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A NOTE ON APPROXIMATION OF HOLOMORPHIC FUNCTIONS BY EXPONENTIALS

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We prove in this paper that, under suitable conditions on the distribution of the points of a set $S \subset \mathbb{C}^N$ and on the density of a family $\mathcal{A}$ of functions, the linear span of a certain family of exponentials related to S and $\mathcal{A}$ is dense in the space of holomorphic functions in a domain G. Some earlier results are derived as consequences.

Key Words : Holomorphic Function; Entire Function; Approximation by Exponentials, Runge Domain; Counting Function; Circumscribed Radius; Laplace Transform

1. INTRODUCTION, NOTATION AND KNOWN RESULTS

In this paper, $\mathbb{R}$ will denote the real line and $\mathbb{C}$ is the complex plane. Q is the field of rational numbers, $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is the one-point compactification of $\mathbb{C} \cdot \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk. If $N \in \mathbb{N}$, then G will stand for a domain in $\mathbb{C}^N$, that is, $G \subset \mathbb{C}^N$, G is nonempty, open and connected. Recall that a subset $S \subset \mathbb{C}^N$ is said to be *rare or nowhere dense* if its closure $\overline{S}$ has an empty interior.

$H(G)$ denotes, as usual, the space of holomorphic functions in G, endowed with the topology of uniform convergence on compact subsets. $H(G)^*$ is its topological dual space. Runge's theorem [Ru, p. 290][7] asserts that, given a domain $G \subset \mathbb{C}$ and a set $A \subset \mathbb{C}_\infty$ which has one point in each connected component of $\mathbb{C}_\infty \setminus G$, then the set of rational functions with poles only in A is dense in $H(G)$. In particular, the set of polynomials is dense in $H(G)$ whenever G is simply connected, i.e., $\mathbb{C}_\infty \setminus G$ is connected. A *hole* of G is a bounded connected component of $\mathbb{C}_\infty \setminus G$. A domain $G \subset \mathbb{C}^N$ is said to be a Runge domain if the set of polynomials is dense in $H(G)$ [V, p. 206]. If $N = 1$, then G is Runge if and only if it is simply connected. For properties and characterizations of Runge domains, see, for instance, [H, pp. 52-59][3], [K, p. 204[4] and [V, pp. 206-208][8].

If f is entire (i.e., $f \in H(\mathbb{C}^N)$) then, trivially, the set $Z(f)$ of its zeros is closed. Assume that, in addition, f is not identically zero. If $N = 1$, $Z(f)$ is discrete. If $N \geq 2$, $Z(f)$ is never a discrete subset. If $r = (r_1, \ldots, r_N) \in (0, +\infty)^N$, we denote

$$M_f(r) = \max_{\substack{|z_j| = r_j \\ j=1,\dots,N}} |f(z_1, \dots, z_N)| = \max_{\substack{|z_j| \le r_j \\ j=1,\dots,N}} |f(z_1, \dots, z_N)|.$$

We consider in $\mathbb{C}^N$ the euclidean norm $|z| = \left(\sum_{j=1}^{N} |z_j|^2 \right)^{1/2}$ for $z = (z_1, \dots, z_N)$. The open ball with center $a \in \mathbb{C}^N$ and radius $s > 0$ is $B(a, s) = \{z \in \mathbb{C}^N : |z - a| < s\}$. We agree that $B(a, \infty) = \mathbb{C}^N$. We define the circumscribed radius of G as $R(G) = \inf_{b \in \mathbb{C}^N} \sup_{z \in G} |z - b| = \inf\{s > 0 :$ there is an open ball B of radius s with $G \subset B\}$. Clearly $0 < R(G) \le \infty$, and $R(G)$ is finite if and only if G is bounded. The infimum in the definition of $R(G)$ is attained, that is, there is $b \in \mathbb{C}^N$ such that $G \subset B(b, R(G))$. It is obvious that $R(G) = s$ whenever G is an open ball with radius s.

A multi-index is a N-tuple $k = (k_1, \dots, k_N) \in \mathbb{N}_0^N$ of non-negative integers. Then we denote $z^k = z_1^{k_1} \dots z_N^{k_N}$ and $zw = z_1 w_1 + \dots + z_N w_N$ whenever $z = (z_1, \dots, z_N)$ and $w = (w_1, \dots, w_N)$.

