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*PARANORMAL COMPOSITION OPERATORS

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In this paper *paranormal and quasi *paranormal composition operators on $L^2(\lambda)$ are characterised. We introduce a new class $(M, k)^*$ of operators and study the relationship between the class (M, k) and $(M, k)^*$.

Key Words : Hilbert Space, *Paranormal; Quasi-*paranormal Operators; Composition Operators

1. INTRODUCTION

During the last twenty years several authors have studied the properties of various classes of composition operators on L^2 spaces. The concept of normality of bounded linear operators on a Hilbert space has been generalized by different authors. Arora and Thukral have introduced and studied *paranormal and quasi- *paranormal operators[1,2]. The aim of this paper is to obtain a necessary and sufficient condition for a bounded linear operator T on a Hilbert space H and composition operators on L^2 spaces to be *-paranormal and quasi-*paranormal. A new class $(M, k)^*$ of operators is introduced and studied and its relationship with the (M, k) class is obtained. Also we discuss the relationship between these classes and *paranormal and quasi-*paranormal operators.

2. PRELIMINARIES

Let H be an infinite dimensional complex separable Hilbert space and T, a bounded linear operator on H. T is said to be normal if $T^* T = TT^*$, quasi-normal if $T(T^*T) = (T^*T)T$, hyponormal if $T^*T \geq TT^*$ or equivalently $\| T^*(x) \| \leq \| T(x) \|$ for all x in H, quasi-hyponormal if $T^{*^2} T^2 \geq (T^*T)^2$ or equivalently $\| T^*T(x) \| \leq \| T^2(x) \|$ for all x in H. T is called *paranormal if $\| T^*(x) \|^2 \leq \| T^2(x) \| \| x \|$ for all x in H and quasi-*paranormal if $\| (T^*T)(x) \|^2 \leq \| T^3(x) \| \| T(x) \|$ for all x in H. Ando[5] has proved that an operator T is paranormal if and only if $T^{*^2} T^2 + 2KT^*T + k^2 \geq 0$ for all real k. In an analogous manner we derive a characterization of *paranormal operators and quasi-*paranormal operators.

T is of (M, k) class if $T^{*k} T^k \geq (T^*T)^k$ for $k \geq 2$. It is known that the $(M, 2)$ class coincides with the class of quasi-hyponormal operators. But, the class of hyponormal operators does not coincide with (M, k) for any k. However, if we define a new class $(M, k)^*$ as $\{T : T^{*k} T^k \geq (TT^*)^k\}$ we find that $(M, 1)^*$ coincides with the class of hyponormal operators. Also we find $(M, k)^* \subseteq (M, k+1)$.

Let (X, S, λ) be a σ-finite measure space. Let T be a measurable transformation from X into itself. $L^2(X, S, \lambda)$ is denoted as $L^2(\lambda)$. The equation $C_T(f) = f \bullet T, f \varepsilon L^2(\lambda)$ defines a composition transformation from $L^2(\lambda)$ to the space of complex valued functions on X. If the measure $\lambda \bullet T^{-1}$ is absolutely continuous with respect to λ and the Radon-Nikodym derivative $d(\lambda \bullet T^{-1})/d\lambda$ is essentially bounded then C_T is bounded and linear and it is called a composition operator on $L^2(\lambda)$ induced by Arora and Thukral[1,2] have defined and discussed the properties of *paranormal and quasi-*paranormal operators on a Hilbert space H. In this paper we discuss the properties of *paranormal and quasi-*paranormal composition operators on $L^2(\lambda)$. We also study the relationship between the composition operators on $L^2(\lambda)$ and the classes of operators (M, k) and $(M, k)^*$.

3. *PARANORMAL OPERATORS

$B(H)$ denotes the set of all bounded linear operators on a Hilbert space H.

