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ASYMPTOTICS FOR THE MAXIMAL NUMBER OF EDGES OF A DIGITAL CONVEX ARC

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Given two natural numbers a and $b(a < b)$, the following optimization problem is considered: Determine a digital convex arc with the cathetes' lengths a and b, which has the maximal possible number $n(a, b)$ of edges.

The family of sequences of pairs of suitably chosen cathetes' lengths is exhibited; each member of the family corresponds to an optimal digital convex arc. This family will serve as a basis for deriving the following asymptotic estimate for $n(a, b)$ $(a < b < a^2)$:

$$n(a, b) = \frac{3}{\sqrt[3]{\pi^2}}\, a^{1/3}\, b^{1/3} + O(b^{1/3} \log b).$$

Key Words: Digital Geometry; Digital Convex Arc; Optimization

1. INTRDOUCTION

There is an increasing interest in optimization problems on the integer grid. Motivation for considering such questions comes from several sources, in particular from integer programming and computer graphics.

The following optimization problem, related to convexity on the integer grid, will be considered here :

Let there be natural numbers a and b $(a \leq b)$. Consider the triangle ABO in the xOy coordiante plane with the vertex coordinates $A(a, 0)$, $B(0, b)$ and $O(0, 0)$.

Problem : Determine a convex path in the plane that consists of concatenated line segments, all the vertices of which have integer coordinates, which connects the points (0, 0) and a, b), and which has the maximal number $n(a, b)$ of edges.

Equivalently, determine a sequence of points (Fig. 1) with integer coordinates $B \equiv V_0(x_0, y_0), V_1(x_1, y_1), \ldots, V_n(x_n, y_n) \equiv A$, which satisfies the following set of conditions :

- $0 = x_0 < x_1 < \ldots < x_n = a; \qquad b = y_0 > y_1 > \ldots > y_n = 0.$
- The points $V_1, V_2, \ldots, V_{n-1}$ belong to the interior of the triangle ABO.
- All the interior angles of the polygon $V_0 V_1 \ldots V_n$ are less than π radians.

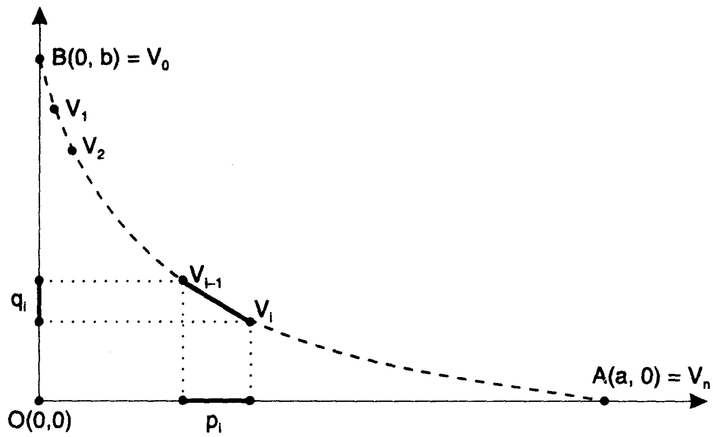

FIG. 1. Digital convex arc with the cathetes' lengths a and b.

- The number n is the largest possible.

The sequence of consecutive edges (V_{i-1}, V_i), $1 \leq i \leq n$, is called digital convex arc. The edges OA and OB are called cathetes of the arc.

The considered problem can be shortly formulated in the form :

- Determine a digital convex arc with given cathetes' lengths a and b, which has the largest possible number $n(a, b)$ of edges.

Thus the point sequence $V_i(x_i, y_i)$, $0 \leq i \leq n$, replaces the hypothenuse of the triangle ABO by a digital convex arc, turned towards the interior of the triangle.

A degenerate case of the considered problem arises for $b \geq 1 + 2 + \ldots + a$. Then $n(a, b) = a$; an obvious optimal solution has a edges with edge slopes of the form $q/1$ with different values of q.