Assume that $S \subset \mathbb{C}^N$. If $N = 1$, we denote by $n(S; t)$ $(t \ge 0)$ the number of points of S lying on the closed disk $\{|\zeta| \le t\}$. The notation is *a fortiori* more involved in the case $N \ge 2$. If $t \ge 0$, $a = (a_1, \dots, a_N) \in \mathbb{C}^N$, $r = (r_1, \dots, r_N) \in (0, +\infty)^N$ and $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ is such that $|\lambda| = 1$, then $n(a, S, \lambda; t)$ will stand for the number of points of S lying on the "complex segment" $L(a, \lambda, t) = \{(a_1 + \lambda_1 \zeta, \dots, a_N + \lambda_N \zeta) : |\zeta| \le t\}$ of the complex affine straight line $L(a, \lambda) = \{(a_1 + \lambda_1 \zeta, \dots, a_N + \lambda_N \zeta) : \zeta \in \mathbb{C}\}$. The rate of growth of the function $n(a, S, \lambda; t)$ obviously characterizes the density of the points of S on the line $L(a, \lambda)$. In this paper, the most used instance will be the case $S \subset Z(f)$, where f is an adequate entire function. The reason for fixing a point a in the case $N \ge 2$ in the definition of the counting function is the simple one that the asymptotic distribution of zeros on a line $L(a, \lambda)$ may be wholy different from their distribution on other line $L(b, \mu)$. Obviously, this cannot happen if $N = 1$. With the same notation, if f is entire and $N = 1$ ($N \ge 2$, resp.), then we put $n(f; t)$ ($n(a, f, \lambda; t)$, resp.) to mean the number of points of $Z(f)$ in $|\zeta| \le t$ (in $L(a, \lambda, t)$, resp.), repeated according to their multiplicity. An elementary application of Hurwitz's theorem [A, p. 178][1] on the number of roots of the limit function shows that if $f(0) \ne 0$, then the function $\lambda \mapsto n(0, f, \lambda; t)$ is bounded for any fixed t (see [Ro, p. 233])[6].

Let f be entire and not identically zero. Jensen's formula relates the values of $\log |f|$ on the boundary of a ball with the distribution of the zeros of f in the ball. For the case $N = 1$, the formula can be written as follows:

$$\int_0^r \frac{n(f; t) - n(f; 0)}{t} \, dt = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta - \log |c| - n(f; 0) \log r$$

for every $r > 0$, where c is the coefficient of order $n\,(f; 0)$ in the Taylor expansion of f around the origin. We must suppose $f(0) \ne 0$ in the case $N = 2$, for if $f\,(0) = 0$, then f cannot generally be expressed around the origin as a product $f(z) = z^k \cdot g(z)$, where g is an entire function such that $g(0) \neq 0$. With this in mind, Jensen's formula becomes the following one (see [Ro, Chapter 4, Sections 2.1-2.4]):[6]

$$\int_0^{|r|} \left(\frac{1}{s} \int_0^{2\pi} \cdots \int_0^{2\pi} n(0, f, \lambda(\alpha, r); s)\, d\alpha_1 \ldots d\alpha_{N-1} \right) ds$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \log |f(r_1 e^{i\theta_1}, \ldots, r_N e^{i\theta_N})|\, d\theta_1 \ldots d\theta_N - \log |f(0)|,$$

for every $r = (r_1, \ldots, r_N) \in (0, +\infty)^N$. We have denoted

$$\lambda(\alpha, r) = \left(\frac{r_1}{|r|} e^{i\alpha_1}, \ldots, \frac{r_{N-1}}{|r|} e^{i\alpha_{N-1}}, \frac{r_N}{|r|} \right).$$

It is well known that the linear span of the set $\{\exp(cz) : c \in \mathbb{C}^N\}$ is dense in $H(\mathbb{C}^N)$ [H, p. 97][3]. In fact, the points c need not fill in the whole space $\mathbb{C}^N$:

Theorem A — *If V is a nonempty open subset of $\mathbb{C}^N$, then the linear span of $\{\exp(cz) : x \in V\}$ is dense in $H(\mathbb{C}^N)$.*

The proof of the latter result can be found in [GS, pp. 258-260][2], where the authors consider the Laplace transform of $H(\mathbb{C}^N)^*$, that is, the linear operator

$$L \in H(\mathbb{C}^N)^* \mapsto \hat{L} \in H(\mathbb{C}^N)$$

defined by $\hat{L}(w) = L(z \mapsto e^{zw})$ $(w \in \mathbb{C}^N)$. This transform is also used in [LR, pp. 61-64 and 78-79][5] to prove the next result, which involves the asymptotic behaviour of a sequence of points.

