let t_1 be the first moment at which the integral curve $(t, \phi_i(t))$ reaches some of the hypersurfaces $\left\{\sigma_k\right\}_{k=1}^{\infty}$. Conditions of Lemma 1 are fulfilled and therefore $\tau_1 = t_1 > t_0$. Moreover, the number of the first hypersurface reached by the integral curve is $i_1 = 1$. Setting $x(t; t_0, x_0, \varphi) = \phi_i(t)$ as $t \in [t_0, t_1]$ we have $\phi(t_1 + 0) = I_1(\phi_1(t_1)) + \phi(t_1) = \phi_i^+$ for $t = t_1$. Now the above mentioned existence theorem, applied to the problem (2) in the interval (t_1, τ_2) ensures that there exists a solution $\phi_2(t)$ such that $\phi_2(t) = \phi_1(t)$ as $t_1 - h \leq t \leq t_1$ and $\phi_2(t_1) = \phi_1^+$ as $t_1 < t \leq \tau_2$. In the same way, let us denote by $\phi_i(t)$ the solutions of problem (2) in the intervals $(\tau_{i-1}, \tau_i]$, i = 3, 4, ..., respectively. On each of the intervals $(\tau_i, \tau_{i+1}]$ we have only one of the next two cases:

a) $\tau_i = t_k$. Then for $t = t_k$ we have

$$\phi_i(t_k + 0) = \phi_i(t_k) + I_k(\phi_i(t_k)) = \phi_i^+ .$$

It follows from the existence theorem for the problem (2) in the interval $(t_k, \tau_{i+1}]$ that there exists a solution $\phi_{i+1}(t)$ such that $\phi_{i+1}(t) = \phi_i(t)$ as $\tau_i - h \leq t \leq t_k$ and $\phi_{i+1}(t_k) = \phi_i^+$. Thus the solution $x(t; t_0, x_0, \varphi)$ of problem (1) can be extended to the moment τ_{i+1}, $i = 2, 3, \ldots$ letting $x(t; t_0, x_0, \varphi) = \phi_{i+1}(t), t \in (t_k, \tau_{i+1})$.

b) $\tau_i \in \left(\left\{ t_l^h \right\}_{l=0}^{\infty} \cup \left\{ \theta_z^h \right\}_{\tau=1}^{s} \right) \Big\backslash \left\{ t_k \right\}_{k=2}^{\infty}$. Then, by virtue of theorem on continuability of solutions of problem (2) (cf. Hale, 1977), the solution $x(t; t_0, x_0, \varphi)$ of the initial value problem (1) can be defined in the interval $(\tau_i, \tau_{i+1}]$ setting

$$x(t; t_0, x_0, \varphi) = \phi_i(t), t \in (\tau_i, \tau_{i+1}].$$

Finally, by means of condition H8, solution $x(t; t_0, x_0, \varphi)$ of problem (1) is defined for $t \in J^+(t_0, x_0, \varphi)$.

Proof of Assertion 2 : Conditions of Lemma 1 are satisfied and $i_1 = 1$. Therefore $i_2 = 2$, $i_3 = 3$, ..., where i_k is the number of the hypersurface that the integral curve $(t, x(t; t_0, x_0, \varphi))$ reaches at the moment t_k, $k = 1, 2, \ldots$

Thus we conclude that $i_k \to \infty$ as $k \to \infty$. Now condition $H8$ leads to

$$\lim_{k \to \infty} t_k = \lim_{k \to \infty} \tau_{i_k}(x_k) = \lim_{k \to \infty} \tau_k(x_k) = \infty, \qquad \ldots (3)$$

where $x_k = x(t_k; t_0, x_0, \varphi)$.

Since the solution $x(t) = x(t; t_0, x_0, \varphi)$ is defined on each of the intervals

$(t_k t_{k+1}]$, $k = 1, 2, \ldots$ by virtue of assertion 1, then (3) implies that the solution is continuable for all $t \geq t_0$.

Proof of Assertion 3 : Validity of condition H_2 ensures that the above defined solutions $\phi_1(t), \phi_2(t), \ldots$ are unique on their definite domains and therefore the solution $x(t; t_0, x_0, \varphi)$ of problem (1) is unique. ■

Let us consider now an initial value problem for the system of differential-difference equations with impulse effects at fixed moments of time :

$$\dot{x}(t) = f(t, x(t), x(t-h)), t \in \tau_k, t > t_0,$$

$$x(t) = \varphi(t), t \in [t_0 - h, t_0],$$

$$x(t_0 + 0) = x_0, \qquad \ldots (4)$$

$$\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) = I_k(x(\tau_k)), \tau_k > 0, \ k = 1, 2, \ldots,$$

where $t_0 = \tau_0 < \tau_1 < \ldots < \tau_k < \tau_{k+1} <$ and $\lim_{k \to \infty} \tau_k = \infty$. In the case under consideration $\tau_k(x) \equiv \tau_k$, $k = 1, 2, \ldots$ and σ are hyper-planes in $I\!R^{n+1}$.

Theorem 2 — *Let the conditions H_1, H_3-H_5 are valid.*

***Theorem* 2** — *Let the conditions H_1, H_3-H_5 are valid.*

Then for each point $(t_0, x_0) \in I\!R \times \Omega$ and for each function $\varphi \in PC(t_0)$ there exists a solution $x(t; t_0, x_0, \varphi)$ of the problem (4) that is defined on the interval $[t_0 - h, \omega)$ and it is continuable in the right of ω.