An edge slope is a fraction of the form

$$\frac{q_i}{p_i} = \frac{y_{i-1} - y_i}{x_i - x_{i-1}}, \qquad \text{for} \quad 1 \leq i \leq n.$$

A projection of an edge of a digital convex arc is the orthogonal projection of that edge to a cathete of the arc. The lengths of projections of an edge with the edge slope q_i/p_i to the cathetes belonging to the y- and x-axis are q_i and p_i respectively (see Fig. 1).

A digital convex polygon is a polygon, all the vertices of which are points of the integer grid and all the interior angles of which are strictly less than π radians.

2. A FAMILY OF OPTIMAL "STAIRCASES"

The purpose of this section is to introduce a family of sequences of pairs of suitably chosen cathetes' lengths, for which the corresponding digital convex arcs are optimal. This family will serve as a basis for deriving an asymptotic estimate for $n(a, b)$ (in Section 3). The parameter of the family will be denoted by k, while the sequences within it will be indexed with the prime parameter t.

Let $O(a, b)$ denote the set of edge slopes chosen for (that is, used within) an optimal solution of the considered problem. Two properties of this set will be proved first :

Lemma 1 — If an edge slope q/p belongs to the set $O(a, b)$, then the natural numbers q and p are relatively prime.

PROOF : Convexity in the considered problem is taken in the strict sense (the third condition). This means that no three consecutive collinear vertices within an arc are allowed.

There is no sense to use an edge slope of the form $(k \cdot q_i)/(k \cdot p_i)$ in $O(a, b)$, for some integer $k > 1$. Such an edge cannot be used in $O(a, b)$ simultaneously with an edge with the slope q_i/p_i, otherwise these two edges in the convex arc would be collinear. On the other hand, it is obvious that the second one of the two edges "requires less room" and allows more other edges to be inserted into the arc. □

Remark : It is practical to allow a small deviation from this rule when choosing the edge slope of the last edge in a digital arc, in order to make the endpoints of the arc coincide with the outer endpoints of the cathetes.

Lemma 2 — The edge slopes within the set $O(a, b)$ satisfy the "staircase condition", that is:

If $q/p \in O(a, b)$ and a pair of natural numbers q' and p' is such that $q' \leq q$, $p' \leq p$ and the numbers q' and p' are relatively prime, then $q'/p' \in O(a, b)$.

PROOF : The edge slope q'/p' of the above form is in accordance with Lemma 1 and (a similar reasoning as the one applied with Lemma 1) "requires less room" and allows more other edges to be inserted into the arc than the edge q/p. Therefore, it must not happen that the edge slope q/p is chosen, but q'/p' is not. □

The sets of edge slopes satisfying the staircase condition will be called *staircases*.

Given a prime number t and a natural number k, let $S(t, k)$ denote a special staircase consisting of all the edge slopes q/p satisfying that :

1. the natural numbers q and p are relatively prime.
2. $q + p \cdot k \leq t$.

Further, let $card(t, k)$ denote the total number of edges with edge slopes q/p in $S(t, k)$, while the total lengths of their projections (q, resp. p) to y- and x-axis are denoted by $B(t, k)$ and $A(t, k)$ respectively.

The following theorem shows that $S(t, k)$ is an optimal solution of the considered problem for the cathetes' lengths $b = B(t, k)$ and $a = A(t, k)$:

***Theorem* 1** — *Given a prime number t and a natural number k, it holds that*

$$n(A(t, k), B(t, k)) = card(t, k).$$

PROOF : Suppose that an optimal staircase $O(A(t, k), B(t, k))$ has some edge slopes than $card(t, k)$. Let $D_1 = O(A(t, k)\, B(t, k)) \setminus S(t, k)$ and $D_2 = S(t, k) \setminus O(A(t, k), B(t, k))$. Then $|D_1| > |D_2|$. Note that each $q/p \in D_1$ satisfies that $q + k \cdot p > t$ and each $q/p \in D_2$ satisfies that $q + k \cdot p \leq t$. Consequently,

$$(*) \sum_{q/p \in D_1} q + k \cdot \sum_{q/p \in D_1} p > |D_1| \cdot t > |D_2| \cdot t \geq \sum_{q/p \in D_2} q + k \cdot \sum_{q/p \in D_2} p.$$

