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SELECTION PROBLEM WITH APPLICATIONS

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A generalized version of Yannelis-Prabhakar's continuous selection theorem is proved. As applications, some equilibrium existence theorems for the version of Shafer-Sonnenschein's abstract economy, some existence theorems of solutions for generalized quasi-variational inequalities, some existence theorems of maximal elements, a fixed point theorem in non-compact infinite product spaces and a non-empty intersection theorem are given.

Key Words : Continuous Selection; Generalized Quasi-Variational Inequality; Maximal Element; Abstract Economics; Fixed Point

1. INTRODUCTION AND PRELIMINARIES

Let X and Y be two topological spaces and $T: X \to 2^Y$ be a multi-valued mapping. The mapping T is said to have a continuous selection if there exists a single-valued mapping $f: X \to Y$ such that $f(x) \in T(x)$ for all $x \in X$.

It is an important problem in nonlinear analysis theory, especially, equilibrium existence problems, variational inequalities, complementarity problems, fixed point theory, etc, that under what conditions a multi-valued mapping has a continuous selection?

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Concerning this problem, in 1983, Yannelis and Prabhakar[20] proved the following result:

***Theorem* A[20]** — *Let X be a nonempty paracompact Hausdorff topological space and Y be a Hausdorff topological vector space. Let $T: X \to 2^Y$ be a multi-valued mapping satisfying the following conditions:*

1. *for each $x \in X$, $T(x)$ is nonempty convex; and*
2. *for each $y \in Y$, $T^{-1}(y)$ is open in X.*

Then T has a continuous selection.

The purpose of this paper is to prove a new version of Yannelis-Prabhakar's continuous selection theorem,[20] which contains Theorem A as its special case. As applications, some existence theorems of equilibrium for the version of Shafer-Sonnenschein's abstract economies, some existence theorems of solutions for generalized quasi-variational inequalities, a fixed point theorem, a nonempty intersection theorem and some maximal element existence theorem are given. The results presented in this paper generalize and improve the corresponding results in [3-8, 12, 13, 15-17, 20, 21].

For the sake of convenience, we first recall some definitions and lemmas.

Definition 1.1[18] — Let X be a nonempty set and Y be a topological space. A mapping $T: X \to 2^Y$ is said to be transfer open-valued if for any given $x \in X$ and $y \in T(x)$, there exists a point $x' \in X$ such that $y \in \text{int } T(x')$.

Remark 1.1 : From the definition, it is easy to see that if $T: X \to 2^Y$ is open-valued, then T is transfer open-valued. In fact, if for any given $x \in X$, $T(x)$ is an open subset in Y, then for any given $y \in T(x)$, taking $x' = x$, we have $y \in \text{int } T(x') = T(x)$. Hence, T is transfer open-valued. Conversely, if T is transfer open-valued, then T needs not to be open-valued. This can be seen from the following example.

Example 1.1 — Let $Y = [0, 1]$, $X = \{x \in (0, 1) : x \text{ is a rational}\}$ and $T: X \to 2^Y$ be a mapping defined by

$$T(x) = (0, x] \cup \{y \in (x, 1) : y \text{ is an irrational}\}, \; x \in X.$$

Then for each $x \in X$, $T(x)$ is not open. Next, we show that T is transfer open-valued. In fact, for any given $x \in X$ and $y \in T(x)$, we know that $0 < y < 1$ and so there exists a rational $x' \in (0, 1)$ such that $y < x'$. Then we have

$$y \in (0, x') \subset (0, x'] \cup \{y \in (x', 1) : y \text{ is an irrational}\}$$

$$= T(x').$$

This implies that $y \in \text{int } T(x')$, i.e., T is transfer open-valued.

Lemma 1.1[6] — Let X be a nonempty set and Y be a topological space. Then $T: X \to 2^Y$ is transfer open-valued if and only if

$$\bigcup_{x \in X} T(x) = \bigcup_{x \in X} \mathit{int}\, T(x). \qquad \dots (1.1)$$

Lemma 1.2 — Let X be a convex subset of a Hausdorff topological vector space, D be a nonempty subset of X and $T: X \to 2^D$ be a multi-valued mapping such that for each

$x \in X$, $coT(x) \subset D$. If $T^{-1}: D \to 2^X$ is transfer open-valued, then $(coT)^{-1}$ is also transfer open-valued, where the mapping $coT: X \to 2^D$ is defined by $(coT)(x) = coT(x)$ for all $x \in X$.

