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MINIMAL DISPLACEMENT FOR CERTAIN CLASSES OF MAPPINGS

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In a recent paper, W. A. Kirk obtained upper bound estimates for the infimum of $d(x, Tx)$ for T a selfmap of a bounded metric space satisfying certain contractive definitions. In this paper, we generalize the results of Kirk by replacing his contractive definitions with more general ones.

Key Words : Hölder Type Mappings; Minimal Displacement; Metric Fixed Point Theory

In a recent paper, Kirk[1] obtained estimates for the infimum of $d(x, Tx)$, and some related results, for a map T satisfying a contractive definition of the form

$$d(Tx, Ty) \leq l \ \max \ \{d(x, y), h\} \quad h > 0, \ 0 \leq l \leq 1.$$

In this paper, we establish his results for a much larger class of maps satisfying

$$d(Tx, Ty) \leq l \ \max \ \{m(x, y), h\} \quad h > 0, \ 0 \leq l \leq 1, \qquad \ldots (1)$$

where $m(x, y) := \max d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)\}/2\}$.

Definition 1 — Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be a w-generalized contraction, $w > 0$, if there exists a $k \in (0, 1)$ such that, for each $x, y \in M$, $d(Tx, Ty) \leq km(x, y) + w$.

Note that if $k \in (0, 1)$ and $l \in (k, 1)$ then, for $x, y \in M$,

$$km(x, y) + w \leq l\, m(x, y) \iff m(x, y) \geq \frac{w}{l-k}.$$

Definition 2 — A map $T : M \to M$ is said to be h_l-generalized contractive if T satisfies (1) for $h > 0$ and $l \in (0, 1)$.

Proposition 1 — *If* $T : M \to M$ *is a w-generalized contraction with contractive constant* $k \in (0, 1)$, *then* T *is* h_l*-generalized contractive for* $h = w/(l-k)$ *and* $l \in (k, 1)$.

PROOF : If $m(x, y) \geq h$, then $d(Tx, T^2x) \leq km(x, y) + w \leq lm(x, y)$. If $m(x, y) < h$, then $d(Tx, Ty) \leq kh + w \leq lh$.

Proposition 2 — *f $T : M \to M$ is a w-generalized contraction with contractive constant k, then*

$$\inf\{d(x, Tx) : x \in M\} \leq \frac{w}{1-k}. \quad \ldots (3)$$

PROOF : From Proposition 1, T is h_l-generalized contractive.

Note that $m(x, Tx) = \max\{d(x, Tx), d(Tx, T^2x), [d(x, T^2x) + 0]/2\}$. Suppose that $m(x, T) \geq h$. From (1), if max $m(x, Tx) = d(Tx, T^2x)$, then we have $d(Tx, T^2x) \leq ld(Tx, T^2x)$, a contradiction. If max $m(x, y) = d(x, T^2x)/2$, then we have $d(Tx, T^2x) \leq (l/2)\,[d)x, Tx) + d(Tx, T^2x)]$, which implies that $d(Tx, T^2x) \leq l(?2 - l)\, d(x, Tx) \leq ld(x, Tx)$.

If $m(x, Tx) < h$, then we have $d(Tx, T^2x) \leq lh$. Thus, in all cases we have

$$d(Tx, T^2x) \leq l \max\{d(x, Tx), h\}. \quad \ldots (2)$$

Let $l \in (k, 1)$, and define $h(l) = w/(l-k)$. Suppose there exist an $x \in M$ such that $d(T^n x, T^{n+1} x) \geq h(l)$, $n = 1, 2, \ldots$. Then from (2),

$$h(l) \leq d(T^n x, T^{n+1} x) \leq ld(T^{n-1} x, T^n x) \leq \cdots \leq l^n\, d(x, Tx) \to 0 \quad \text{as} \quad n \to \infty,$$

a contradiction. Therefore, there exists an n for which $d(T^n x, T^{n+1} x) \leq h(l)$. Since l is arbitrary, (3) follows.

Definition 3 : An h_1-generalized contractive mapping will be called h-generalized nonexpansive.

***Theorem* 3** — *Let K be a nonempty bounded closed convex subset of a Banach space X, T an h-generalized nonexpansive selfmap of K. Then* $\inf\{\| x - Tx \| : x \in K\} \leq h$.

PROOF : Define, for $t \in (0, 1)$

$$T_t(x) = (1-t)z + tT(x), x\, z \in K.$$

Then

$$\| T_t(x) - T_t(y) \| = t \| Tx - Ty \| \leq t \max\{m(x, y), h\},$$

and T_t is an h_t-generalized contraction. From Proposition 2, there exists a $x_t \in K$ such that $\| x_t - T_t(x_t) \| \leq th$. But

$$\| x_t - T(x_t) \| \leq \| x_t - T_t(x_t) \| + \| T_t(x_t) - T(x_t) \| = \| x_t - T_t(x_t) \| + (1-t) \| z - T(x_t) \|.$$

Since K is bounded, letting $t \to 1$ — gives $\inf\{\| - Tx \| : x \in K\} \leq h$.