On the other hand, it must hold that

(**) $\Sigma_{q/p \in D_1} q \leq \Sigma_{q/p \in D_2} q$ and $\Sigma_{q/p \in D_1} p \leq \Sigma_{q/p \in D_2} p$. Namely, after the edge slopes from D_2 are removed, the sums of numerators and denominators of edge slopes from D_1 must not exceed the liberated space on the cathetes. The inequalities from (*) and (**) make a contradiction. □

A collection of optimal values $n(A(t, k), B(t, k)$ is given in Table I :

Table I : Some small values of functions $A(t, k), B(t, k), card(t, k)$

t	$A(t, 2)$	$B(t, 2)$	$card(t, 2)$	$A(t, 3)$	$B(t, 3)$	$card(t, 3)$
5	5	7	4	2	3	2
7	12	20	8	6	11	5
11	42	78	20	20	48	13
13	69	126	28	31	79	18
17	144	278	47	66	180	31
19	202	387	59	93	249	39
23	343	674	85	153	442	56
29	674	1331	134	303	878	89
31	826	1612	153	366	1059	101
37	1361	2702	215	610	1790	143

3. ASYMPTOTICS

In this section is given an asymptotic formula for the maximal number $n(a, b)$ of edges within an arc with the cathetes' lengths a and b. This formula is obtained by considering the optimal staircase $S(t, k)$. It is shown that this formula is accordance with the asymptotic formula for the maximal number of edges of a digital convex polygon with given diameter, which has been derived in [2].

The following lemma is concerned with monotonicity (in t) of the sequences $A(t, k), B(t, k)$ and $card(t, k)$ for k fixed :

Lemma 3 — If two prime numbers t_1 and t_2 satisfy that $t_1 < t_2$ then $A(t, k) < A(t_2, k)$, $B(t_1, k) < B(t_2, k)$ and $n(t_1, k) < n(t_2, k)$.

PROOF : Follows from the fact that $S(t_1, k)$ is a proper subset of $S(t_2, k)$. □

Let $\phi(n)$ and $F_n(x)$ denote the well-known Euler function and the number of relatively prime numbers with n which are not larger than x, respectively. Lemma from Appendix says that $F_n(x) \approx \frac{\phi(n)}{n} \cdot x$..

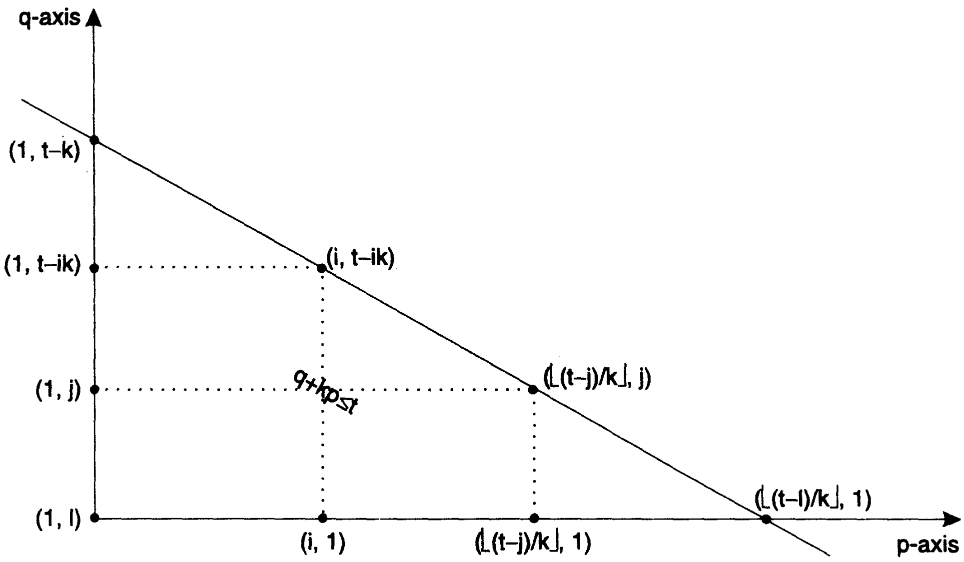

FIG. 2. Domain of $S(t,\ k)$ in the pq-plane.