PROOF : For any $d \in D$ and $x \in (coT)^{-1}(d)$, i.e., $d \in coT(x)$, there exists a finite set $\{d_1, ..., d_n\} \subset T(x)$ such that

$$d = \sum_{i=1}^{n} \beta_i d_i, \ \beta_i \geq 0, \ \sum_{i=1}^{n} \beta_i = 1.$$

Hence, we have $x \in T^{-1}(d_i)$, $i = 1, 2, ..., n$. Since T^{-1} is transfer open-valued, there exists a point $d_i' \in D$ such that $x \in \operatorname{int} T^{-1}(d_i')$, $i = 1, ..., n$, and so there exists a neighbourhood $N_i(x) \subset X$ of x such that $N_i(x) \subset T^{-1}(d_i')$, $i = 1, ..., n$. Denote $N(x) = \bigcap_{i=1}^{n} N_i(x)$. Then $N(x)$ is also a neighbourhood of x and

$$x \in N(x) \subset I^{-1}(d_i') = \{u \in X : d_i' \in T(u)\} \subset \{u \in X : d_i' \in (coT)(u)\}$$

$$= (coT)^{-1}(d_i'), \ i = 1, ..., n.$$

This implies that $x \in \operatorname{int}(coT)^{-1}(d_i')$, $i = 1, ..., n$, i.e., $(coT)^{-1}$ is transfer open-valued. This completes the proof.

Lemma 1.3[6] — Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $T: X \to 2^X$ be a multi-valued mapping such that

(1) for any $x \in X$, $T(x)$ is nonempty convex; and

(2) T^{-1} is transfer open-valued.

Then T has a fixed point in X.

Lemma 1.4[11] — Let X be a convex subset of a locally convex Hausdorff topological vector space, D be a nonempty compact subset of X and $T: X \to 2^D$ be a upper semi-continuous mapping (briefly, u.s c.) with nonempty closed convex values. Then T has a fixed point in D, i.e., there exists a point $\bar{x} \in D$ such that $\bar{x} \in T(\bar{x})$.

2. Continuous Selection Theorems of Yannelis-Prabhakar Version

In this section, we shall give some new continuous selection theorems of Yannelis-Prabhakar's version, which generalize Theorem A to more general cases.

***Theorem* 2.1** — *Let E be a Hausdorff topological space, F be a Hausforff topological vector space, X be a nonempty paracompact subset of E and Y be a nonempty subset of F. Let*

$S, T : X \to 2^Y$ be two multi-valued mappings such that

(1) *for each $x \in X$, $S(x)$ is nonempty and $coS(x) \subset T(x)$; and*

(2) *S^{-1} is transfer open-valued.*

Then T has a continuous selection.

PROOF : By the conditions (1) and (2), we know that for each $x \in X$, $S(x) \neq 0$ and S^{-1} is transfer open-valued. By Lemma 1.1, we have

$$X = \bigcup_{y \in Y} S^{-1}(y) = \bigcup_{y \in Y} \operatorname{int} S^{-1}(y).$$

This implies that $\mathcal{D} : = \{\operatorname{int} S^{-1}(y) : y \in Y\}$ is an open covering of X. Since X is paracompact, there exists a locally finite open refinement $\mathcal{F} : = \{U_\alpha : \alpha \in I\}$ of $\mathcal{D}$ and a partition of the unity subordinated to $\mathcal{F}$, $\{f_\alpha : \alpha \in I\}$, such that for each $\alpha \in I$,

(a) $f_\alpha : X \to [0, 1]$ is continuous;

(b) $\{x\ X; f_\alpha(x) > 0\} \subset U_\alpha$; and

(c) $\sum_{\alpha \in I} f_\alpha(x) = 1$ for all $x \in X$.