***Theorem* 3.1** — *An operator $T \in B(H)$ is *paranormal if and only if $T^{*^2} T^2 + 2kTT^* + k^2 \geq 0$ for all $k \in R$.*

PROOF : For all $x \in H$,

$$T^{*^2} T^2 + 2kTT^* + k^2 \geq 0 \text{ for all } k \in R.$$

$$\Leftrightarrow \langle (T^{*^2} T^2 + 2kTT^* + k^2)(x), x \rangle \geq 0 \text{ for all } k \in R$$

$$\Leftrightarrow \langle (T^{*^2} T^2 (x), x \rangle + 2k \langle TT^*(x), x \rangle + k^2 \langle x, x \rangle \geq 0 \text{ for all } k \in R$$

$$\Leftrightarrow \langle T^2(x), T^2(x) \rangle + 2k \langle T^*(x), T^*(x) \rangle + k^2 \langle x, x \rangle \geq 0 \text{ for all } k \in R$$

$$\Leftrightarrow \| T^2(x) \|^2 + 2k \| T^*(x) \|^2 + k^2 \| x \|^2 \geq 0 \text{ for all } k \in R.$$

We know that by elementary properties of real quadratric forms, if $a > 0$, b and c are real numbers then $at^2 + bt + c \geq 0$ for every real t if and only if $b^2\text{-}4ac \leq 0$.

Hence, $$T^{*^2} T^2 + 2kTT^* + k^2 \geq 0 \text{ for all } k \in R.$$

$$\Leftrightarrow 4 \| T^*(x) \|^4 \leq 4 \| T^2(x) \|^2 \| x \|^2 \text{ for all } x \in H$$

$$\Leftrightarrow \| T^*(x) \|^2 \leq \| T^2(x) \| \| x \| \text{ for all } x \in H$$

$$\Leftrightarrow T \text{ is } {}^*\text{paranormal.}$$

***Theorem* 3.2** — *An operator T is quasi-*paranormal if and only if $T^{*^3} T^3 + 2k(T^*T)^2 + K^2T^*T \geq 0$ for all $k \in R$.*.

PROOF : For any $x \in H$,

$$T^{*^3}T^3 + 2k(T^*T)^2 + k^2T^*T \geq 0 \text{ for all } k \in R.$$

$$\Leftrightarrow \langle (T^{*^3}T^3 + 2k(T^*T)^2 + k^2T^*T)(x), x \rangle \geq 0 \text{ for all } k \in R.$$

$$\Leftrightarrow \langle T^3(x), T^3(x) \rangle + 2k \langle T^*T(x), T^*T(x) \rangle + k^2 \langle T(x), T(x) \rangle \geq 0 \text{ for all } k \in R.$$

$$\Leftrightarrow \| T^3(x) \|^2 + 2k \| T^*T(x) \|^2 + k^2 \| T(x) \|^2 \geq 0 \text{ for all } k \in R.$$

By elementary properties of real quadratic forms we get

$$T^{*^3}T^3 + 2k(T^*T)^2 + k^2T^*T \geq 0 \text{ for all } k \in R$$

$$\Leftrightarrow 4 \| T^*T(x) \|^4 \leq 4 \| T^3(x) \|^2 \| T(x) \|^2$$

$$\Leftrightarrow \| T^*T(x) \|^2 \leq \| T^3(x) \| \| T(x) \|$$

$\Leftrightarrow$ T is quasi-*paranormal.

Now we define a new class of operators $(M, k)^*$ as follows:

Definition 3.3 — An operator T on a Hilbert space H is of class $(M, k)^*$ if $T^{*k}T^k \geq (TT^*)^k$.

From this definition we conclude that the class of all hyponormal operators coincides with the $(M, 1)^*$ class.

Example 3.4 — The right shift operator on l_2 is of class $(M, k)^*$ for all $k \geq 1$. For any $x = (x_1, x_2, \ldots)$, $T(x) = (0, x_1, x_2, \ldots)$ and $T^*(x) = (x_2, x_3, \ldots)$. From this we observe that $\langle T^{*k}T^k - (TT^*)^k (x), x \rangle = x_1^2 \geq 0$ for any $k \geq 1$ and any $x \in l_2$. Hence, by definition T is of class $(M, k)^*$ for any $k \geq 1$.