It can be verified (see Fig. 2 and Appendix) that

$$A(t,k) = \sum_{q/p \in S(t,k)} p = \sum_{p=1}^{\lfloor (t-1)/k \rfloor} p \cdot F_p(t-pk)$$

$$= \sum_{p=1}^{\lfloor (t-1)/k \rfloor} p \cdot \frac{\phi(p)}{p} \cdot (t-pk)$$

$$= t \cdot \sum_{p=1}^{\lfloor (t-1)/k \rfloor} \phi(p) - k \cdot \sum_{p=1}^{\lfloor (t-1)/k \rfloor} p \cdot \phi(p)$$

$$B(t,k) = \sum_{q/p \in S(t,k)} q = \sum_{q=1}^{t-1} q \cdot F_q \cdot \left(\left\lfloor \frac{t-1}{k} \right\rfloor \right)$$

$$= \sum_{q=1}^{t-k} q \cdot \frac{\phi(q)}{q} \left\lfloor \frac{t-q}{k} \right\rfloor$$

$$\approx \frac{1}{k} \left(t \cdot \sum_{q=1}^{t-k} \phi(q) - \sum_{q=1}^{t-k} q \cdot \phi(q) \right)$$

$$card(t,k) = \sum_{p=1}^{\lfloor (t-1)/k \rfloor} \cdot F_p(t-pk) = \sum_{p=1}^{\lfloor (t-1)/k \rfloor} \left(\frac{t}{p} - k \right) \cdot \phi(p).$$

Taking into account that

$$\sum_{s=1}^{t} \phi(s) = \frac{3t^2}{\pi^2} + O(t \log t) \qquad \text{([5], Th. 330), as well as that}$$

$$\sum_{s=1}^{t} s \cdot \phi(s) = \frac{2t^2}{\pi^2} + O(t^2 \log t),$$

and

$$\sum_{s=1}^{t} \frac{\phi(s)}{s} = \frac{6t}{\pi^2} + O(\log t) \quad \text{([4])}$$

and approximizing the expressions $t - 1$ and $t - k$ by t (note that $t \to \infty$, while k is fixed), one obtains that :

$$A(t, k) = \frac{t^3}{k^2\pi^2} + O(t^2 \log t)$$

$$B(t, k) = \frac{t^3}{k\pi^2} + O(t^2 \log t)$$

$$card(t, k) = \frac{3t^2}{k\pi^2} + O(t \log t).$$

Substituting $k \approx B(t, k)/A(t, k)$ and $t = ((\pi^2 B^2(t, k))/A(t, k))^{1/3} + O(\log t)$ into the expression for $card(t, k)$, it follows that

$$card(A, B) = \frac{3}{\pi^{2/3}} \cdot B^{1/3} \cdot A^{1/3} + O(B^{1/3} \log B).$$

Starting from arbitrary cathetes' lengths b and a ($a^2 \geq b \geq a$), one can take $k = \lfloor b/a \rfloor$ and determine two closest (in most cases consecutive) prime numbers t_1 and t_2 such that

$t_1 < t_2$ and $A(t_1, k) \leq a \leq A(t_2, k)$ and $B(t_1, k) \leq b \leq B(t_2, k)$. This leads to the following result:

***Theorem* 2** — *The maximal number n(a, b) of edges of digital arc with the cathetes' lengths a and b* ($a^2 \geq b \geq a$) *has the following behaviour* :

$$n(a, b) = \frac{3}{\pi^{2/3}} \cdot b^{1/3} \cdot a^{1/3} + O(b^{1/3} \log b).$$

It has been shown in [2] that the maximal number $n(m)$ of edges of a digital convex polygon which can be inscribed into $m \times m$ – grid square (with the edges parallel to the coordinate axes) has the behaviour :

$$n(m) = \frac{12}{(4\pi^2)^{1/3}} \cdot m^{2/3} + O(m^{1/3} \log m).$$

The derivation of $n(m)$ was based on the use of a monotonous family of symmetric (with four equal arcs) optimal polygons (such a family is an analogue of the above $S(t, k)$). This formula is in accordance with Theorem 2; it can be derived from the formula for $n(a, b)$ by replacing $a = b = m/2$ and by multiplying with 4.