Since $\mathcal{F}$ is a refinement of $\mathcal{D}$, for each $\alpha \in I$, there exists a point $y_\alpha \in Y$ such that $U_\alpha \subset \operatorname{int} S^{-1}(y_\alpha)$. Next, we define a mapping $f : X \to \operatorname{co}(Y)$ by

$$f(x) = \sum_{\alpha \in I} f_\alpha(x)\, y_\alpha. \qquad \text{... (2.1)}$$

Since $\mathcal{F}$ is locally finite, for each $x \in X$ there exists a neighbourhood $N(x)$ of x and a finite set $I_0 = \{\alpha_1, ..., \alpha_n\} \subset I$ such that for each $x_0 \in N(x)$, we have $f_\alpha(x_0) = 0$ whenever $\alpha \in I \setminus I_0$ and $\sum_{i=1}^{n} f_{\alpha_i}(x_0) = 1$. Hence, f is continuous. For each $x \in X$, let

$$I(x) = \{\alpha \in I : f_\alpha(x) > 0\}.$$

Then $I(x)$ is a finite subset of I. For any $\alpha \in I(x)$, from (b), it follows that $x \in U_\alpha \subset \operatorname{int} S^{-1}(y_\alpha)$ and so $y_\alpha \in S(x)$ for all $\alpha \in I(x)$. It follows from the condition (1) that

$$f(x) \in \operatorname{co} \{y_\alpha : \alpha \in I(x)\} \subset \operatorname{co}S(x) \subset T(x),$$

which means that f is a continuous selection of T. This completes the proof.

Remark 2.1 : From Remark 1.1, Theorem A is a special case of Theorem 2.1.

Theorem 2.2 — *Let $\{E_i : i \in I\}$ be a family of locally convex Hausdorff topological vector spaces. For each $i \in I, X_i$ is a convex subset of E_i and D_i is a nonempty compact subset of X_i. Denote $X = \prod_{i \in I} X_i$ and $D = \prod_{i \in I} D_i$. Suppose further that for each $i \in I, S_i, T_i : X \to 2^{D_i}$ are two multivalued mappings such that*

(1) *for each $x \in X, S_i(x) \neq 0$ and $coS_i(x) \subset T_i(x)$; and*

(2) *S_i^{-1} is transfer open-valued.*

Then there exists a point $\bar{x} \in D$ such that $\bar{x} \in \prod_{i \in I} T_i(\bar{x})$.

PROOF : Since D is a compact subset of X, it follows from [8, Lemma 1] that co$D$ is a paracompact set in X. By Theorem 2.1, for each $i \in I$, the mapping $T_i|_{\text{co}\,D} : \text{co}\,D \to 2^{D_i}$ has a continuous selection $f_i : \text{co}D \to D_i$.

Next, we define a mapping $f : \text{co}D \to D$ by

$$f(x) = \prod_{i = \in I f(x)} \quad , \quad x \in \text{co}\,D.$$

By Fan [9, Lemma 3], f is a single-valued u.s.c. mapping. By Lemma 1.4, there exists a point $\bar{x} \in D$ such that $\bar{x} = f(\bar{x})$. Therefore we have

$$\bar{x} = \prod_{i \in I} f_i(\bar{x}) \in \prod_{i \in I} T_i(\bar{x}).$$

This completes the proof.

From Theorem 2.2, we can obtain the following corollary immediately.

Corollary 2.3 — Let E be a locally convex Hausdorff topological vector space, X be a convex subset of E, D be a nonempty compact subset of X and $T : X \to 2^D$ be a multi-valued mapping such that

(1) for each $x \in X, T(x)$ is nonempty convex;

(2) T^{-1} is transfer open-valued.

Then there exists a point $\bar{x} \in D$ such that $\bar{x} \in T(\bar{x})$.

Remark 2.2 : Corollary 2.3 is a generalization of the famous Fan-Browder fixed point theorem ([4], [10]).

3. APPLICATIONS TO GENERALIZED QUASI-VARIATIONAL INEQUALITIES

In this section, we shall use the continuous selection theorems and fixed point theorems obtained in Section 2 to study the existence problem of solutions for generalized quasi-variational inequalities.