Example 3.5 — An operator of class $(M, k)^*$ which is not in (M, k).

Let T be a weighted shift operator on l_2 with weights $\alpha_1 = 1/2$, $\alpha_2 = 2$, $\alpha_3 = 1/4$, $\alpha_n = 16$ for all $n \geq 4$. That is $T(x) = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots)$ and $T^*(x) = (\alpha_1 x_2, \alpha_2 x_3, \ldots)$. Let e_n be the vector (0, 0, ..., 1, 0, ...) with 1 at the nth place and zero elsewhere. A simple computation shows that

$$(T^{*^2}T^2 - (TT^*)^2)(e_n) = \alpha_n^2 \alpha_{n+1}^2 - (\alpha_{n-1})^4 \text{ for all } n > 1$$

Also $(T^{*2}T^2 - (TT^*)^2)(e_1) = \alpha_1^2 \alpha_2^2 \geq 0$ which implies that

$\langle T^{*^2}T^2 - (TT^*)^2 (e_n), e_n \rangle \geq 0$ for all n. Therefore, T is of class $(M, 2)^*$. But $(T^*T)^2 (e_2) - T^{*^2}T^2(e_2) = 63/4\, e_2$ which implies that T is not of class $(M, 2)$ and also not hyponormal.

Example 3.6 — An operator in (M, k) but not in $(M, k)^*$.

Suppose H is a direct sum of denumerable copies of two dimensional Hilbert space $R \times R$. Let A and B be any two positive operators on $R \times R$. For any fixed positive integer n, define an operator $T = T_{A, B, n}$ on H as

$$T((x_1, x_2, \ldots)) = (0, A(x_1), A(x_2), \ldots, A(x_n), B(x_{n+1}), \ldots)$$

A simple computation yields that its adjoint T^* is

$$T^*((x_1, x_2, ...)) = (A(x_2), A(x_3), ..., A(x_{n+1}), B(x_{n+2}), ...)$$

and T is quasi-hyponormal if and only if $AB^2A - A^4 \geq 0$.

If we take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then we can easily observe that $T_{A,B,n}$ is quasi-hyponormal which implies that $T_{A,B,n}$ is of class $(M, 2)$.

Now fix an element $x = (0, 0, ..., (-8/7, 1), 0, 0, ...)$ where $(-8/7, 1)$ is in the $n + 1$th place.

A simple computation yields that $\langle (T^{*2}T^2 - (TT^*)^2)(x, x)$ is given by

$$\langle (T^{*^2}T^2 - (TT^*)^2)(x), x \rangle = \| B^2(x_{n+1}) \|^2 - \| A^2(x_{n+1}) \|^2$$

$$= \left\langle \begin{pmatrix} 7 & 8 \\ 8 & 8 \end{pmatrix} (-8/7, 1), (-8/7, 1) \right\rangle$$

$$= \langle (0, -8/7), (-8/7, 1) \rangle$$

$$= -8/7 < 0$$

which implies that $T_{A,B,n}$ is not of class $(M, 2)^*$.

***Theorem* 3.7** — *If an operator T is of class (M, 2)* then it is *paranormal .*

PROOF : T is of class $(M, 2)^* \Rightarrow T^{*2}T^2 \geq (TT^*)^2$

$$\Rightarrow \langle T^{*^2}T^2(x) - (TT^*)^2(x), x \rangle \geq 0 \text{ for all } x \in H$$

$$\Rightarrow \langle T^2(x), T^2(x) \rangle - \langle (TT^*)(x), TT^*(x) \rangle \geq 0 \text{ for all } x \in H$$

$$\Rightarrow \| T^2(x) \|^2 \geq \| TT^*(x) \|^2 \text{ for all } x \in H.$$

$$\Rightarrow \| TT^*(x) \| \leq \| T^2(x) \| \text{ for all } x \in H. \quad \text{... (1)}$$

Now

$$\| T^*(x) \|^2 = | \langle T^*(x), T^*(x) \rangle |$$

$$= | \langle TT^*(x), x \rangle | \leq \| TT^*(x) \| \| x \|$$

$$\leq \| T^2x \| \| x \| \text{ by (1).}$$

Therefore, T is *paranormal.

