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PARABOLIC VARIATIONAL INEQUALITIES AND THEIR LIPSCHITZ PERTURBATIONS IN HILBERT SPACES*

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The existence and the regular problem for solutions of the nonlinear functional differential equation

$$x'(t) + Ax(t) + \partial\phi(x(t)) \ni f(t, x(t)) + h(t)$$

is studied where A is a linear nonmonotone operator and generates an analytic semigroup. The norm estimates of solutions of the equation in the functional spaces are also given.

Key Words : Parabolic Variational Inequalities; Maximal Monotone Operator; Subdifferential Operator; Regularity

1. INTRODUCTION

Let H and V be two real Hilbert spaces. Assume V is dense subspace in H and that the injection of V into H is continuous. The norm on V (resp. H) will be denoted by $\|\cdot\|$ (resp. $|\cdot|$) respectively. Let $\phi : H \to (-\infty, +\infty])$ be a lower semicontinuous, proper convex function.

The first purpose in this paper is to consider the existence of solution of the following nonlinear functional differential equation on H

$$\text{(NE)} \quad \begin{cases} \dfrac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) \ni k(t),\ 0 < t \le T, \\ x(0) = x_0 \end{cases}$$

where $\partial\phi : V \to V^*$ denotes the subdifferential operator of ϕ. Let A be symmetric operator from V into V^* and satisfy the coercive condition :

$$\text{(A1)} \qquad (Au, u) \ge \omega_1 \|u\| - \omega_2 |u|,\ \omega_1, \omega_2 \in I\!R,$$

for every $u \in V$ where the duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$. If $\omega_1 > 0$ then

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$\omega_2 + A + \partial\phi$ is maximal monotone and the equation (NE), which is called the linear parabolic variational inequality, was widely developed as seen in section 4.3.2 of Barbu[1]. It is well known that the solution x belongs to $L^2(0, T; V) \cap C([0, T]; H)$, where $x_0 \in H$ and $k \in L^2(0, T; H)$ for the given time $T > 0$. With the condition (A1) (in particular, $\omega_1 > 0$), the norm estimation of a solution in $L^2(0, T; V)$ was also established. If the operator A is not monotone in general then the existence of solution of the equation (NE) is not assured.

In this paper without the condition (A1) we establish the existence of solutions of the equation (NE) and give the norm estimations of a solution for the equation (NE), which is the work [1], in case where A is not a symmetric monotone operator under initial condition $x_0 \in (D(A), H)_{1/2,2}$ where $(D(A), H)_{1/2,2}$ denotes the real interpolation space between $D(A)$ and H. Next, we also consider the following perturbation of nonlinear term :

$$\text{(NNE)} \qquad \begin{cases} \dfrac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) \ni k(t), \; 0 < t \le T, \\ x(0) = x_0 \end{cases}$$

Here, the nonlinear mapping f is a Lipschitz continuous from V into H. The equation (NNE) is caused by the following nonlinear variational inequational problem:

$$\begin{cases} \left(\dfrac{dx(t)}{dt} + Ax(t)\, , x(t) - z \right) + \phi(x(t)) - \phi(z) \\ \qquad \le f(x(t)) + k(t), x(t) - z), \quad \text{a.e.,} \quad 0 < t \le T, \quad z \in V \\ x(0) = x_0. \end{cases}$$

We also intend to investigate the existence and the norm estimates of a solution of the above nonlinear equation on $L^2(0, T; V) \cap W^{1,2}(0, T; H)$, which is applicable to the optimal control control problem as dealt in [2, 3].

2. Nonlinear Variational Inequalities

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. For the sake of simplicity, we may consider

$$\| u \|_* \le | u | \le \| u \|, \; u \in V$$

where $\| \cdot \|_*$ is the norm of the element of V^*.

Remark 1 : If an operator A is bounded linear from V to V^* and generates an analytic semigroup, then it is easily seen that

$$H = \left\{ x \in V^* : \int_0^T \| Ae^{tA} x \|_*^2 \, dt < \infty \right\},$$

for the time $T > 0$. Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{1/2,2} = H.$$

For the sake of simplicity, we also assume that there exists a constant C_1 such that

$$\| u \| \leq C_1 \| u \|_{D(A)}^{1/2} | u |^{1/2} \qquad \text{... (2.1)}$$

for every $u \in DA_0$, where

$$\| u \|_{D(A)} = (| Au |^2 + | u |^2)^{1/2}$$

is the graph norm of $D(A)$. Thus, in what follows we will write $V = (D(A), H)_{1/2,2}$ as a matter of convenience.

Let $-A : D(A) \subset H \to H$ generate an analytic semigroup in H and be bounded linear from V to V^*. The following L^2-regularity for abstract parabolic equation

$$\text{(LE)} \qquad \begin{cases} \dfrac{dx(t)}{dt} + Ax(t) = k(t), \, 0 < t \leq T, \\ x(0) = x_0 \end{cases}$$

has a unique solution x in $[0, T]$ for each $T > 0$ if $x_0 \in V$ and $f \in L^2(0, T; H)$. Moreover, we have

$$\| x \|_{L^2(0,T;D(A)) \cap W^{1,2}(0,T,H)} \leq C_T (\| x_0 \|_V + \| f \|_{L^2(0,T;H)}), \qquad \text{... (2.2)}$$

where C_T depends on T and M (see Theorem 2.3 of [4] or Theorem 3.2 of [5, Vol 2]).