Theorem 3.1 — *Let E be a locally convex Hasudorff topological vector space, X be a paracompact convex subset of E, D be a nonempty compact subset of X, F be a Hausdorff topological vector space and Y be a nonempty subset of F. Suppose further that*

(1) *$T: X \to 2^Y$ is a multi-valued mapping with nonempty convex values and $T^{-1}: Y \to 2^X$ is transfer open-valued;*

(2) *$S: X \to 2^D$ is a multi-valued mapping with nonempty closed convex values; and*

(3) *$\psi: X \times Y \times X \to I\!R$ is a continuous functional such that*

(a) *$\psi(x, y, x) \geq 0$ for all $x \in X$ and $y \in T(x)$;*

(b) *the function $z \mapsto \psi(x, y, z)$ is quasi-convex on $S(x)$.*

Then there exists $\bar{x} \in D, \bar{x} \in S(\bar{x}), \bar{y} \in T(\bar{x})$ such that

$$\psi(\bar{x}, \bar{y}, x) \geq 0 \quad \text{for all} \quad x \in S(\bar{x}).$$

PROOF : Since X is paracompact, by the condition (1) and Theorem 2.1, T has a continuous selection $f: X \to Y$. Now, we define a mapping $G: X \to 2^D$ by

$$G(x) = \{z \in S(x) : -\psi(x, f(x), z) = \max_{u \in S(x)} (-\psi(x, f(x), u))\}. \qquad \text{.. (3.1)}$$

It follows from the condition (2) and the Aubin-Ekeland [1, Proposition 23] that G is a u.s.c. mapping. Besides, by the conditions (2) and (3)-(b), G is nonempty compact and convex-valued. By Lemma 1.4, there exists a $\bar{x} \in D$ such that $\bar{x} \in G(\bar{x})$. This implies that

$$\bar{x} \in S(\bar{x}) \quad \text{and} \quad \psi(\bar{x}, f(\bar{x}), \bar{x}) = \min_{u \in S(\bar{x})} \psi(\bar{x}, f(\bar{x}), u). \qquad \text{... (3.2)}$$

Letting $\bar{y} = f(\bar{x})$, then $\bar{y} \in T(\bar{x})$. From (3.2) and the condition (3)-(a), we have

$$\psi(\bar{x}, \bar{y}, x) \geq \psi(\bar{x}, \bar{y}, \bar{x}) \geq 0 \quad \text{for all} \quad x \in S(\bar{x}).$$

This completes the proof.

Remark 3.1 : Theorem 3.1 extends and improves Theorem 6.1.1 in Chang[5]. In addition, if take $F = E^*$ (: the dual space of E) and $\psi(x, y, z) = Re\langle y, z - x\rangle$, $(x, y, z) \in X \times Y \times X$ in Theorem 3.1, it follows from Theorem 3.1 that there exist $\bar{x} \in D, \bar{x} \in S(\bar{x}), \bar{y} \in T(\bar{x})$ such that

$$Re\langle \bar{y}, z - \bar{x}\rangle \geq 0 \quad \text{for all} \quad z \in S(\bar{x}).$$

Therefore, Theorem 3.1 also extends and improves the main result in Kim[13].

Theorem 3.2 — *Let E, F, X, D, Y be the same as in Theorem 3.1. Suppose further that*

(1) *$T: X \to 2^Y$ is a multi-valued mapping with nonempty convex values and T^{-1} is transfer open-valued;*

(2) *$S: X \to 2^D$ is a multi-valued mapping with nonempty convex values, S^{-1} is open-valued and the mapping $\bar{S}$ defined by $\bar{S}(x) = \overline{S(x)}$ is u.s.c.;*

(3) $\psi : X \times Y \times X \to I\!R$ *is a functional satisfying*

(a) *the function* $(x, y) \mapsto \psi(x, y, u)$ *is u.s.c.;*

(b) *the function* $u \mapsto \psi(x, y, u)$ *is quasi-convex on* $S(x)$*;*

(c) *for any continuous selection* $g : X \to Y$ *of* T*, the set*

$$\bigcup_{u \in X} S^{-1}(u) \cap \{x \in X : \psi(x, g(x), u) < 0\}$$

is paracompact; and

(d) *for any* $x \in X, y \in T(x), \psi(x, y, x) \geq 0.$

Then the conclusion of Theorem 3.1 still holds.