Theorem B — *Let $G \subset D$ be a simply connected domain and $S = \{c_n : n \in \mathbb{N}\} \subset \mathbb{C} \setminus \{0\}$ a sequence of distinct complex numbers. If*

$$\limsup_{r \to \infty} \frac{1}{r} \int_0^r \frac{n(S; t)}{t}\, dt \geq 1,$$

then the linear span of $\{\exp(c_n z) : n \in \mathbb{N}$ is dense in $H(G)$.

In particular, if the sequence $\{c_n : n \in \mathbb{N}\}$ has a finite accumulation point (i.e., $c_n \not\to \infty$ as $n \to \infty$), then the conclusion of the theorem holds. Indeed, in such a case there would be $t_0 > 0$ such that $n(S; t) = \infty$ for all $t \geq t_0$.

In this note, we generalize all the above results about approximation by exponentials to holomorphic functions on general domains of $\mathbb{C}^N$. Our results depend upon the "density" of certain set of values. For a Runge domains, sufficient conditions for approximation lie on its circumscribed radius.

2. APPROXIMATION BY EXPONENTIALS

We first deal with the case $N \geq 2$, and then consider the special case $N = 1$. For future references, we say that a subset $S \subset \mathbb{C}^N$ such that for every $a \in A$ there exists $t = t(a) > 0$ satisfying that given $m \in \mathbb{N}$ there is $\lambda \in \mathbb{C}^N$ with $|\lambda| = 1$ and $n(a, S, \lambda; t) > m$.

Theorem 1 — *Let S, G be nonempty subsets of $\mathbb{C}^N$ such that $N \geq 2$ and G is a domain. Assume that there are N families of functions $\mathcal{A}_1, \ldots, \mathcal{A}_N$ in $H(G)$* such that the linear span of

$$\{g_1^{k_1} \cdot \ldots \cdot g_N^{k_N} : k_j \in \mathbb{N}_0, g_j \in \mathcal{A}_j, j = 1, \ldots, N\}$$

is dense in H (G). Denote by M(S) the linear span of the set of functions

$$\{\exp\ (c_1 g_1 + \ldots + c_N g_N) : (c_1, \ldots, c_N) \in S, g_i \in \mathcal{A}_j, j = 1, \ldots, N\},$$

and by V(a, r) the quantity

$$\int_0^{|r|} \left(\frac{1}{s} \int_0^{2\pi} \ldots \int_0^{2\pi} n(0, S, \lambda(\alpha, r); s)\, d\alpha_1 \ldots d\alpha_{N-1} \right) ds$$

for every $r \in (0, +\infty)^N$ and every $a \in \mathbb{C}^N$. Then the following properties are satisfied:

a) If S has the property (P), then M(S) is dense in H(G);

b) If there are a non-rare subset $A \subset \mathbb{C}^N$ and a point $b = (b_1, \ldots, b_N) \in \mathbb{C}^N$ such that

$$\limsup_{|r| \to \infty} \frac{V(a, r)}{|r|} \geq (2\pi)^{N-1} \cdot \sup_{\substack{g_j \in \mathcal{A}_j \\ j=1,\ldots,N}} \sup_{z \in G} \left(\sum_{j=1}^{N} |g_j(z) - b_j|^2 \right)^{1/2}$$

for all $a \in A$, then M(S) is dense in H(G); and

c) If S is non-rare, then M(S) is dense in H(G).