***Theorem* 3.8** — *If an operator T on H is of class (M, k)* then it is of class (M, k + 1). The converse is also true when T has dense range in H.*

PROOF : Let T be an operator of class $(M, k)^*$. Then

$$T^{*^k} T^k \geq (TT^*)^k, \text{ i.e., } \langle (T^{*^k} T^k - (TT^*)^k)(x), x \rangle \geq 0 \text{ for all } x \in H.$$

Now,

$$\langle (T^{*^{k+1}} T^{k+1} - (T^*T)^{k+1})(x), x \rangle = \langle T^*(T^{*^k} T^k - (TT^*)^k)\, T(x), x \rangle$$

$$= \langle (T^{*^k} T^k - (TT^*)^k)(Tx)), T(x)) \rangle$$

$$\geq 0$$

Hence, $T^{*^{k+1}} T^{k+1} \geq (T^*T)^{k+1}$ so that T is of class $(M, k+1)$.

Conversely, if T is an operator of class $(M, k+1)$ with dense range, then we claim that T is of class $(M, k)^*$.

Let x be an arbitrary element in H. Since $\overline{R[T]} = H$ there exists a sequence (x_n) in H such that $T(x_n) \rightarrow x$.

For each 'n' $\langle (T^{*^k} T^k - (TT^*)^k)(T(x_n)), T(x_n) \rangle$

$$= \langle (T^{*^{k+1}} T^{k+1} - T^*T)^{k+1} (x_n), x_n \rangle \geq 0.$$

Taking limit as $n \rightarrow \infty$ its limit is also non-negative. But by Theorem 1 in [4, pp. 42] if $(x_n) \rightarrow x$ and $(y_n) \rightarrow y$ then $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$. Here, $T(x_n) \rightarrow x$ and $T^{*^k} T^k - (TT^*)^k$ is continuous. Hence, $(T^{*^k} T^k - (TT^*)^k)(T(x_n)) \rightarrow (T^{*^k} T^k - (TT^*)^k)(x)$ so that by Theorem 1 in [4, pp. 42], we have

$$\langle (T^{*^k} T^k (TT^*)^k)(T(x_n)), T(x_n) \rangle \rightarrow \langle T^{*^k} T^k - (TT^*)^k (x), x \rangle$$

It follows, therefore, that $\langle T^{*^k} T^k - (TT^*)^k)(x), x \rangle \geq 0$ for all $x \in H$

$$\Rightarrow T^{*^k} T^k \geq (TT^*)^k$$

$$\Rightarrow T \text{ is of class } (M, k)^*.$$

Corollary 3.9 — Every quasi-hyponormal operator with dense range is hyponormal.

PROOF : Let T be a quasi-hyponormal operator with dense range. Then it is of class $(M, 2)$ with dense range. Hence, by Theorem 3.8, T is of class $(M, 1)^*$ which implies that T is hyponormal.

Example 3.10 — Consider the example 3.4. It is of class $(M, k)^*$ for all $k \geq 1$. Hence, by Theorem 3.8 it is of class (M, k) for all $k \geq 2$. But the range of T is not dense in H. Therefore, the condition of having dense range is sufficient but not necessary for the above theorem.

***Theorem* 3.11** — *If T is a quasinormal operator on H, then T is hyponormal if and only if T is of class $(M, k)^*$ for all $k \geq 1$.*

PROOF : If T is of class $(M, k)^*$ for all $k \geq 1$ then it is of class $(M, 1)^*$ which implies that T is hyponormal.