PROOF : By the condition (1) and Theorem 2.1, T has a continuous selection $f : X \to Y$. Next, we define a mapping $G : X \to 2^D$ by

$$G(x) = \{u \in S(x) : \psi(x, f(x), u) < 0\}, \quad x \in X.$$

By the condition (2), S is nonempty convex-valued and, by the condition (3)-(b), $u \mapsto \psi(x, y, u)$ is quasi-convex on $S(x)$ and so G is convex-valued. Again, for any u D, we have

$$\begin{aligned} G^{-1}(u) &= \{x \in X : u \in G(x)\} \\ &= \{x \in X : u \in S(x) \text{ and } \psi(x, f(x), u) < 0\} \\ &= S^{-1}(u) \cap \{x \in X : \psi(x, f(x), u) < 0\} \end{aligned}$$

and by the assumption, S^{-1} is open-valued and $(x, y) \mapsto \psi(x, y, u)$ is u.s.c. Therefore, G^{-1} is also open-valued and so G^{-1} is transfer open-valued. We denote

$$\begin{aligned} W = &\bigcup_{u \in X} S^{-1}(u) \cap \{x \in X : \psi(x, f(x), u) < 0\} \\ = &\bigcup_{u \in X} G^{-1}(u). \end{aligned} \qquad \dots (3.3)$$

(i) If $W = 0$, then for each $u \in X, G^{-1}(u) = 0$ and so $G(x) = 0$ for all $x \in X$. Thus we have

$$\psi(x, f(x), u) \geq 0 \text{ for all } x \in X, u \in S(x). \qquad \dots (3.4)$$

Besides, by the assumption that S is nonempty convex-valued and S^{-1} is open-valued, it follows from Corollary 2.3 there exists a point $\bar{x} \in D$ such that $\bar{x} \in S(\bar{x})$. Letting $\bar{y} = f(\bar{x}) \in T(\bar{x})$, then, from (3.4), it follows that

$$\psi(\bar{x}, \bar{y}, x) \geq 0 \quad \text{for all} \quad x \in S(\bar{x}).$$

(ii) If $W \neq \phi$, by the condition (3)-(c), W is a paracompact subset of X and it is easy to show that for each $x \in W$, $G(x)$ is nonemtpy convex. By Theorem 2.1, it follows that the mapping $G|_W : W \to 2^D$ has a continuous selection $h : W \to D$ such that $h(x) \in G(x)$ for all $x \in W$. Now, we define a mapping $H : X \to 2^D$ by

$$H(x) = \begin{cases} (h(x)) & \text{if } x \in W, \\ \overline{S(x)} & \text{if } x \in X \setminus W. \end{cases}$$

It follows from the condition (2) and the continuity of h that H is a u.s.c. mapping with nonempty closed convex valued. By Lemma 1.4, there is a point $\bar{x} \in D$ such that $\bar{x} \in H(\bar{x})$. If $\bar{x} \in W$, we have $\bar{x} = h(\bar{x}) \in G(\bar{x})$ and so $\psi(\bar{x}, f(\bar{x}), \bar{x}) < 0$. This contradicts the condition (3)-(d). Hence, we have $\bar{x} \in X \setminus W$. This implies that $\bar{x} \in \overline{S(x)}$ and for each $u \in X, u \notin G(\bar{x})$, i.e., $G(\bar{x}) = 0$. Therefore, we have

$$\bar{x} \in \overline{S(x)} \quad \text{and} \quad \psi(\bar{x}, \bar{y}, x) \geq 0 \quad \text{for all} \quad x \in S(\bar{x}),$$

where $\bar{y} = f(\bar{x}) \in T(x)$. This completes the proof.

Remark 3.2 : Theorem 3.2 extends and improves Theorem 3 in Shinh and Tan[16].

Definition 3.1[21] — Let X be a convex subset of a locally convex Hausdorff topological vector space. A function $\psi(x, y) : X \times X \to \mathbb{R}$ is said to be O-diagonally convex (resp., O-diagonally concave) in y, if for any finite set $\{y_1, \ldots, y_m\} \subset X$ and $y_0 = \sum_{i=1}^{m} \beta_i y_i, \beta_i \geq 0, \sum_{i=1}^{m} \beta_i = 1$, we have

$$0 \leq \sum_{i=1}^{m} \beta_i \, \psi(y_0, y_i) \left(\text{resp., } \sum_{i=1}^{m} \beta_i \, \psi(y_0 m\, y_0) \leq 0 \right).$$

Theorem 3.3 — *Let X be a nonempty compact convex set of a locally convex Hausdorff topological vector space and Y be a nonempty set of a Hausdorff topological vector space F. Let $T : X \to 2^Y$, $S : X \to 2^X$ be two multi-valued mappings such that*

(1) *T is nonempty convex-valued and T^{-1} is transfer open-valued;*

(2) *S is a continuous mapping with nonempty closed convex values;*

(3) *$\psi : X \times Y \times \to \mathbb{R}$ is u.s.c. and*

(a) *$\psi(x, y, u)$ is O-diagonally convex in u; and*

(b) *$\psi(x, y, x) \geq 0$ for all $x \in X$ and $y \in T(x)$.*

Then the conclusion in Theorem 3.1 still holds.