4. Conclusion

The maximization of the number of edges of a digital convex arc (with given cathetes' lengths) is a step from the square case to the rectangle case, when the maximization of the number of edges of digital convex polygons inscribed into a quadrangle is considered. The square case, a formulation of which can be found in [7], has been completely solved. For example, an exact general solution is given in [6], while the asymptotical and algorithmical aspects have been discussed in [2] and [1], respectively.

The solution of asymptotical aspects of the square case[2] is analogous to the above solution of the considered problem at the following two points :

1. The concept of "sparing the projection lengths" is used. Namely, convexity of an arc implies that the projections of its edges onto the adjacent cathetes cannot overlap each other. Thus the sums of projection lengths are upper-bounded by the cathetes' lengths. The heuristic goal function $\min_{q/p} (p+q)$ leads to the optimal (greedy) solutions of the square case. However, when the cathetes' lengths are not equal, then the sparing should be in accordance with their difference.
2. An auxiliary monotonous family of optimal solutions is used. Exact optimal solution in the general case can be interpolated between some two consecutive optimal solutions from the family, by using some more sophisticated methods.

The solution of the problem of maximizing the number of edges of a digital convex arc with given different cathetes' lengths can be directly applied for the symmetric variant of the rectangle case, that is, for determination of a symmetric (with four equal arcs) digital convex polygon, with the maximal possible number of vertices, which approximizes an ellipse with given axes in the integer grid. This is done by copying the above solution into each one of the four *arcs*: SE-arc (south-east), NE-arc, NW-arc and SW-arc.

However, it is not guaranteed that the optimal solution of the rectangle case, that is, of the following more general problem, must be symmetric (with four arcs solved in the same manner):

• Given the edge sizes a and b of a rectangle R, determine a digital convex polygon with the maximal possible number of edges, which can be inscribed into R.

Some motivations for considering both the symmetric and the general variant of the rectangle problem, can be respectively found in the following two practical questions :

How precisely can be an ellipse drawn on a computer screen with square pixels, if the ellipse is represented by a symmetric convex polygon, while the measure of precision is the number of vertices of such a polygon ?

What is the lower bound for the code size, which is sufficient to represent any convex shape on the rectangular grid of a given size ?

APPENDIX

Lemma — If $\omega(n)$ is the number of distinct prime divisors of n, then

$$F_n(x) = \frac{\varphi(n)}{n} \cdot x + O(2^{\omega(n)}),$$

where $\omega(n)$ is the number of different prime factors of n.

PROOF : We use the well-known Moebius function $\mu(n)$ defined as :

$$\mu(1) = 1;$$

if $n > 1$, and $n = p_1^{a_1} \ldots p_k^{a_k}$ is the prime decomposition of n, then

$$\mu(n) = \begin{cases} (-1)^k & \text{if } a_1 = a_2 = \ldots = a_k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_n(x) = \sum_{m \le x} \left(\sum_{d \mid m,\, d \mid n} \mu(d) \right) = \sum_{d \mid n} \mu(d) \sum_{m \le x/d} 1$$

$$= \sum_{d \mid n} \mu(d) \left[\frac{x}{d} \right] = \sum_{d \mid n} \mu(d) \left(\frac{x}{d} + O(1) \right)$$

(square brackets denote integer part)

$$= x \sum_{d \mid n} \frac{\mu(d)}{d} + Q_n(x),$$

is satisfied. The proof is completed by using the identity.

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