Conversely, suppose T is hyponormal so that $T^*T \geq TT^*$. We will prove that $T^{*k}T^k - (TT^*)^k \geq 0$ by induction on k. Since T is hyponormal the result is true when $k = 1$.

Now suppose that the result is true when $k = n$ so that $T^{*n}T^n - (TT^*)^n \geq 0$. A simple calculation shows that T^*T commutes with $T^{*n}T^n - (TT^*)^n$ since T is quasi-normal. Then by problem 7(xi) in [4, pp 149] we get $(T^{*n}T^n)(T^*T) \geq (TT^*)^n (T^*T)$. Since T^*T commutes with T and T^* we get

$$(T^{*^n} T^n)(T^*T) = T^{*^{n+1}} T^{n+1}. \quad \ldots (1)$$

Also

$$(TT^*)^n (T^*T) = (TT^*)^{n-1} (TT^*)(T^*T) = (TT^*)^{n-1} (TT^*TT^*) = (TT^*)^{n+1}.$$

Applying these two results in (1) we get $T^{*^{n+1}} T^{n+1} \geq (TT^*)^{n+1}$ which implies that T is of class $(M, k+1)^*$.

Thus the result is true when $k = n + 1$. Then by induction T is of class $(M, k)^*$ for all $k \geq 1$.

4. *PARANORMAL AND QUASI-*PARANORMAL COMPOSITION OPERATORS

In this section we derive some characterisation of *paranormal and quasi-*paranormal composition operators on $L^2(\lambda)$ and obtain their relationship with the classes (M, k) and $(M, k)^*$. Throughout this section $f_0^{(k)}$ indicates Radon Nikodym derivative of $\lambda(T^k)^{-1}$ with respect to λ and f_0 indicates the Radon Nikodym derative of $\lambda(T^{-1})$ with respect to λ.

***Theorem* 4.1** — *A composition operator C_T with dense range is *paranormal if and only if $f_{0(2)} + 2k(f_0 \bullet T) + k^2 \geq 0$ (a.e.) for all $k \in R$.*

PROOF : By Theorem 3.1 C_T is *paranormal

$$\Leftrightarrow \langle ((C_T^{*^2} C_T^2 + 2kC_TC_T^* + k^2)(g), g \rangle \geq 0 \text{ for all } g \in L^2(\lambda) \text{ and for all } k \in R.$$

$$\Leftrightarrow \langle (C_T^{*^2} C_{T_2} + 2kC_TC_T^* + k^2)(\chi_E), \chi_E \rangle \geq 0 \text{ for all } \chi E \in L^2(\lambda) \text{ and for all } k \in R.$$

But by theorem 1 in [3, pp [126], if C_T has dense range then $C_TC_T^* = M_{f_0}$ and $C_T^{*^2} CT_2 = M_{f_0}(2)$. Applying these two results we get,

C_T is *paranormal

$$\Leftrightarrow \langle M_{f_0}(2) + 2kM_{f_0} \bullet_T + k^2)(\chi_E), \chi_E \rangle \geq 0 \text{ for all } \chi_E \in L^2(\lambda) \text{ and for all } k \in R.$$

$$\Leftrightarrow \int_E (M_{f_0}(2) + 2kM_{f_0} \bullet T + k^2)(\chi_E)\, d\lambda \geq 0 \text{ for all } k \in R.$$

$$\Leftrightarrow \int_E (f_0^{(2)} + 2k(f_0 \bullet T) + k^2)\, d\lambda \geq 0 \text{ for all } k \in R.$$

$$\Leftrightarrow (f_0^{(2)} + 2k(f_0 \bullet T) + k^2) \geq 0 \text{ (a.e.) for all } k \in R.$$