If T satisfies a Hölder type condition, then we obtain strict inequality for a small class of mappings.

Theorem 4 — *let K be a nonempty bounded closed convex subset of a Banach space X, T a selfmap of K satisfying*

$$\| Tx - Ty \| \leq h(\max\{\| x - y \|, \| x - Tx \|, \| y - Ty \|\})^p, x, y, \in K \qquad \text{... (4)}$$

for some $h, p \in (0, 1)$. Then

$$\inf\{\| x - Tx \| : x \in K\} < h.$$

PROOF : Since $h < 1$, by Theorem 3 there exists an $x \in K$ such that $\| x - Tx \| < 1$. From (4),

$$\| Tx - T^2x \| \leq h\,(\max\{\| x - Tx \|, \| Tx - T^2x \|\})^p.$$

If the maximum is $\| Tx - T^2x \|$, then we have $\| Tx - T^2x \| \leq h \| Tx - T^2x \|^p$, which implies that $\| Tx - T^2x \| \leq h^{1/(1-p)} < h$, since $h, p \in (0, 1)$.

If the maximum is $\| x - Tx \|$, then, from (4), $\| Tx - T^2x \| \leq h \| x - Tx \|^p < h$.

Let $B(x; r)$ denote the closed ball centered at x with radius $r \geq 0$. A subset D of a metric space M is called admissible if $D = \text{cov}(D)$, where

$$\text{cov}(D) := \bigcap \{B : B \text{ is a ball centered in } M \text{ and } B \supseteq D\}.$$

The collection of all admissible subsets of M will be denoted by $G(M)$. For $D \in G(M)$ let

$$\text{diam}(D) = \sup \{d(x, y : x, y \in D\}$$

and

$$r(D) := \inf \{r > 0 \text{ there exists an } x \in D \text{ with } D \supseteq B(r; r)\}.$$

The family $G(M)$ is said to be compact if any family of nonempty subsets of $G(M)$ which has the finite intersection property has nonempty intersection. $G(M)$ is said to be normal if $f(D) < \text{diam}(D)$ for each $D \in G(M)$ for which $\text{diam}(D) > 0$.

Theorem 5 — *Let (M, d) be a metric space for which G(M) is compact and normal. Suppose that T is an h-generalized nonempansive selfmap of M. Then there exists a* $z \in M$ *such that* $d(z, Tz) < h$.

PROOF : By Zorn's Lemma there exists a $D \in G(M)$ which is minimal with respect to being nonempty and invariant under T. Then $D = \text{cov}(T(D))$. Let $r = r(D)$ and let C denote the Chebyshev centre of D. If $\text{diam}(D) = 0$, then D consists of a single point which is fixed under T and there is nothing to prove. Assume that $\text{diam}(D) > 0$ and let $z \in C$. If $r < h$, then $m(z, Tz) \leq r < h$.

Assume that $h \leq r$.

Let $x \in D$. If $m(z, x) \geq h$, then $d(Tx, Tz) \leq m(z, x) \leq r$. On the other hand, if $m(z, x) \leq h$, then $d(Tz, Tx) \leq h \leq r$. Thus, in either case, $T(x) \in B(T(z); r)$.

Hence, $T(D) \subseteq B(T(z); r)$, which implies that $D = \text{cov}(T(D)) \subseteq B(T(z); r)$, which in turn implies that $T(z) \in C$, and therefore, T is a selfmap of C. But, since $C \in G(M)$, this contradicts the minimality of D.

The set $G(M)$ is said to be *uniformly normal* if there exists $c \in (0, 1)$ such that $r(D) \leq c$ $\text{diam}(D)$ for each $D \in G(M)$ with $\text{diam}(D) > 0$.

Theorem 6 — *Let (M, d) be a metric space for which G(M) is compact and uniformly normal with normality constant* $c \in (0, 1)$, *and suppose that T is an h-generalized nonexpansive selfmap of M. Then there exists a* $z \in M$ *such that* $d(x, TZ) \leq ch$.

PROOF : Again assume that $D \in G(M)$ is minimal with respect to being nonempty and invariant under T. Then, as before, $D = \text{cov}(T(D))$. Let r and C be as in Theorem 4. If $\text{diam}(D) = 0$, then D consists of a single point and there is nothing to prove. Assume $\text{diam}(D) > 0$, and let $z \in C$. If $\text{diam}(D) \leq h$, then $m(z, x) \leq r = c\ \text{diam}(D) \leq ch$. Suppose that $h < \text{diam}(D)$. Define $h' = \max\{h, r\}$, and define $C' = \{z \in D : D \subseteq B(z; h')\}$.