Let $\phi : H \to (-\infty, +\infty)$ be a lower semicontinuous, proper convex function. Then the subdifferential operator $\partial\phi : H \to H^*$ of ϕ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \leq \phi(y) + (x^*, x - y), y \in H\}.$$

First, let us concern with the following perturbation of subdifferential operator :

$$\text{(NE)} \qquad \begin{cases} \dfrac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) \ni k(t), \, 0 < t \leq T, \\ x(0) = x_0 \end{cases}$$

For every $\varepsilon > 0$, define

$$\phi_\varepsilon(x) = \inf \{\| x - J_\varepsilon x \|_*^2 / 2\varepsilon + \phi(J_\varepsilon x) : x \in H\},$$

where $J_\varepsilon = (I + \varepsilon A)^{-1}$. If $A = \partial\phi$ then the function $\partial\phi_\varepsilon$ is Fréchet differentiable on V and its Frechet differential $\partial\phi_\varepsilon = A_\varepsilon$ is Lipschitz continuous on H with Lipschitz constant ε^{-1} where

$A_\varepsilon = \varepsilon^{-1}(I-(I+\varepsilon A)^{-1})$ as is seen in Corollary 2.2 in [6; Chapter II]. They are also well-known results that $\lim_{\varepsilon\to 0} \phi_\varepsilon = \phi$ and $\lim_{\varepsilon \to 0} \partial\phi_\varepsilon(x) = (\partial\phi)^0(x)$ for every $x \in D(\partial\phi)$ where $(\partial\phi)^0 : V \to V^*$ is the minimum element of $\partial\phi$. Now, we introduce the smoothing system corresponding to (NE) as follows:

$$\text{(SE)} \qquad \begin{cases} \dfrac{dx(t)}{dt} + Ax(t) + \partial\phi_\varepsilon(x(t)) = k(t), \ 0 < t \le T, \\ x(0) = x_0 \end{cases}$$

***Theorem* 2.1** — *Let $k \in L^2(0, T; H)$ and $x_0 \in H$. Then the equation (NE) has a unique solution*

$$x \in L^2(0, T; V) \cap C[0, T]; H),$$

which satisfies

$$x'(t) = k(t) - Ax(t) - \partial\phi^0(x(t))$$

and

$$\| x \|_{L^2 \cap C} \le C_2(1 + \| x_0 \| + \| k \|_{L^2(0,t;H)}) \qquad \ldots (2.3)$$

where C_2 is a constant and $L^2 \cap C = L^2(0, T; V) \cap C([0, T]); H)$.

2) Let A be symmetric and let us assume that there exist $h \in H$ such that for every $\varepsilon > 0$ and any $y \in D(\phi)$

$$\text{(A2)} \qquad J_\varepsilon(y + \varepsilon h) \in D(\phi) \text{ and } \phi(J_\varepsilon(y + \varepsilon h)) \le \phi(y).$$

Then for $k \in L^2(0, T; H)$ and $x_0 \in \overline{D(\phi)} \cap V$ the equation (NE) has a unique solution

$$x \in L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \cap C([0, T]; H),$$

which satisfies

$$\| x \|_{L^2 \cap W^{1,2} \cap C} \le C_2(1 + \| x_0 \| + \|k\|_{L^2(0,T;H)}) \qquad \ldots (2.4)$$

PROOF : 1). Let us fix $T > 0$ such that

$$\varepsilon^{-1} C_1 C_T (T/\sqrt{2})^{1/2} < 1. \qquad \ldots (2.5)$$

For $i = 1, 2$, we consider the following equation :

$$\frac{d}{dt} x_i(t) + Ax_i(t) + \partial\phi_\varepsilon(y_i(t)) = k(t), \quad t \in (0, T],$$

where

$$x_i(0) = 0.$$

Then

$$\frac{d}{dt}(x_1(t)-x_2(t))+A(x_1(t)-x_2(t))+\partial\phi_\varepsilon(y_1(t))-\partial\phi_\varepsilon(y_2(t)) = 0$$

and

$$x_1(0)-x_2(0) = 0.$$

We are going to show that $y \mapsto x$ is strictly contractive from $L^2(0, T; V)$ to itself if the condition (2.5) is satisfied. From (2.2) and noting the Lipschitz continuity $\partial\phi_\varepsilon$ with Lipschitz constant ε^{-1}, it follows that

$$\| x_1 - x_2 \|_{L^2(0,T;D(A))} \le C_T \| \partial\phi_\varepsilon(y_1) - \partial\phi_\varepsilon(y_2) \|_{L^2(0,T;H)}$$

$$\| \partial\phi_\varepsilon y_1 - \partial\phi_\varepsilon y_2 \|_{L^2(0,T;H)} \le \varepsilon^{-1} \| y_1 - y_2 \|_{L^2(0,T;V)}.$$