If, moreover, condition H_2 holds then the solution of (4) is unique.

The proof of Theorem 2 is a simple consequence of Theorem 1.

Now we consider an initial value problem for the linear system of differential-difference equations with impulse effects at fixed moments.

$$\begin{aligned} &\dot{x}(t) = A(t) \times (t) + B(t) \times (t-h), t \neq \tau_k, t > t_0, \\ &x(t) = \varphi(t), t \in [t_0 - h, t_0], \\ &x(t_0 + 0) = x_0, \\ &\Delta x(\tau_k) = B_k \times (\tau_k), \ k = 1, 2, ..., \ \tau_k > t_0. \end{aligned} \quad \text{... (5)}$$

where $A(t, B(t)$ and B_k, $k = 1, 2, ...$ are matrices of the type $(n \times n)$.

***Theorem* 3** — *Assume the matrices $A(t)$ and $B(t)$ to be continuous for $t > t_0, t \neq \tau_k$, $k = 1, 2, ...$, with points of discontinuity $\tau_1, \tau_2, \ldots$ where they are continuous from the left.*

Then for each point $(t_0, x_0) \in I\!R \times \Omega$ and for each function $\varphi \in PC(t_0)$ there exists a unique solution $x(t) = x(t; t_0, x_0, \varphi)$ of the problem (5) that is defined for all $t > t_0$.

Theorem 3 is a consequence of the theorem on existence and uniqueness for the solutions of a linear system of differential difference equations.

The problem on left-continuability of solutions will be considered now for systems of the type (4) only.

Assume $x(t)$ to be a solution of the problem (4) defined on the interval (γ, ω).

If $\gamma \neq \tau_k$ then the problem on continuability of $x(t)$ in the left of γ can be solved in the same way as for differential-difference equations without impulses. In this case the solution $x(t)$ is continuable in the left of γ and $J^- = J^-(t_0, x_0, \varphi) = (d, t_0)$. Moreover, careful calculations show that the solution $x(t)$ of problem (4) satisfies the equation :

$$x(t) = \left. \begin{array}{l} x_0 + \sum\limits_{t_0 \le \tau_k < t} I_k(x(\tau_k)) + \int\limits_{t_0}^{t} f(s, x(s), x(s-h))ds, \ t \in J^+ \\ x_0 - \sum\limits_{t_0 \le \tau_k < t} I_k(x(\tau_k)) + \int\limits_{t_0}^{t} f(s, x(s), x(s-h))\,ds, \ t \in J^- \end{array} \right\}$$

The solution of the linear system (5) can be extended in the left of τ_k if the next conditions are fulfilled :

$$\det (E + B_k) \neq 0, \ k = 1, 2, ..., \quad \text{... (6)}$$

where E is the identity matrix of the type $(n \times n)$.

Let $U_k(t, s)$ $(t, s \in (\tau_{k-1}, \tau_k])$ be the Cauchy matrix for the linear system :

$$\dot{x}(t) = A(t) \times (t),\ \tau_{k-1} < t < \tau_k,\ k = 1, 2, ...,$$

Then by virtue of Theorem 3 the solution of the initial problem (5) can be decomposed as :

$$x(t; t_0, x_0, \varphi) = x(t) = W(t, t_0 + 0)\, x_0 + \int_{t_0}^{t} W(t, s)\, B(s) \times (s - h) dS, \qquad \text{... (7)}$$

where

$$W(t, s) = \begin{cases} U_k(t, s) \text{ as } t, S \in (\tau_{k-1}, \tau_k], \\ U_{k+1}(t, \tau_k + 0)\,(E + B_k\, U_k(\tau_k, s) \\ \qquad \text{as } \tau_{k-1} < S \le \tau_k < t \le \tau_{k+1}, \\ U_k(t, \tau_k)\,(E + B_k)^{-1}\, U_{k+1}(\tau_k + 0, s) \\ \qquad \text{as } \tau_{k-1} < t \le \tau_k < s \le \tau_{k+1}, \\ U_{k+1}(t, \tau_k + 0) \prod_{j=k}^{i+1} (E + B_j)\, U_j(\tau_j(\tau_j, \tau_j + 0)\,(E + B_i)\, U_i(\tau_i, S) \\ \qquad \text{as } \tau_{i-1} < S \le \tau_i < \tau_k < t < \tau_{k+1}, \\ U_i(t, \tau_i) \prod_{j=i}^{k-1} (E + B_j)^{-1}\, U_{j+1}(\tau_j + 0, \tau_{j+1})\,(E + B_k)^{-1}\, U_{k+1}(\tau_{k+0}, s) \\ \tau_{i-1} < t \le \tau_i < \tau_k < S \le \tau_{k+1}. \end{cases}$$

is the solving operator of the system

$$\dot{x}(t) = A(t)\, x(t),\ t \ne \tau_k,$$

$$\Delta x(\tau_k) = B_k \times (\tau_k).$$

Now (7), (8) as well as the linearity of the operator $U_k(t, s)$ imply that the space L of all solutions of the problem (5) is n-dimensional linear space.

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