PROOF : In order that $M(S)$ can be dense in $H(G)$, it is enough to prove that if $L \in H(G)^*$ is such that $L(\exp\ (c_1 g_1 + \ldots + c_N g_N)) = 0$ for all $c = (c_1, \ldots, c_N) \in S$ and all $g_j \in \mathcal{A}_j$ $(j = 1, \ldots, N)$, then $L = 0$. This is a consequence of the Hahn-Banach theorem. But, in turn, it suffices to show that $L(g_1^{k_1} \cdot \ldots \cdot g_N^{k_N}) = 0$ for all $k_j \in \mathbb{N}_0$ and all $g_j \in \mathcal{A}_j$ $(j = 1, \ldots, N)$, since the linear span of the functions $g_1^{k_1} \cdot \ldots \cdot g_N^{k_N}$ is dense in $H(G)$. We may suppose without loss of generality that none of the functions of the families $\mathcal{A}_j$ $(j = 1, \ldots, N)$ is constant.

With this in mind, fix a point $b = (b_1, \ldots, b_N) \in \mathbb{C}^N$ and a function g_j in every $\mathcal{A}_j$ $(j = 1, \ldots, N)$. Denote $G_j(z) = g_j(z) - b_j$ $(\forall j \in \{1, \ldots, N\}, \forall z \in G)$. Define the "generalized" Laplace transform of L as

$$\hat{L}(w) = L(\exp(w_1 G_1 + \ldots + w_N G_N)) \ (w = (w_1, \ldots, w_N) \in \mathbb{C}^N).$$

It is a standard exercise to prove that $\hat{L}$ is an entire function on $\mathbb{C}^N$ with partial derivatives

$$D^k \hat{L}(w) = L(G_1^{k_1} \ldots G_N^{k_N} \cdot \exp(w_1 G_1 + \ldots + w_N G_N))$$

for every multi-index k and every $w \in \mathbb{C}^N$. We have denoted $D^k = D_1^{k_1} \ldots D_N^{k_N}$, where D_j is the partial differentiation operator with respect to the variable w_j. If we got to demonstrate that $\hat{L} \equiv 0$, we would have

$$D^k \hat{L}(0) = 0 \ \ \forall k,$$

so $L(G_1^{k_1} \cdot \ldots \cdot G_N^{k_N}) = 0$ for every multi-index k. But this implies that $L(g_1^{k_1} \cdot \ldots \cdot g_N^{k_N}) = 0$ for all k, since the linear manifolds spanned by $\{g_1^{k_1} \cdot \ldots \cdot g_N^{k_N} : k \in \mathbb{N}_0^N\}$ and $\{G_1^{k_1} \cdot \ldots \cdot G_N^{k_N} : k \in \mathbb{N}_0^N\}$ are the same. Hence, our goal is to prove that $\hat{L} \equiv 0$.

Observe that $S \subset Z(\hat{L})$ and that

$$n(a, S, \lambda; t) \leq n(a, Z(\hat{L}), \lambda; t) \leq n(a, \hat{L}, \lambda; t) = n(0, f_a, \lambda; t) \ldots \quad (1)$$

for all $a \in \mathbb{C}^N$, all $t \geq 0$ and all $\lambda \in \mathbb{C}^N$ with $|\lambda| = 1$, where we have set $f_a(w) = \hat{L}(w + a)$. Now, we prove parts *a*), *b*) and *c*).

a) From the hypothesis, given $a \in A$ there exists $t > 0$ such that the function $\lambda \mapsto n(a, S, \lambda; t)$ is not bounded, so the function $\lambda \mapsto n(0, f_a, \lambda; t)$ neither is. Then $f_a(0) = 0$ by a property mentioned in Section 1, so $\hat{L}(a) = 0$ for every $a \in A$, i.e., $A \subset Z(\hat{L})$. But $Z(\hat{L})$ is closed, so $Z(\hat{L}) \supset \overline{A}$ and $\overline{A}$ contains a nonempty open set. By the Analytic Continuation Principle, $\hat{L} \equiv 0$. Thus $M(S)$ is dense.

b) Consider the point b of the hypothesis, fix $g_j \in \mathcal{A}_j$ $(j = 1, \ldots, N)$ and construct the corresponding entire function $\hat{L}$. Note that the choice of the point b was unimportant in part *a*), but clearly not here. Assume, by the way of contradiction, that $\hat{L} \not\equiv 0$. The same argument of the proof of a) tells us that there must be at least one point $a \in A$ which is not a zero for $\hat{L}$, i.e., $\hat{L}(a) \neq 0$ or $f_a(0) \neq 0$. By Jensen's formula for several variables,

$$\int_0^{|r|} \left(\frac{1}{s} \int_0^{2\pi} \ldots \int_0^{2\pi} n(0, f_a, \lambda(\alpha, r); s)\, d\alpha_1 \ldots d\alpha_{N-1} \right) ds$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \ldots \int_0^{2\pi} \log |f_a(r_1 e^{i\theta_1}, \ldots, r_N e^{i\theta_N})|\, d\theta_1 \ldots d\theta_N - \log |f_a(0)| \quad \ldots (2)$$

for all $r \in (0, +\infty)^N$.