Using the Hölder inequality we also obtain that

$$\| x_1 - x_2 \|_{L^2(0,T;H)} = \left\{ \int_0^T | x_1(t) - x_2(t) |^2 \, dt \right\}^{1/2}$$

$$= \left\{ \int_0^T \left| \int_0^t (\dot{x}_1(\tau) - \dot{x}_2(\tau)) \, d\tau \right|^2 dt \right\}^{1/2}$$

$$\le \left\{ \int_0^T t \int_0^t | \dot{x}_1(\tau) - \dot{x}_2(\tau) |^2 \, d\tau \, dt \right\}^{1/2}$$

$$\le \frac{T}{\sqrt{2}} \| x_1 - x_2 \|_{W^{1,2}(0,T;H)}. \qquad \text{... (2.6)}$$

Therefore, it terms of (2.1) and (2.6) we have

$$\| x_1 - x_2 \|_{L^2(0,T;V)} \le C_1 \| x_1 - x_2 \|^{1/2}_{L^2(0,T;D(A))} \| x_1 - x_2 \|^{1/2}_{L^2(0,T;H)}$$

$$\le C_1 \| x_1 - x_2 \|^{1/2}_{L^2(0,T;D(A))} \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| x_1 - x_2 \|^{1/2}_{W^{1,2}(0,T;H)}$$

$$\le C_1 \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| x_1 - x_2 \|_{L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)}$$

$$\le C_1 C_T \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| \partial\phi_\varepsilon(y_1) - \partial\phi_\varepsilon(y_2) \|_{L^2(0,T;H)}$$

$$\leq \varepsilon^{-1} C_1 C_T \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| y_1 - y_2 \|_{L^2(0, T; V)}.$$

So by virtue of the condition (2.5) the contraction principle gives that the equation of (SE) has a unique solution in $[0, T]$. Since $\lim_{\varepsilon \to 0} \partial\phi_\varepsilon (x) = (\partial\phi)^0 (x)$ for every $x \in D(\partial\phi)$ we have

$$x' (t) = k(t) - Ax(t) - \partial\phi^0 (x(t))$$

From now on, we establish the norm estimations of solutions. Let $x(\cdot)$ be a solution of (SE) and $y(\cdot)$ be a solution of the following linear functional differential equation :

$$\frac{d}{dt} y(t) + Ay(t) = k(t) \qquad t \in (0, T]$$

and

$$y(0) = x_0.$$

Consider the following problem :

$$\frac{d}{dt} (x(t) - y(t)) + A(x)(t) - y(t)) + \partial\phi_\varepsilon (x(t)) = 0$$

and

$$x(0) - y(0) = 0.$$

In virtue of (2.2) we have

$$\| x - y \|_{L^2(0, T; D(A_0)) \cap W^{1,2}(0, T; H)} \leq C_T \| \partial\phi_\varepsilon (x) \|_{L^2(0, T; H)}$$

$$\leq \varepsilon^{-1} C_T \| x \|_{L^2(0, T; V)} + C_T \| \partial\phi_\varepsilon (0) \|_{L^2(0, T; H)}$$

$$\leq \varepsilon^{-1} C_T (\| x - y) \|_{L^2(0, T; V)} + \| y \|_{L^2(0, T; V)} + C_T \| \partial\phi_\varepsilon (0) \|_{L^2(0, T; H)}.$$

Combining (2.1), (2.6) and the above inequality we have

$$\| x - y \|_{L^2(0, T; V)} \leq C_1 \| x - y \|^{1/2}_{L^2(0, T; D(A))} \| x - y \|^{1/2}_{L^2(0, T; H)}$$

$$\leq C_1 \| x - y \|^{1/2}_{L^2(0, T; D(A))} \left\{ \frac{T}{\sqrt{2}} \| x - y \|_{W^{1,2}(0, T; H)} \right\}^{1/2}$$

$$\leq C_1 \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| x - y \|_{L^2(0, T; D(A)) \cap W^{1,2}(o, T; H)}$$

$$\leq \varepsilon^{-1} C_1 \left(\frac{T}{\sqrt{2}} \right)^{1/2} C_T (\| x - y \|_{L^2(0, T; V)} + \| y \|_{L^2(0, T; V)})$$

$$+ C_1 \left(\frac{T}{\sqrt{2}} \right)^{1/2} C_T \| \partial\phi_\varepsilon (0) \|_{L^2(0, T; H)}.$$

Therefore, we have

$$\| x - y \|_{L^2(0,T;V)} \le \frac{\varepsilon^{-1} C_1 C_T \left(\frac{T}{\sqrt{2}}\right)^{1/2}}{1 - \varepsilon^{-1} C_1 C_T \left(\frac{T}{\sqrt{2}}\right)^{1/2}} \| y \|_{L^2(0,T;V)}$$

$$+ \frac{C_1 C_T \left(\frac{T}{\sqrt{2}}\right)^{1/2}}{1 - \varepsilon^{-1} C_1 C_T \left(\frac{T}{\sqrt{2}}\right)^{1/2}} \| \partial\phi_\varepsilon(0) \|_{L^2(0,T;H)},$$

noting $\| \cdot \| \le \| \cdot \|_{D(A)}$, which yields

$$\| x \|_{L^2(0,T;V)} \le \frac{1}{1 - \varepsilon^{-1} C_1 C_T \left(\frac{T}{\sqrt{2}}\right)^{1/2}} \| \partial\phi_\varepsilon(0) \|_{L^2(0,T;D(A))}$$

$$+ \frac{C_1 C_T \left(\frac{T}{\sqrt{2}}\right)^{1/2}}{1 - \varepsilon^{-1} C_1 C_T \left(\frac{T}{\sqrt{2}}\right)^{1/2}} \| \partial\phi_\varepsilon(0) \|_{L^2(0,T;H)}.$$