***Theorem* 4.2** — *A composition operator C_T on $L^2(\lambda)$ with dense range is *paranormal if and only if it is of class (M, 2)*.*

PROOF : By Theorem 3.7 any operator of class $(M, 2)^*$ is *paranormal. Conversely, suppose C_T is a composition operator with dense range which is *paranormal. Then by Theorem 4.1 we get $(f_0^{(2)} + 2k(f_0 \bullet T) + k^2) \geq 0$ (a.e.) for all $k \in R$. Applying the elementary properties of real quadratic forms we get,

$(f_0 \bullet T)^2 \leq f_0^{(2)}$ (a.e.) which implies

$$\int_E (f_0^{(2)} - (f_0 \bullet T)^2)\, d\lambda \geq 0 \text{ for all } E \text{ such that } \lambda(E) < \infty.$$

$$\int_X (M_{f_0}(2) - M_{f_0} \bullet T)^2)(\chi_E)\, d\lambda \geq 0 \text{ for all } \chi_E\ L^2(\lambda)$$

$$< C_T^{*^2} C_{T_2} - (C_T C_{T^*})^2)(\chi_E), \chi_E \rangle \geq 0$$

C_T is of class $(M, 2)^*$.

***Theorem* 4.3** — *A composition operator C_T on $L^2(\lambda)$ is quasi-*paranormal if and only if $f_0^3 \leq f_0^{(3)}$ a.e.*

PROOF : By Theorem 3.2, C_T is quasi-*paranormal

$$\Leftrightarrow \langle ((C_T^{*^3} C_T^3 + 2k(C_T^* + C_T)^2 + k^2 C_T^* C_T)(g), g \rangle \geq 0 \text{ for all } k \in R \text{ and } g \in L^2(\lambda).$$

$$\Leftrightarrow \langle C_T^{*^3} C_T^3 + 2k(C_T^* C_T)^2 + k^2 C_T^* C_T)(\chi_E), \chi_E \rangle \geq 0 \text{ for all } k \in R \text{ and } \chi E \in L^2(\lambda)$$

$$\Leftrightarrow \langle (M_{f_0}^{(3)} + 2kM_{f_0}^2 + k^2 M_{f_0})(\chi_E), \chi_E \rangle \geq 0 \text{ by Theorem in [3, p 126].}$$

$$\Leftrightarrow \int_E (f_0^{(3)} + 2kf_0^2 + k^2 f_0)\, d\lambda \geq 0 \text{ for all } k \in R.$$

$$\Leftrightarrow f_0^{(3)} + 2kf_0^2 + k^2 f_0 \geq 0 \text{ a.e. for all } k \in R.$$

By elementary properties of real quadratic forms we get C_T is quasi *paranormal $\Leftrightarrow f_0^4 \leq f_0^{(3)} f_0$ (a.e.) $\Leftrightarrow f_0^3 \leq f_0^{(3)}$ (a.e.)

***Theorem* 4.4** — *A composition operator C_T on $L^2(\lambda)$ is quasi-*paranormal if and only if C_T is of class (M, 3).*

PROOF : For any operator T, which is of class $(M, 3)$ we have $T^{*^3} T^3 \geq (T^*T)^3$ which implies

$$\langle (T^{*^3} T^3 - (T^*T)^3)(x), x \rangle \geq 0 \quad \text{for all } x \in L^2(\lambda),$$

i.e.,

$$\| TT^*T(x) \| \leq \| T^3(x) \|. \qquad \ldots (1)$$

Hence,

$$\| T^*T(x) \|^2 = \langle (T^*T)(x), T^*T(x) \rangle = \langle TT^*T(x), T(x) \rangle$$

$$\leq \| TT^*T(x) \| \, \| T(x) \|$$

$$\leq \| T^3(x) \| \, \| T(x) \| \text{ by (1).}$$

Theorefore, T is quasi-*paranormal.