We now proceed to show that $\hat{L}$ satisfies an exponential-type inequality. To this end, by using the continuity condition for L, we note that there is a compact subset $K \subset G$ and a positive constant C such that

$$L(f) \leq C \cdot \sup_{z \in K} |f(z)| \quad \forall f \in H(G).$$

Therefore, by definition of $\hat{L}$,

$$|\hat{L}(w)| = |L(\exp(w_1 G_1 + \ldots + w_N G_N))|$$

$$\leq C \cdot \sup_{z \in K} \exp(w_1 G_1(z) + \ldots + w_N G_N(z))|$$

$$\leq C \cdot \sup_{z \in K} \exp(|w_1||G_1(z)| + \ldots + |w_N||G_N(z)|)|$$

$$\leq C \cdot \exp\left(\sup_K \left(\sum_{j=1}^{N} |G_j|^2 \right)^{1/2} \left(\sum_{j=1}^{N} |w_j|^2)^{1/2} \right) \right),$$

for every $w \in \mathbb{C}^N$. Hence

$$M_{\hat{L}}(r) \leq C \cdot \exp\left(|r| \cdot \sup_K \left(\sum_{j=1}^{N} |G_j|^2 \right)^{1/2} \right) \qquad \ldots (3)$$

for every $r \in (0, +\infty)^N$. From (1) and (2) we obtain, for all $r \in (0, +\infty)^N$, that

$$V(a, r) \leq \frac{1}{2\pi} \int_0^{2\pi} \ldots \int_0^{2\pi} \log |f_a(r_1 e^{i\theta_1}, \ldots r_N e^{i\theta_N})| \, d\theta_1 \ldots d\theta_N - \log |\hat{L}(a)|.$$

It is evident that $M_{f_a}(r) \leq M_{\hat{L}}(s(r))$ for every $r \in (0, +\infty)^N$, where $s(r) = (r_1 + |a_1|, \ldots, r_N + |a_N|)$. Then

$$V(a, r) \leq (2\pi)^{N-1} \cdot \log M_{\hat{L}}(s(r)) - \log |\hat{L}(a)|.$$

This inequality, together with (3), drives us to

$$V(a, r) \leq (2\pi)^{N-1} \cdot (\log C + \sup_K \left(\sum_{j=1}^{N} |G_j|^2)^{1/2} \cdot s(r) \right) - \log |\hat{L}(a)|$$

Since $s(r)/|r| \to 1$ as $|r| \to \infty$, we derive that

$$\limsup_{|r|\to\infty} \frac{V(a,r)}{|r|} \leq (2\pi)^{N-1}\cdot\sup_K \left(\sum_{j=1}^{N} |G_j|^2\right)^{1/2} < (2\pi)^{N-1}\cdot\sup_{z\in G}\left(\sum_{j=1}^{N} |g_j(z)-b_j|^2\right)^{1/2},$$

which clearly contradicts the hypothesis. Therefore, $\hat{L}\equiv 0$, as required.