Thus from (2.2) it follows

$$\| x \|_{L^2(0,T;V)} \le C_2 (1 + \| x_0 \| + \| k \|_{L^2(0,T;H)} \quad \text{... (2.7)}$$

for some constant $C_2 > 0$. Since the condition (2.5) is independent of initial values, the solution of (SE) can be extended to the interval $[0, nT]$ for every natural number n, that is, we can prove the estimate mentioned above also in the interval $[T, 2T]$ with initial data $x(T)$. If we multiply equation (NE) by $x(t) - x_0$ we have that

$$\frac{1}{2} \frac{d}{dt} | x(t) - x_0 |^2 + A(x(t) - x_0), x(t) - x_0) \le (k(t) - \xi, x(t) - x_0)$$

where $\xi \in Ax_0 + \partial\phi(x_0)$, and noting that $| \cdot | \le \| \cdot \|$,

$$| x(t) - x_0 |^2 \le 2 \int_0^t (\| A(x(s) - x_0) \|_* + | k(t) - \xi |) \, | x(s) - x_0 | \, ds. \quad \text{... (2.8)}$$

Combining (2.7) and (2.8), we obtain the inequality (2.3).

2). Under the assumption (A2), by the same method of Theorem 4.3.2 in Ref.[6] the equation (NE) has a unique solution

$$x \in L^2(0, T; D(A)) \cap C([0, T]; H),$$

which satisfies

$$\| x \|_{L^2 \cap C} \le C_2(1 + \| x_0 \| + \| k \|_{L^2(0, T; H)}).$$

To prove the norm estimate of x in $W^{1,2}(0, T; H)$, acting on both sides of (NE) by $x'(t)$ and integrating over $[0, t]$ we obtain that

$$\frac{1}{2}\int_0^t | x'(s) |^2 ds + \frac{1}{2}(Ax(t), x(t)) + \phi(x(t)) \le \frac{1}{2}(A(x_0), x_0)$$

$$+ \phi(x_0) + \frac{1}{2}\int_0^t | k(s) |^2 ds.$$

Here, we used

$$\frac{d}{dt}\phi(x(t)) = (g(t), \frac{d}{dt}x(t)), \qquad \text{a.e. } 0 < t,$$

for all $g(t) \in \partial\phi(x(t))$. We may assume without loss of generality that

$$\min\{\phi(u) : u \in H\} = 0.$$

Therefore, from (2.7) it follows that

$$\int_0^t | x'(s) |^2 ds \le C(1 + \| x_0 \| + \| k \|_{L^2(0, T; H)})$$

that the assertion (2.4) holds. □

Remark 2 : Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$Re\, a(u, u) \ge c_0 \| u \|^2 - c_1 | u |^2, \qquad c_0 > 0, \quad c_1 \ge 0.$$

Let A be the operator associated with the sesquilinear form $-a(\cdot, \cdot)$:

$$(Au, v) = -a(u, v), \quad u, \quad v \in V.$$

A is a bounded linear operator from V to V^*, and its realization in H which is the restriction of A to

$$D(A) = \{u \in V;\ Au \in H\}$$

is also denoted by A. Then A generates an analytic semigroup in both H and V^*, and hence, we can derive the results of Theorem 2.1 regarding term by term to deduce the estimate the solution x on $L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$. That is, if $(x_0, k) \in V \times L^2(0, T; V^*)$, then for each $T > 0$, we have that

$$\| x \|_{L^2 \cap W^{1,2}} \leq C_2(1 + \| x_0 \| + \| k \|_{L^2(0, T; V^*)}).$$

If V is compactly embedded in H, the following embedding

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset L^2(0, T; H))$$

is compact in view of Remark 1 and Theorem 2 of J. P. Aubin [7]. Hence, the mapping $k \mapsto x$ is compact from $L^2(0, T; H)$ to $L^2(0, T; H)$.

Theorem 2.2: *Let us assume that the hypotheses as (A2) of Theorem 2.1. Then* $x \in L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$, *and the mapping*

$$V \times L^2(0, T; H) \ni (x_0, k) \mapsto x \in L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$$

is continuous.