PROOF : By the condition (1) and Theorem 2.1, T has a continuous selection $f : X \to Y$. By the condition (3), the function $(x, u) \mapsto -\psi(x, f(x), u)$ is lower semi-continuous (briefly, l.s.c). From

[1, Proposition 19], it follows that the function $x \mapsto \sup_{u \in S(x)} -\psi(x, f(x), u)$ is also l.s.c. Hence, the set

$$\left\{ x \in X : \sup_{u \in S(x)} -\psi(x, f(x), u) \leq 0 \right\}$$

is closed in X. Besides, by the condition (3)-(a), the function $-\psi(x, f(x), u)$ is O-diagonally concave in u. It follows from Theorem 3.1 in Zhou and Chen[21] that there exists a point $\bar{x} \in X, \bar{x} \in S(\bar{x})$, such that

$$\sup_{u \in S(\bar{x})} -\psi((\bar{x}), f(\bar{x}), u) \leq 0.$$

Taking $\bar{y} = f(\bar{x})$, we have $\bar{y} \in T(\bar{x})$ and $\psi(\bar{x}, \bar{y}, u) \geq 0$ for all $u \in S(\bar{x})$. This completes the proof.

From Theorem 3.3, we can obtain the following.

Corollary 3.4 — Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E and $f : X \to E^*$ be a single-valued mapping such that, for each $y \in X$, the set

$$\{x \in X : \langle f(x), x - y \rangle \leq 0\} \quad \text{... (3.5)}$$

is closed. Then the following variational inequality

$$\langle f(x), u - x \rangle \geq 0 \quad \text{for all} \quad u \in X \quad \text{... (3.6)}$$

has a solution in X.

PROOF : Define mappings $S, T : X \to 2^X$ and $\psi : X \times E^* \times X \to I\!R$ by

$$S(x) = X, x \in X,$$

$$T(x) = \{u \in X : \langle f(x), x - u \rangle > 0\}, \ x \in X,$$

$$\psi(x, y, z) = \langle y, z - x \rangle, (x, y, z) \in X \times E^* \times X,$$

respectively. Hence, for each $x \in X, T(x)$ is convex and, from (3.5), for each $u \in X$ the set

$$\begin{aligned} T^{-1}(u) &= \{x \in X : u \in T(x)\} \\ &= \{x \in X : \langle f(x), x - u \rangle > 0\} \\ &= X \setminus \{x \in X : \langle f(x), x - u \rangle \leq 0\} \end{aligned}$$

is open in X. Hence, T^{-1} is transfer open-valued.

If for each $x \in X, T(x) \neq 0$, then, by Corollary 2.3, there exists a point $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$. Therefore, we have

$$0=\langle f(\bar{x}), \bar{x}-\bar{x}\rangle>0,$$

which is a contradiction. Thus there exists a point $x^* \in X$ such that $T(x^*)=0$ and so we have

$$\langle f(x^*), x^*-u\rangle \leq 0 \quad \text{for all} \quad u \in X,$$

i.e., $\langle f(x^*), u-x^*\rangle \geq 0$ for all $u \in X$. This completes the proof.

Remark 3.3 : Corollary 3.4 extends and improves the corresponding results in Browder[3] and Yao and Guo[20].

4. Applications to Maximal Element Problems

Definition 4.1 — Let X be a nonempty subset of a topological space and $P: X \to 2^X$ be a multi-valued mapping. If there exists a point $\bar{x} \in X$ such that $P(\bar{x})=\phi$, then the point $\bar{x}$ is called a maximal element in X.

The existence problem of maximal elements is an important problem in the study of economic equilibrium theory. In this section, we shall use the results given in Section 2 to study the existence problem of maximal elements.

Theorem 4.1 — *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $P: X \to 2^X$ be a mapping such that*

(1) *for each $x \in X, x \notin coP(x)$;*

(2) *P^{-1} is transfer open-valued.*

Then there exists a point $\bar{x} \in X$ such that $P(\bar{x})=\phi$, i.e., there exists a maximal element in X.