c) It is evident that $\overline{M(S)}\supset M(\overline{S})\supset M(S)$, so $M(S)$ is dense in $H(G)$ if and only if $M(\overline{S})$ is. Thus, we can suppose that S is closed, so S contains an open ball $B = B(d, R)$. Given $a \in \mathbb{C}^N$, $\{d\}$, it is clear that for $s > |d-a|$ and $\lambda = \frac{d-a}{|d-a|}$ we have $n(a, B, \lambda; s) = \infty$, so $n(a, S, \lambda; s) = \infty$. It now suffices to apply a) with $A = \mathbb{C}^N \setminus\{d\}$. There is another way of proof: Given $a \in A$, we can write $d_j - a_j = |d_j - a_j| e^{i\beta_j}$ ($j = 1, \ldots, N$). It is obvious that for each $j \in \{1, \ldots, N-1\}$, there is an open angular interval $I_j \subset [0, 2\pi]$ containing $\beta_j - \beta_N$ such that $n(a, B, \lambda(\alpha, r); s) = \infty$ for all $\alpha = (\alpha_1, \ldots, \alpha_{N-1}) \in I_1 \times \ldots \times I_{N-1}$, all $s > |d-a|$ and all r of the form $r(t) = (t|d_1 - a_1|, \ldots, t|d_N - a_N|)$ ($t > 0$). Therefore, $n(a, S, \lambda(\alpha, r); s) = \infty$ for the same values of α, s and r, so $V(a, r(t)) = \infty$ whenever $t > 1$. Then

$$\limsup_{|r|\to\infty} \frac{V(a,r)}{|r|} = \infty$$

and *b)* can also be applied. The theorem is totally proved. □

Corollary 1 — If $G \subset \mathbb{C}^N$ is a Runge domain, $S \subset \mathbb{C}^N$ and there is a non-rare subset $A \subset \mathbb{C}^N$ such that

$$\limsup_{|r|\to\infty} \frac{V(a,r)}{|r|} \geq (2\pi)^{N-1} R(G)$$

for all $a \in A$, then the linear span of $\{\exp(cz) : c \in S\}$ is dense in $H(G)$. In particular, the same conclusion holds if either S is non-rare or has the property (P).

PROOF : Polynomials are dense in $H(G)$. But {polynomials} = span $\{z^k : k \in \mathbb{N}_0^N\}$.

So it suffices to take $\mathcal{A}_j = \{z_j\}$ ($j = 1, \ldots, N$) in Theorem 1. The second part of the result follows readily from parts *a)* and *c)* of Theorem 1. The first part is derived from the definition of circumscribed radius by taking a point $b \in \mathbb{C}^N$ in part *b)* of Theorem 1 such that $B(b, R(G)) \supset G$. □

Note that Theorem A follows from Corollary 1 just by taking $S = V$, $G = \mathbb{C}^N$.

Theorem 2 — *Suppose that $G \subset \mathbb{C}$ is a domain and that $S = \{c_n : n \in \mathbb{N}\} \subset \mathbb{C} \setminus \{0\}$ is a sequence of distinct complex numbers. If there is a point $b \in \mathbb{C}$ such that*

$$\limsup_{r\to\infty} \frac{1}{r}\int_0^r \frac{n(S;t)}{t}\,dt \geq \sup_{g\in\mathcal{A}}\ \sup_{z\in G} |g(z) - b|,$$

where $\mathcal{A} \subset H(G)$ *is a family such that the linear span of* $\{g^k : g \in \mathcal{A}, k \in \mathbb{N}_0\}$ *is dense in* $H(G)$*, then the linear span* $M(S)$ *of* $\{\exp(c_n g) : g \in \mathcal{A}, n \in \mathbb{N}\}$ *is also dense in* $H(G)$*. In particular, the same conclusion holds if* $c_n \not\to \infty$ $(n \to \infty)$.

PROOF : The proof runs over the same steps as Theorem 1 (specially, part *b*)), but it is much easier: take $N = 1$ and $\mathcal{A}_1 = \mathcal{A}$. The details are left to the reader. The only delicate point is the fact that in Jensen's formula for $N = 1$ the case $f(0) = 0$ is possible (so $n(f; 0) \neq 0$), where we have taken this time $f(w) = L(\exp((g(z) - b)\, w))$ $(w \in \mathbb{C})$. Here $g \in \mathcal{A}$ and $L \in H(G)^*$ are fixed and L is such that $L(\exp(c_n h)) = 0$ for every $n \in \mathbb{N}$ and every $h \in \mathcal{A}$.