PROOF : From 2) of Theorem 2.1, we know that x belongs to $L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$. Let $(x_{0i}, k_i) \in V \times L^2(0, T; H)$, and x_i be the solution of (SE) with (x_{0i}, k_i) in place of (x_0, k) for $i = 1, 2$. Then in view of (2.2) we have

$$\begin{aligned} \| x_1 - x_2 \|_{L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)} &\leq C_T \ \{\| x_{01} - x_{02} \| \\ &+ \| \partial\phi_\varepsilon(x_1) - \partial\phi_\varepsilon(x_2) \|_{L^2(0, T; H)} + \| k_1 - k_2 \|_{L^2(0, T; H)}\} \\ &\leq C_1 \ \{\| x_{01} - x_{02} \| + \varepsilon^{-1} \| x_1 - x_2 \|_{L^2(0, T; H)} + \| k_1 - k_2 \|_{L^2(0, T; H)}\}. \quad \text{... (2.9)} \end{aligned}$$

Since

$$x_1(t) - x_2(t) = x_{01} - x_{02} + \int_0^t (\dot{x}_1(s) - \dot{x}_2(s))\, ds,$$

we get

$$\| x_1 - x_2 \|_{L^2(0, T; H)} \leq \sqrt{T} \| x_{01} - x_{02} \| + \frac{T}{\sqrt{2}} \| x_1 - x_2 \|_{W^{1,2}(0, T; H)}.$$

Hence, from (2.1) we get

$$\begin{aligned} \| x_1 - x_2 \|_{L^2(0, T; V)} &\leq C_1 \| x_1 - x_2 \|^{1/2}_{L^2(0, T; D(A))} \| x_1 - x_2 \|^{1/2}_{L^2(0, T; H)} \\ &\leq C_1 \| x_1 - x_2 \|^{1/2}_{L^2(0,T; D(A))} \end{aligned}$$

$$\times \left\{ T^{1/4} \| x_{01} - x_{02} \|^{1/2} + \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| x_1 - x_2 \|^{1/2}_{W^{1,2}(0, T; H)} \right\}$$

$$\leq C_1 T^{1/4} \| x_{01} - x_{02} \|^{1/2} \| x_1 - x_2 \|^{1/2}_{L^2(0, T; D(A))}$$

$$+ C_1 \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| x_1 - x_2 \|_{L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)}$$

$$\leq 2^{-7/4} C_1 \| x_{01} - x_{02} \|$$

$$+ 2C_1 \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| x_1 - x_2 \|_{L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)} \quad \text{... (2.10)}$$

Combining (2.9) and (2.10) we obtain

$$\| x_1 - x_2 \|_{L^2(0, T; D(A)) \cap W^{1,2}(0 T; H)}$$

$$\leq C_T \{ \| x_{01} - x_{02} \| + \varepsilon^{-1} (2^{-7/4} C_1 \| x_{01} - x_{02} \|$$

$$+ 2C_1 \left(\frac{2}{\sqrt{2}} \right)^{1/2} \| x_1 - x_2 \|_{L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)})$$

$$+ \| k_1 - k_2 \|_{L^2(0, T; H)} \}. \quad \text{... (2.11)}$$

Suppose that $(x_{0n}, k_n) \to (x_0, k)$ in $V \times L^2(0, T; H)$, and let x_n and x be the solutions (NE) with (x_{0n}, k_n) and (x_0, k) respectively. Let $0 < T_1 \leq T$ be such that

$$\varepsilon^{-1} C_1 C_T (2T_1)^{1/2} < 1.$$

Then by virtue of (2.11) with T replaced by T_1 we see that $x_n \to x$ in $L^2(0, T_1; D(A)) \cap W^{1,2}(0, T_1; H) \subset C([0, T_1]; V)$. This implies that $x_n(T_1) \mapsto x(T_1)$ in V. Hence, the same argument shows that $x_n \to x$ in

$$L^2(T_1, \min\{2T_1, T\}; D(A)) \cap W^{1,2}(T_1, \min\{2T_1, T\}; H).$$

Repeating this process we conclude that $x_n \to x$ in $L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$. □

3. PERTURBATION OF NONLINEAR TERM

Let f be a nonlinear single valued mapping from V into H. We assume that

(F) $$| f(x_1) - d(x_2) | \leq L \| x_1 - x_2 \|,$$

for every $x_1, x_2 \in V$.

We are interested in the following nonlinear variational inequality problem :

$$\begin{cases} \left(\dfrac{dx(t)}{dt} + Ax(t), x(t) - z\right) + \phi\,(x(t) - \phi\,(z) \\ \qquad \le (f\,(x(t)) + k(t), x(t) - z), \quad a.e., \quad 0 < t \le T, \quad z \in V \\ x(0) = x_0. \end{cases}$$

In terms of the subgradient mapping $\partial\phi$ the above problem can be represented as

$$\text{(NNE)} \qquad \begin{cases} \dfrac{dx(t)}{dt} + Ax(t) + \partial\phi\,(x(t)) \ni f\,(x(t)) + k(t), \qquad 0 < t \le T, \\ x(0) = x_0. \end{cases}$$

Now, we introduce smoothing system corresponding to (NNE) as follows :

$$\text{(SNE)} \qquad \begin{cases} \dfrac{dx(t)}{dt} + Ax(t) + \partial\phi_\varepsilon\,(x(t)) = f\,(x(t)) + k(t), \qquad 0 < t \le T, \\ x(0) = x_0. \end{cases}$$

Since — A generates a semigroup $S(t)$ on H, the mild solution of (SNE) can be represented by

$$x_\varepsilon\,(t) = S(t)x_0 + \int_0^t S(t-s)\ \{f(x_\varepsilon\,(s) + k(s) - \partial\phi_\varepsilon\,(x_\varepsilon\,(s))\}\ ds. \qquad \text{... (3.1)}$$

For the sake of simplicity we assume that $S(t)$ is uniformly bounded; there exists a constant $M \ge 1$ such that

$$\| S(t) \| \le M.$$

We establish the following result on the solvability of (NNE).