In particular if C_T is of class $(M, 3)$ then it is quasi-*paranormal. Conversely suppose C_T is a composition operator on $L^2(\lambda)$ which is quasi-*paranormal. Then by Theorem 4.3.

$$f_0^3 \leq f_0^{(3)} \text{ (a.e) which implies}$$

$$\int_E (f_0^{(3)} f_0^3)\, d\lambda \geq 0 \text{ for all } E \text{ such that } \lambda(E) < \infty$$

$$\int_X (M_{f_0}(3) - (M_{f_0})^3)(\chi_E)\, \chi_E \, d\lambda \geq 0$$

$$(C_{T^*}^3 C_{T_3} - (C_{T^*} C_T)^3 (\chi_E), \chi_E) \geq 0 \text{ for all } \chi_E \in L^2(\lambda)$$

Hence, C_T is of class $(M, 3)$.

Corollary 4.5 — [2, p. 83] Any quasi-*paranormal composition operator on $L^2(\lambda)$ with dense range is *paranormal.

PROOF : By Theorem 4.4, any quasi-*paranormal composition operator on $L^2(\lambda)$ is of class $(M, 3)$. If the range of C_T is also dense in $L^2(\lambda)$ then by Theorem 3.6 it follows that C_T is of class $(M, 2)^*$. By Theorem 4.2, we conclude that C_T is *paranormal.

Corollary 4.6 — A composition operator C_T on $L^2(\lambda)$ is quasi-*paranormal if and only if $\| C_T M_{f_0}(x) \| \leq \| C_T^3(x) \|$ for all x in $L^2(\lambda)$.

PROOF : By Theorem 4.4, C_T is quasi-*paranormal

$$\Leftrightarrow C_T \text{ is of class } (M, 3)$$

$$\Leftrightarrow C_T^{*^3} C_T^3 \geq (C_T^* C_T)^3$$

$$\Leftrightarrow \langle (C_T^* C_T)^3 (x), x \rangle \leq \| C_T^3 (x) \|^2 \text{ for all } x \in L^2(\lambda)$$

$$\Leftrightarrow \| C_T C_T^* C_T(x) \| \leq C_T^* C_T^3 (x) \| \text{ for all } x \in L^2(\lambda)$$

$$\Leftrightarrow \| C_T M_{f_0} x \| \leq \| C_T^3 (x) \| \text{ for all } x \in L^2(\lambda) \text{ (by [3, p. 126])}$$

***Theorem* 4.7** — *If C_T is a composition operator on $L^2(\lambda)$, then C_T is of class $(M, k)^*$ if and only if $\| f_0^{(k)1/2} \chi_E \| \geq (f_0 \bullet T)^{k/2} p(\chi_E) \|$ for all $\chi_E \in L^2(\lambda)$ where p is the projection on to $\overline{R(C_T)}$.*

PROOF : A simple calculation yields the result $(C_T C_{T^*})^k = M_{(f_0 \bullet T)^k} p$ by using the results in [3, pp 126].

Now C_T is of class $(M, k)^* \Leftrightarrow C_T^{*k} C_T^k \geq (C_T C_T^*)^k$

$$\Leftrightarrow \langle C_T^{*^k} C_T^k - (C_T C_T^*)^k (\chi_E), \chi_E \rangle \geq 0 \text{ for all } \chi E \in L^2(\lambda)$$

$$\Leftrightarrow \langle (M_{f_0} (k) - M_{(f_0 \bullet T)^k} p) (\chi_E), \chi_E \rangle \geq 0.$$

$$\Leftrightarrow \int_E f_0 (k) \chi_E d\lambda - \int_E (f_0 \bullet T)^k p(\chi_E) \, d\lambda \geq 0$$

$$\Leftrightarrow \int_E | f_0^{(k)1/2} \chi_E |^2 \, d\lambda - \int_E | [(f_0 \bullet T)^{k/2} p(\chi_E) \|^2 \, d\lambda \geq 0 \text{ since } p^2 = p.$$

$$\Leftrightarrow \| f_0^{(k)1/2} \chi_E \|^2 \geq \| (f_0 \bullet T)^{k/2} p(\chi_E) \|^2$$

$$\Leftrightarrow \| f_0^{(k)1/2} \chi_E \| \geq \| f_0 \bullet T)^{k/2} p(\chi_E) \|.$$