It should be proved that $f \equiv 0$. This happens when S has a finite accumulation point, because $S \subset Z(f)$. It remains to show that $f \equiv 0$ if S has no finite accumulation point. But in this case there is $r_0 > 0$ such that $|c_n| > r_0$ for all $n \in \mathbb{N}$, because $0 \notin S$. By the way of contradiction, assume that $f \not\equiv 0$. Then, by Jensen's formula,

$$\frac{1}{r}\int_0^r \frac{n(S;t)}{t}\,dt = \frac{1}{r}\int_{r_0}^r \frac{n(S;t)}{t}\,dt \leq \frac{1}{r}\int_{r_0}^t \frac{n(f;t)}{t}\,dt$$

$$= \frac{1}{r}\int_{r_0}^r \frac{n(f;t) - n(f;0)}{t}\,dt + n(f;0)\cdot\frac{\log(r/r_0)}{r}$$

$$= \frac{1}{2\pi r}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta - \frac{\log|c| + \log r_0}{r},$$

for every $r > 0$, where c is a nonzero constant. Now, it suffices to employ an exponential-type inequality for f similar to (3) and take "limsup" as $r \to \infty$, obtaining the contradiction

$$\limsup_{r\to\infty} \frac{1}{r}\int_0^r \frac{n(S;t)}{t}\,dt < \sup_{z\in G} |g(z) - b|.$$

Consequently, $f \equiv 0$ and the closure of $M(S)$ is $H(G)$. □

Corollary 2 — If $G \subset \mathbb{C}$ is a simply connected domain and $S = \{c_n : n \in \mathbb{N}\} \subset \mathbb{C} \setminus \{0\}$ is a sequence of distinct complex numbers such that

$$\limsup_{r\to\infty} \frac{1}{r}\int_0^r \frac{n(S;t)}{t}\,dt \geq R(G),$$

then the linear span of $\{\exp(c_n z) : n \in \mathbb{N}\}$ is dense in $H(G)$. In particular, the same conclusion holds if $c_n \not\to \infty$ $(n \to \infty)$.

PROOF : Take $\mathcal{A} = \{z\}$ in Theorem 2 and a point $b \in \mathbb{C}$ such that $B(b, R(G)) \supset G$, and we are done. □

Corollary 2 evidently extends Theorem B, since $R(G) \leq 1$ whenever $G \subset \mathbb{D}$.

Next, we take out another consequence for general plane domains.

Corollary 3 — Assume that $G \subset \mathbb{C}$ is a domain, A is a set which has one point in each hole of G and $S = \{c_n : n \in \mathbb{N}\} \subset \mathbb{C} \setminus \{0\}$ is a sequence of distinct points. Let us suppose that there are points $b, d \in \mathbb{C}$ with $d \neq 0$ and a mapping $m : A \to \mathbb{C} \setminus \{0\}$ satisfying the inequality

$$\limsup_{r \to \infty} \frac{1}{r} \int_0^r \frac{n(S;t)}{t} dt \geq \max \left\{ \sup_{z \in G} |dz - b|, \sup_{a \in A} \sup_{z \in G} \left| \frac{m(a)}{z-a} - b \right| \right\}.$$

Then the linear span of $\{\exp(dc_n z) : n \in \mathbb{N}\} \bigcup \left\{ \exp\left(\frac{m(a)c_n}{z-a} \right) : a \in A, n \in \mathbb{N} \right\}$ is dense in $H(G)$.

PROOF : Take $\mathcal{A} = \{dz\} \bigcup \left\{ \frac{m(a)}{z-a} : a \in A \right\}$ in Theorem 2 and apply Runge's theorem together with the decomposition of a rational function into a sum of a polynomial plus finitely many partial fractions. □

To finish, we provide three examples. Firstly, with $N = 2j$ assume that $\varepsilon > 0$ and that $\{q_n : n \in \mathbb{N}\}$ is an enumeration of the set $A = B(0, \varepsilon) \bigcap [(\mathbf{Q} + i\mathbf{Q}) \bigcap (\mathbf{Q} + i\mathbf{Q})]$.