Lemma 3.1 : Let $f \in L^2(0, T; H)$ and $x(t) = \int_0^t S(t-s) f(s)ds$. Then there exists a constant C_T in (2.2) such that

$$\| x \|_{L^2(0,\,T;\,D(A_0))} \le C_T \| f \|_{L^2(0,\,T;\,H)}, \qquad \text{... (3.2)}$$

$$\| x \|_{L^2(0,\,T;\,H)} \le MT \| f \|_{L^2(0,\,T;\,H)}, \qquad \text{... (3.3)}$$

and

$$\| x \|_{L^2(0,\,T;\,H)} \le \sqrt{C_T MT}\, \| f \|_{L^2(0,\,T;\,H)}. \qquad \text{... (3.4)}$$

PROOF : From (2.2) (cf. see also [6]) it follows (3.2) immediately and (3.3) is a consequence of the estimate

$$\| x \|^2_{L^2(0,T;H)} = \int_0^T \left| \int_0^t S(t-s) f(s)\, ds \right|^2 dt$$

$$\leq M^2 \int_0^T \left(\int_0^t | f(s) |\, ds \right)^2 dt$$

$$\leq M^2 \int_0^T t \int_0^t | f(s) |^2\, ds\, dt$$

$$\leq M^2 \frac{T^2}{2} \int_0^T | f(s) |^2\, ds.$$

From (2.1), (3.2) and (3.3) it follows that

$$\| x \|_{L^2(0,T;V)} \leq \sqrt{C_T M T}\, \| f \|_{L^2(0,T;H)}.$$ □

***Theorem* 3.1** : *Let $x_0 \in V$, $k \in L^2(0, T; H)$ and the assumption (F) be satisfied. Then the equation (NNE) has a unique solution*

$$x \in L^2(0, T; V) \cap C([0, T]; H).$$

which satisfies

$$x'(t) = f(x(t)) + k(t) - Ax(t) - \partial\phi^0 (x(t))$$

and there exists a constant C_3 depending on T such that

$$\| x \|_{L^2 \cap C} \leq C_3 (1 + \| x_0 \| + \| k \|_{L^2(0,T;V^*)}). \quad \text{... (3.5)}$$

PROOF : Let $y \in L^2(0, T; V)$. Then $f(y(\cdot)) \in L^2(0, T; H)$ by assumption (F). Thus, in virtue of Theorem 2.1 we know that the problem

$$\begin{cases} \dfrac{dx(t)}{dt} + Ax(t) + \partial\phi (x(t)) \ni f(y(t)) + k(t), & 0 < t \leq T, \\ x(0) = x_0. \end{cases} \quad \text{... (3.6)}$$

has a unique solution $x_y \in L^2(0, T; V) \cap C([0,T]; H)$ corresponding to y in (3.6).

Let us fix the time T such that

$$\sqrt{C_T MT} < (1 + \varepsilon^{-1})^{-1}. \qquad \text{... (3.7)}$$

Let

$$J(y)(t) = S(t)x_0 + \int_0^t S(t-s)\ \{f(y(s)) + k(s) - \partial\phi_\varepsilon(x(s))\}ds.$$

We are going to show that the mapping $y \to J(y) = x$ is strictly contractive from $L^2(0, T; V)$ to itself is the condition (3.7) is satisfied. Let x_1, x_2 be the solutions of (3.6) with y replaced by $y_1, y_2 \in L^2(0, T; V)$ respectively. Since

$$J(y_1)(t) - J(y_2)(t) = \int_0^t S(t-s)\ \{(f(y_1(s)) - f(y_2(s)))$$

$$+ (\partial\phi_\varepsilon(x_1(s)) - \partial\phi_\varepsilon(x_2(s)))\}\ ds$$

we have

$$\| J(y_1) - J(y_2) \|_{L^2(0,T;V)} \le \sqrt{C_T MT}\ \{f(y_1(\cdot)) - f(y_2(\cdot)) \|_{L^2(0,T;H)}$$

$$+ \varepsilon^{-1} \sqrt{C_T MT} \| x_1(\cdot) - x_2(\cdot) \|_{L^2(0,T;H)}$$

$$\le \sqrt{C_T MT} L\ \{\| (y_1(\cdot)) - (y_2(\cdot)) \|_{L^2(0,T;V)}$$

$$+ \varepsilon^{-1} \sqrt{C_T MT} \| x_1(\cdot) - x_2(\cdot)) \|_{L^2(0,T;V)}.$$

Here we used that $\partial\phi_\varepsilon$ is Lipschitz continuous on H with Lipschitz constant ε^{-1}. By using the contraction mapping the solution of (SNE) exists uniquely, and since the constant is independent of initial values, the solution can be extended to the interval $[0, nT]$ for every natural number n. Noting that $\lim_{\varepsilon \to 0}(\partial\phi_\varepsilon(x)) = \phi^0(x)$ for every $x \in H$, the proof of existence of solutions of equation (NNE) is complete.