***Theorem* 4.8** — *If C_T is a composition operator on $L^2(\lambda)$ and C_T^2 has dense range then C_T^* is *paranormal if and only if $f_0^2 \leq f_0^{(2)} \bullet T^2$ (a.e.).*

PROOF : By Theorem 3.1, C_T^* is *paranormal if and only if

$$\langle (C_T^2 C_T^{*^2} + 2k C_T^* C_T + k^2) (g), g \rangle \geq 0 \text{ for all } k \in R \text{ and } g \in L^2(\lambda).$$

Since

$$\langle C_T^2 (C_T^2)^* (g), g \rangle = \langle (f_0^{(2)} \bullet T^2) g, g \rangle$$

and

$$C_T^* C_T = M_{f_0}$$

By Theorem 1 in [3, pp 126] we get, C_T^* is *paranormal if and only if

$$\langle ((f_0^{(2)} \bullet T^2) + 2kf_0 + k^2)(g), g \rangle \geq 0 \text{ for all } k \in R \text{ and } g \in L^2(\lambda).$$

$$\Leftrightarrow f_0^{(2)} \bullet T^2 + 2kf_0 + k^2 \geq 0 \text{ (a.e.) for all } k \in R.$$

$$\Leftrightarrow f_0^2 \geq f_0^{(2)} \bullet T^2 \text{ (a.e.) (by elementary properties of real quadratic forms).}$$

***Theorem* 4.9** — *If C_T is a composition operator on $L^2(\lambda)$ having dense range, then C_T^* is quasi-*paranormal if and only if $(f_0 \cdot T)^3 \leq f_0^{(3)} \bullet T^3$ (a.e.)*

PROOF : By Theorem 3.2, C_{T^*} is quasi-*paranormal if and only if

$$\langle ((C_T^3 C_T^{*^3} + 2k (C_T C_T^*)^2 + k^2 C_T C_T^*)(g), g \rangle \geq 0 \text{ for all } k \in R \text{ and } g \in L^2(\lambda).$$

But if C_T has dense range then C_T^2 and C_T^3 also have dense range which implies that

$$\langle C_T^3 C_{T^*}^3 (g), g \rangle = \langle f_0^{(3)} \bullet T^3 (g), g \rangle$$

$$\langle (C_T C_T^*)^2 (g), g \rangle = \langle (f_0^2 \bullet T)(g), g \rangle$$

$$\langle C_T C_{T^*} (g), g \rangle = \langle f_0 \bullet T(g), g \rangle$$

Applying these results in (1), we get C_T^* is quasi-*paranormal

$$\Leftrightarrow \langle (f_0^{(3)} \bullet T^3 + 2kf_0^2 \bullet T + k^2 f_0 \bullet T)(g), g \rangle \geq 0.$$

$$\Leftrightarrow f_0^{(3)} \bullet T^3 + 2kf_0^2 \bullet T + k^2 f_0 \bullet T \geq 0 \text{ (a.e.) for all } k \in R.$$

$$\Leftrightarrow (f_0^2 \bullet T)^2 \leq (f_0^{(3)} \bullet T^3)(f_0 \bullet T) \geq 0 \text{ (a.e.).}$$

$$\Leftrightarrow (f_0^3 \bullet T)(f_0 \bullet T) \leq (f_0^{(3)} \bullet T^3)(f_0 \bullet T) \text{ (a.e.).}$$

$$\Leftrightarrow (f_0 \bullet T)^3 \leq f_0^{(3)} \bullet T^3 \text{ (a.e.).}$$

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