Then $C' \in G(M)$. Since $C \subseteq C', C' \neq \phi$. Let $z \in C'$ and let $x \in D$. If $m(z, x) \geq h'$, then $m(z, x) \geq h, d(Tz, Tx) \leq m(z, x) \leq h'$ and $T(x) \in B(T(z); h')$. If $m(z, x) \leq h'$, then $d(Tz, Tx) \leq h \leq h'$, and again $T(x) \in B(T(z); h')$. Thus $T(D) \subseteq B(T(z); h')$, so that $D = \text{cov}(T(D)) \subseteq B(T(z); h')$, which implies that $T(z) \in C'$. Therefore, T is a selfmap of C'. But, since $C' \in G(M)$ is nonempty and since C' is a proper subset of D, we have a contradiction to be minimality of D.

Definition 4 — A metric space (M, d) is said to be hyperconvex if

$$\bigcap_{\alpha} B(x_\alpha; r_\alpha) \neq \phi$$

for any collection $\{B(x_\alpha; r_\alpha)\}$ of closed balls in M for which $d(x_\alpha; r_\beta) \leq x_\alpha + r_\beta$.

Corollary 7 : Let (M, d) be a hyperconvex space and suppose that T is an h-generalized nonexpansive selfmap of M for $h > 0$. Then there exists a $z \in M$ such that $d(z, Tz) \leq h/2$.

PROOF : For a hyperconvex space M the admissible sets are compact and uniformly normal for $c = 1/2$.

Theorem 8 — *Suppose that K is a weakly compact convex subset of a Banach space X. Suppose that K has normal structure and that T is an h-generalized nonexpansive selfmap of K. Then there exists a point $z \in K$ such that $|z - Tz| < h$.*

Since K is weakly compact and has normal structure, the family of all admissible subsets of K is compact, and the result follows from Theorem 6.

Corollary 9 — If K is a compact convex subset of a Banach space X and T is an h-generalized nonexpansive selfmap of K, then there exists a point $z \in K$ such that $|z - Tz| < h$.

It is well known that compact convex sets have normal structure.

Definition 5 — A nonempty bounded convex set K in a Banach space X is said to have uniform normal structure if there exists a constant $c \in (0, 1)$ such that

$$\inf_{z \in K} \quad \sup\{\| z - y : y \in K\} \le c \quad \sup\{\| x - y \| : x, y \in K\}.$$

If each such set in X has this property for a given $c \in (0, 1)$, then X is said to have uniform normal structure. The smallest number c for which the above inequality holds for all bounded convex $K \subseteq X$ is called the uniform normal structure coefficient of X.

Theorem 10 — *Let X be a Banach space which has uniform normal structure with coefficient c, K a bounded closed convex subset of X. Let T be an h-generalized nonexpansive selfmap of K. Then there exists a $z \in K$ such that $\| z - Tz \| \le ch$.*

If K is a bounded closed convex subset of a space X with uniform normal structure, then the family of all admissible subsets of K is compact, since K is weakly compact, and uniformly normal in the sense of Theorem 6.

Theorem 11 : *Let X be a Banach space which has uniform normal structure with coefficient c, K a bounded closed convex subset of X and T a selfmap of K satisfying a generalized Hölder condition, with $h, p \in (0, 1)$. Then, if p is sufficiently near to 1, then*

$$\inf\{\| x - Tx \| : x \in K\} \; < \; ch.$$

PROOF : By Theorem 9, there exists a $z \in K$ such that $\| z - Tz \| \le ch$.

Thus

$$\| Tz - T^2 z \| \le h \quad (\max\{\| z - Tz \|, \| Tz - T^2 z \|\})^p.$$

If the maximum is $\| z - Tz \|$, then we have

$$\| Tz - T^2 z \| \le h \| z - Tz \|^p \le c^p h^{p+1}.$$

If the maximum is $\| Tz - T^2 z \|$, then we have

$$\| Tz - T^2 z \| \le h \| Tz - T^2 z \|^p,$$

which implies that $\| Tz - T^2 z \| \le h^{1/(1-p)}$. The result now follows by observing that, since $c < 1$, $c^p h^{p+1} < ch$ for all p sufficiently near 1, and that $\lim_{p \to 1-} h^{1/(1-p)} = 0$.

Theorem *12 — Let K be a closed convex admissible subset of l_∞, and let T be an h-generalized nonexpansive selfmap of K. Then there exists a $z \in K$ such that $\| z - Tz \| \leq h/2$.*

Since it is known that l_∞ is hyperconvex, the result follows from Corollary 7.

Corollary 13 — Let $K = [a, b] \subset \mathbb{R}$ and suppose that T is an h-generalized nonexpansive selfmap of K. Then there exists a $z \in K$ such that $| z - Tz | \leq 1/2$.

REFERENCE

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