Next, we proof the norm estimates of solutions of (NNE). Let $x(\cdot)$ be a solution of (SNE) and $y(\cdot)$ be a solution of the following equation :

$$\frac{d}{dt} y(t) + Ay(t) = k(t) \qquad t \in (0, T]$$

and

$$y(0) = x_0.$$

Then, we consider the following problem :

$$\frac{d}{dt}(x(t)-y(t))+A(x(t)-y(t))+\partial\phi_\varepsilon(x(t))=f(x(t))$$

and

$$x(0)-y(0)=0.$$

In virtue of (2.2) we have

$$\begin{aligned}\|x-y\|_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} &\le C_T\|\partial\phi_\varepsilon(x)-f(x)\|_{L^2(0,T;H)}\\ &\le C_T(\varepsilon^{-1}+L)\|x\|_{L^2(0,T;V)}\\ &\quad + C_T\{\|\partial\phi_\varepsilon(0)\|_{L^2(0,T;H)}+\|f(0)\|_{L^2(0,T;H)}\}\\ &\le C_T(\varepsilon^{-1}+L)(\|x-y\|_{L^2(0,T;V)}+\|y\|_{L^2(0,T;V)})\\ &\quad + C_T\{\|\partial\phi_\varepsilon(0)\|_{L^2(0,T;H)}+\|f(0)\|_{L^2(0,T;H)}\}.\end{aligned}$$

Combining (2.1), (2.6) and the above inequality we have

$$\begin{aligned}\|x-y\|_{L^2(0,T;V)} &\le C_1\left(\frac{T}{\sqrt{2}}\right)^{1/2}\|x-y\|_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}\\ &\le C_1\left(\frac{T}{\sqrt{2}}\right)^{1/2} C_T(\varepsilon^{-1}+L)(\|x-y\|_{L^2(0,T;V)}+\|y\|_{L^2(0,T;V)})\\ &\quad + C_1\left(\frac{T}{\sqrt{2}}\right)^{1,2} C_T\{\|\partial\phi_\varepsilon(0)\|_{L^2(0,T;H)}+\|f(0)\|_{L^2(0,T;H)}\}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|x-y\|_{L^2(0,T;V)} &\le \frac{C_1C_T\left(\frac{T}{\sqrt{2}}\right)^{1-2}(\varepsilon^{-1}+L)}{1-C_1C_T\left(\frac{2}{\sqrt{2}}\right)^{1/2}(\varepsilon^{-1}+L)}\|y\|_{L^2(0,T;V)}\\ &\quad + \frac{C_1C_T\left(\frac{T}{\sqrt{2}}\right)^{1-2}}{1-C_1C_T\left(\frac{2}{\sqrt{2}}\right)^{1/2}(\varepsilon^{-1}+L)}(\|\partial\phi_\varepsilon(0)\|_{L^2(0,T;H)}\\ &\quad + \|f(0)\|_{L^2(0,T;H)}),\end{aligned}$$

and, hence

$$\|x\|_{L^2(0,T;V)} \leq \frac{1}{1 - C_1 C_T \left(\frac{T}{\sqrt{2}}\right)^{1/2} (\varepsilon^{-1} + L)} \|y\|_{L^2(0,T;D(A))}$$

$$+ \frac{C_1 C_T \left(\frac{T}{\sqrt{2}}\right)^{1-2}}{1 - C_1 C_T \left(\frac{T}{\sqrt{2}}\right)^{1/2} (\varepsilon^{-1} + L)} (\|\partial\phi_\varepsilon(0)\|_{L^2(0,T;H)}$$

$$+ \|f(0)\|_{L^2(0,T;H)}).$$

If $T > 0$ such that

$$C_1 C_T \left(\frac{T}{\sqrt{2}}\right)^{1/2} (\varepsilon^{-1} + L) < 1 \qquad \ldots (3.8)$$

then from (2.2) it follows

$$\|y\|_{L^2(0,T;D(A))} \leq C_T (\|x_0\| + \|k\|_{L^2(0,T;H)}),$$

and hence

$$\|x\|_{L^2(0,T;V)} \leq C(1 + \|x_0\| + \|k\|_{L^2(0,T;H)}) \ldots \quad (3.9)$$

for some constant $C > 0$. Since the condition (3.8) is independent of initial value, the solution of (SNE) can be extended to the interval $[0, nT]$ for every natural number n. If we multiply equation (NNE) by $x(t) - x_0$ we have that

$$\frac{1}{2}\frac{d}{dt}|x(t) - x_0|^2 + (A(x(t)) - x_0), x(t) - x_0) \leq (k(t) - \xi, x(t) - x_0) + (f(x(t)), x(t) - x_0)$$

where $\xi \in Ax_0\, \partial\phi(x_0)$, and hence,

$$|x(t) - x_0|^2 \leq 2 \int_0^t (\|A(x(s) - x_0)\|_* + |k(s) - \xi| + |f(x(s))|\, \|x(s) - x_0\|\, ds.$$

Therefore, after some calculations involving Hölder inequality, we obtain the inequality (3.5) by combining (3.9) and the above inequality. □

