Indian J. pure appl. Math., **31**(6) : 621-624, June 2000

THE HEIGHT OF THE FIRST STIEFEL-WHITNEY CLASS OF THE REAL FLAG MANIFOLDS1

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Key Words : Height; Stiefel-Whitney Class; Flag Manifold; Fibre bundle; Immersion; Lusternik-Schrinelmann Category

1. INTRODUCTION

In this note, we obtain a result on the height of the first Stiefel-Whitney class, $\omega_1(F(n_1, ..., n_r))$, in the cohomology of the real flag manifold

$$F(n_1, ..., n_r) = \frac{O(n)}{O(n_1) \times \ldots \times O(n_r)}, n = \sum_{i=1}^{r} n_i$$

where height $(\omega_1(M)) = \sup\{p : \omega_1^p(M) \neq 0 \text{ in } H^*(M ; Z_2)\}$.

The result is of interest for several reasons. The height gives a lower bound for cat $(F(n_1, ..., n_r))$, the Lusternik-Schnirelmann (L.-S.) category of $F(n_1, ..., n_r)$. This is due to the fact that

$$\text{cuplength}_{Z_2}(X) \leq \text{cat}(X) \leq \dim(X)$$

for any CW-complex X (cf. [2], [3]), where $\text{cuplength}_\Lambda(X)$ is the largest integer q such that there exist q cohomology classes $u_i \in H^i(X : \Lambda)$ $(i > 0)$ for which $u_1 \cup \ldots \cup u_q \neq 0$. Cat$(X)$ is the least integer m such that X can be covered by $m + 1$ open subsets, each of which is contractible to a point in X. Cat(X) is used to determine the existence of critical points of functions on X, if X is a smooth closed manifold.

Another application is a necessary condition obtained for the immersion of $F(n_1, ..., n_r)$ in a Riemannian manifold with trivial normal bundle.

Stong in [7] found the height of the first Stiefel-Whitney class of the camonical bundle over Grassmannians.

We shall prove the following :

Theorem : Suppose that $\Pi_{i=1}^{r-1}$ is odd, $n-k$ is even $(k = \Sigma_{i=1}^{r-1} n_i)$, and $4 \leq 2k \leq n$, with $2^s < n \leq 2^{s+1}$. Then

$$\text{height } (\omega_1(F(n_1, ..., n_{r-1}, n-k))) = \begin{cases} 2^{s+1} - 2, & \text{if } k = 2, \text{ or if } k = 3 \text{ and } n = 2^s + 1 \\ 2^{s+1} - 1, & \text{otherwise.} \end{cases}$$

Corollary : With the notation of Theorem, the L.-S. category of $F(n_1, ..., n_{r-1}, n-k)$ has the following lower bound :

$$\text{cat } (F(n_1, ..., n_{r-1}\ n-k)) \geq \begin{cases} 2^{s+1} - 2, & \text{if } k = 2, \text{ or if } k = 3 \text{ and } n = 2^s + 1 \\ 2^{s+1} - 1, & \text{otherwise.} \end{cases}$$

Proposition — Suppose that $\Pi_{i=1}^{r-1}$ is odd, $n-k$ (where $k = \Sigma_{i=1}^{r-1} n_i$) is even, and $4 \leq 2k \leq n$ with $2^s < n \leq 2^{s+1}$. Let

$$t := \begin{cases} 2^{s+1} - 2, & \text{if } k = 2, \text{ or if } k = 3 \text{ and } n = 2^s + 1 \\ 2^{s+1} - 1, & \text{otherwise.} \end{cases}$$

Then if $F(n_1, ..., n_{r-1}, n-k)$ immerses in a Riemannian manifold N with trivial normal bundle, $\omega_1^t(N) \in H^t(N; Z_2)$ must be non-zero.

2. Proof of Theorem

Note that from [4; 2.2], the conditions that $\prod_{i=1}^{r-1} n_i$ is odd and $n - k$ is even, imply that $\omega_1(F(n_1, ..., n_{r-1}, n-k))$ is non-zero. The natural map

$$\pi : F(n_1, ..., n_{r-1}, n-k) \to F(k, n-k)$$

defines a fibre bundle, with fibre $F(n_1, ..., n_{r-1})$, (cf. [6; 7.4]). If γ is the cononical k-plane bundle over the Grassmann manifold $F(k, n-k)$, then the induced bundle $\pi^*(\gamma)$ is isomorphic to the Whitney sum $\gamma_1 \oplus ... \oplus \gamma_{r-1}$ of the canonical vector bundles over the flag manifold $F(n_1, ..., n_{r-1}, n-k)$ with $\dim(\gamma_i) = n_i$. Thus from the basic properties of the Stiefel-Whitney classes, we have

$$\pi^*(\omega_1(\gamma)) = \omega_1(\gamma_1) + ... + \omega_1(\gamma_{r-1}).$$

From Korbaš formula in [4],

$$\pi^* (\omega_1(\gamma)) = \omega_1(F(n_1, ..., n_{r-1}, n-k)).$$

Using [1], the induced map in cohomology

$$\pi^* : H^* (F(k, n-k); Z_2) \to H^* (F(n_1, ..., n_{r-1}, n-k); Z_2)$$

is injective. It is then easy to see that

$$\pi^* (\omega_1^{\text{height}(\omega_1(\gamma))} = \omega_1^{\text{height}(\omega_1(\gamma))} (F(n_1, ..., n_{r-1}, n-k))$$

is non-zero, but

$$\pi^* (\omega_1^{1+\text{height}(\omega_1(\gamma))} (\gamma)) = \omega_1^{1+\text{height}(\omega_1(\gamma))} (F(n_1, ..., n_{r-1}, n-k))$$

is zero. The proof now follows, since by Stong[7] we have, for $4 \leq 2k \leq n$

$$\text{height } (\omega_1(\gamma)) = \begin{cases} 2^{s+1} - 2, & \text{if } k = 2, \text{ or if } k = 3 \text{ and } n = 2^s + 1 \\ 2^{s+1} - 1, & \text{otherwise.} \end{cases}$$

Proof of Corollary : The proof of Corollary follows from the fact that

$$\text{height } (\omega_1(X)) \leq \text{cuplength}_{Z_2} (X) \leq \text{cat}(X)$$

for any CW-complex X, (cf. [2], [3]).

3. Proof of Proposition

Suppose $f: F(n_1, ..., n_{r-1}, n-k) \to N$ is an immersion with trivial normal bundle. Then by [5; 3.5], the tangent bundles are related in the following way :

$$f^* (TN) \cong TF(n_1, ..., n_{r-1}, n-k) \oplus \varepsilon,$$

where ε is the trivial bundle of dimension $\dim(N)$-$\dim(F(n_1, ..., n_{r-1}, n-k))$ over $F(n_1, ..., n_{r-1}, n-k)$. Hence

$$f^* (\omega_1(N)) = \omega_1(F(n_1, ..., n_{r-1}, n-k)).$$

By Theorem proved in section 2 above,

$$\omega_1^t (F(n_1, ..., n_{r-1}, n-k)) \neq 0.$$

Therefore also, $\omega_1^t (N) \neq 0$.

ACKNOWLEDGEMENT

The authors wish to express their appreciation to the referee for his very useful suggestions, which led to an improvement of the original paper.

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