Consider the sets

$$S_1 = A,$$

$$S_2 = \left\{ q_n + \left(n - \frac{1}{j} \right) (e^{2\pi i/m}, 1) : n, m, j \in \mathbb{N}; j \leq m \right\}$$

and

$$S_3 = \left\{ q_n + \frac{k}{\sqrt{2}} (e^{i\alpha}, 1) : n, k \in \mathbb{N}; k \geq n; \alpha \in [0, 2\pi] \right\}.$$

The open ball $G = B(0, 1)$ of $\mathbb{C}^2$ is a Runge domain. Denote by $M(S_j)$ the linear span of $\{\exp(cz) : c \in S_j\}$ $(j = 1, 2, 3)$. Let us try to apply Corollary 1. Since $\overline{A} \supset B(0, \varepsilon)$, the set A is non-rare. Then $M(S_1)$ is dense in $H(G)$. Note that each vector $\lambda = \frac{1}{\sqrt{2}} (e^{i\alpha}, 1)$ $(\alpha \in [0, 2\pi])$ satisfies $|\lambda| = 1$. From the definition of S_2, we have that $n\left(q_n, S_2, \frac{1}{\sqrt{2}} (e^{2\pi i/m}, 1); \sqrt{2} n \right) \geq m$ for all $n, m \in \mathbb{N}$, so $M(S_2)$ is dense in $H(G)$. Finally observe that for all $n \in \mathbb{N}$, all $\alpha \in [0, 2\pi]$, all $s \geq 0$ and all $t > 0$, $n\left(q_n, S_3, \frac{1}{\sqrt{2}t} (te^{i\alpha}, t); s \right) \geq [s] - (n-1)$, because $q_n + \frac{k}{\sqrt{2}} (e^{i\alpha}, 1) \in L\left(q_n, \frac{1}{\sqrt{2}} (e^{i\alpha}, 1), s \right)$ for every $k \in \{n, n+1, ..., [s]\}$ (we have denoted by $[s]$ the entire part of s). Therefore, for vectors

$r(t) = (t, t)$ $(t > 0)$, we obtain that

$$\frac{V(q_n, r(t))}{|r(t)|} = \frac{1}{(t^2+t^2)^{1/2}} \cdot \int_0^{(t^2/t^2)^{1/2}} \frac{1}{s}\left(\int_0^{2\pi} n\left(q_n, S_3, \frac{1}{\sqrt{2}t}(te^{i\alpha}, t); s \right) d\alpha \right) ds$$

$$\geq \frac{1}{\sqrt{2}t} \int_{\sqrt{2}n}^{\sqrt{2}t} \frac{1}{s} \cdot 2\pi \cdot (s-n)\, ds = \frac{2\pi}{\sqrt{2}t}\left(\sqrt{2}\,(t-n) - n \log \frac{t}{n} \right) \to 2\pi \, (t \to \infty).$$

Thus

$$\limsup_{|r| \to \infty} \frac{V(a, r)}{|r|} \geq 2\pi = (2\pi)^{2-1} R(G) \ \ \forall\, a \in A.$$

Consequently, $M(S_3)$ is also dense in $H(G)$.

As for the second example, let us consider the pierced annulus

$$G = \{ z \in \mathbb{C} : 1 < |z| < 3, |z-2| > 1/2 \}.$$

Then every function $f \in H(G)$ can be uniformily approximated on compacta by finite linear combinations of functions of the kind $e^{nz/3}, e^{n/z}, e^{n/(2z-4)}$ $(n \in \mathbb{N})$. Indeed, it suffices to put $c_n = n (n \in \mathbb{N}), A = \{0, 2\}$, $b = 0$, $d = 1/3$, $m(0) = 1$ and $m(2) = 1/2$ in Corollary 3.

Finally, let ρ be any positive real number satisfying $\rho e^{\rho} \leq 1$. Then $\rho < 1$ and, from Theorem B (or Corollary 2), the linear span of $\{\exp(nz) : n \in \mathbb{N}\}$ is dense in $H(B, 0, \rho))$. But $\exp(nz) = h(z)^n$ for each n, where $h(z) \equiv e^z$. Observe that

$$|e^z - 1| \leq |z| e^{|z|} < \rho e^{\rho} \leq 1 \ \ \forall\, z \in B(0, \rho),$$

so $\lim_{r \to \infty} \frac{1}{r} \int_0^r \frac{n(s; t)}{t} dt = 1 \geq \sup_{g \in \mathcal{A}} \sup_{z \in G} |g(z) - b|$ if we take $G = B(0, \rho)$, $\mathcal{A} = \{h\}$, $b = 1$ and $S = \mathbb{N}$. Thus, by Theorem 2, the linear span of $\{\exp(n \exp z) : n \in \mathbb{N}\}$ is dense in $H(B(0, \rho))$.

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