Theorem 3.2 : *Assume (A2) and the hypotheses as in Theorem 3.1. Then the equation (NNE) has a unique solution*

$$x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \cap C(0,T];H),$$

which satisfies

$$\|x\|_{L^2 \cap W^{1,2} \cap C} \leq C_2(1 + \|x_0\| + \|k\|_{L^2(0,T;H)})$$

PROOF : As is seen in Theorem 4.3.2 in Ref.[5], we have

$$(Ax(t), x(t)) + \int_0^t |Ax(t)|^2 \leq C\{1 + \|x_0\|^2 + \int_0^t (|f(x(s))|^2 + |k(s)|^2)\, ds\}.$$

Since

$$\|f(x)\|_{L^2(0,T;H)} \leq L\|x\|_{L(0,T;V)} + \|f(0)\|_{L^2(0,T;H)},$$

it follows from (3.5) that the equation (NNE) has a unique solution

$$x \in L^2(0, T; D(A)) \cap C([0, T]; H),$$

which satisfies

$$\|x\|_{L^2 \cap C} \leq C_2(1 + \|x_0\| + \|k\|_{L^2(0,T;H)}).$$

Since

$$\int_0^t |(f(x(s)), x'(s)) + (k(s), x'(s))|\, ds$$

$$\leq \|f(x))|^2_{L^2(0,T;H)} + \|k\|^2_{L^2(0,T;H)} + \frac{1}{2}\|x'\|^2_{L^2(0,T;H)}$$

$$\leq L\|x\|^2_{L^2(0,T;V)} + \|k\|^2_{L^2(0,T;H)} + \frac{1}{2}\|x'\|^2_{L^2(0,T;H)},$$

acting on both sides of (NNE) by $x'(t)$ and integrating over $[0, t]$ we obtain that

$$\frac{1}{2}\int_0^t |x'(s)|^2\, ds + \frac{1}{2}(Ax(t), x(t)) + \phi(x(t))$$

$$\leq \frac{1}{2}(A(x_0), x_0) + \phi(x_0) + L\int_0^t |x(s)\|^2\, ds + \int_0^t |K(s)|^2\, ds.$$

Therefore, from (2.7) and (3.5) it follows that

$$\int_0^t |x'(s)|^2\, ds \leq C(1 + \|x_0\| + \|k\|_{L^2(0,T;H)}),$$

thus the assertion holds. □

Theorem 3.3 : *Let $x_0 \in V$, $k \in L^2(0, T; H)$ and the hypothesis (A2) be satisfied. Then the solution x of the equation (NNE) belongs to $L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$, and the mapping*

$$V \times L^2(0, T; H) \ni (x_0, k) \mapsto x \in L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$$

is continuous.

PROOF : From Theorem 3.2 we know that x belongs to $L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$. Let $(x_{0i}, k_i) \in F \times L^2(0, T; H)$, and x_i be the solution of (SNE) with (x_{0i}, k_i) in place of (x_0, k) for $i = 1, 2$. Then in view of (2.2), we have

$$\begin{aligned}\| x_1 - x_2 \|_{L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)} &\le C_T \{\| x_{01} - x_{02} \| \\ &+ (\| \partial\phi_\varepsilon(x_1) - \partial\phi_\varepsilon(x_2) \| + \| f(x_1) - f(x_2) \| \\ &+ \| k_1 - k_2 \|)_{L^2(0,T;H)}\} \\ &\le C_T \{\| x_{01} - x_{02} \| + (\varepsilon^{-1} + L) \| x_1 - x_2 \|_{L^2(0,T;V)} \\ &+ \| k_1 - k_2 \|_{L^2(0,T;H)}\}. \qquad \text{... (3.10)}\end{aligned}$$

Combining (3.10) and (2.10) we obtain

$$\begin{aligned}&\| x_1 - x_2 \|_{L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)} \\ &\le C_T \{\| x_{01} - x_{02} \| + (\varepsilon^{-1} + L)(2^{-7/4} C_1 \| x_{01} - x_{02} \| \\ &+ 2C_1 \left(\frac{T}{\sqrt{2}}\right)^{1/2} \| x_1 - x_2 \|_{L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)}) \\ &+ \| k_1 - k_2 \|_{L^2(0,T;H)}\}. \qquad \text{... (3.11)}\end{aligned}$$

Suppose that $(x_{0n}, k_n) \to (x_0, k)$ in $V \times L^2(0, T; H)$, and let x_n and x be the solutions (NNE) with (x_{0n}, k_n) and (x_0, k) respectively. Let $0 < T_1 \le T$ be such that

$$(\varepsilon^{-1} + L)\, C_1 C_T (2T_1)^{1/2} < 1.$$

Then by virtue of (3.11) with T replaced by T_1 we see that $x_n \to x$ in $L^2(0, T_1; D(A)) \cap W^{1,2}(0, T_1, H) \subset C([0, T_1]; V)$. This implies that $x_n(T_1) \mapsto x(T_1)$ in V. Hence, the same argument shows that $x_n \to x$ in

$$L^2(T_1, \min\{2T_1, T\}; D(A)) \cap W^{1,2}(T_1, \min\{2T_1, T\}; H).$$

Repeating this process we conclude that $x_n \to x$ in $L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$. □

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