PROOF : Suppose the contrary. Then, for any $x \in X, P(x) \neq \phi$ and hence, for each $x \in X$, $coP(x)$ is nonempty convex. In addition, it follows from the condition (2) and Lemma 1.2 that $(coP)^{-1}$ is transfer open-valued. By Lemma 1.3, there exists a point $x^* \in X$ such that $x^* \in \text{co}\, P(x^*)$. This contradicts the condition (1). Therefore, the conclusion is true. This completes the proof.

Definition 4.2[14] — A vector topological space X is said to be quasi-complete if each bounded closed subset in X is complete.

Lemma 4.2[5] — Let E be a quasi-complete locally convex Hausdorff topological vector space and D be a nonempty compact subset of E. Then the convex closure $\overline{co}\,(D)$ is compact.

Theorem 4.3 — *Let X be a paracompact convex subset of a quasi-complete locally convex Hausdorff topological vector space, D be a nonempty compact subset of X and $P: X \to 2^D$ be a multi-valued mapping such that*

(1) *for each $x \in coD, x \notin coP(x)$; and*

(2) *P^{-1} is transfer open-valued.*

Then there exists a point $\bar{x} \in X$ such that $P(\bar{x})=\phi$.

PROOF : Suppose the contrary. Then for each $x \in X, P(x) \notin 0$ and so the mapping $\text{co}P: X \to 2^{\text{co}D}$ is nonempty convex-valued. By the condition (2) and Lemma 1.2, $(\text{co}P)^{-1}$ is transfer open-valued. It follows from Theorem 2.1 that the mapping $\text{co}P$ has a continuous selection $f: X \to \text{co}D$ such that $f(x) \in \text{co}P(x)$ for all $x \in X$. Since D is a compact subset of X, by Lemma 4.2, $\overline{\text{co}}(D)$ is also compact subset of X. Hence, f is a continuous mapping from X into $\overline{\text{co}}(D)$. By Lemma

1.4, there exists a point $x^* \in \operatorname{co} D$ such that $x^* = f(x^*) \in \operatorname{co} P(x^*)$, which contradicts the condition (1). This completes the proof.

5. Applications to Nonempty Intersection Problems and Economic Equilibrium Problems

In this section, we need the following notations:

Let $\{X_i : i \in I\}$ be a family of topological spaces and denote

$$X = \prod_{i \in I} X_i, \quad \hat{X}_i = \prod_{\substack{j \in I \\ j \neq i}} X_j, \ i \in I.$$

Let $\Pi_i : X \to X_i$ and $\hat{\Pi}_i : X \to \hat{X}_i$ be the projections. If $x \in X$, then we have $\Pi_i(x) = x_i$, $\hat{\Pi}_i(x) = \hat{x}_i$ and $x = (x_i, \hat{x}_i)$. Let A be a subset of X, $x_i \in X_i$ and $\hat{x}_i \in \hat{X}_i$. We denote

$$A[x_i] = \{\hat{y}_i \in \hat{X}_i, (x_i, \hat{y}_i) \in A\},$$

$$A[\hat{x}_i] = \{y_i \in X_i, (y_i, \hat{x}_i) \in A\}.$$

If $A_i \subset X_i$ and $\hat{A}_i \subset \hat{X}_i$ are two subsets, then we denote

$$A_i \times \hat{A}_i = \{(x_i, \hat{x}_i) \in X : x_i \in A, \hat{x}_i \in \hat{A}_i\}.$$

Theorem 5.1 — *Let $\{E_i : i \in I\}$ be a family of locally convex Hausdorff topological vector spaces, X_i be a convex subset of E_i for each $i \in I$ and D_i be a nonempty compact subset of X_i. Let $\{A_i : i \in I\}$ $\{B_i : i \in I\}$ be two families of subsets of X such that*

(1) *for each $i \in I$ and each $x \in X$, $S_i(x) \neq 0$ and $\operatorname{co} S_i(x) \subset T_i(x)$, where the mappings $S_i, T_i : X \to 2^{D_i}$ are defined by*

$$S_i(x) = B_i[\hat{x}_i] \cap D_i, x \in X,$$

$$T_i(x) = A_i[\hat{x}_i] \cap D_i, x \in X; \text{ and}$$

(2) *for each $i \in I$, $g_i : D_i \to 2^{B_i}$ is transfer open-valued, where g_i is defined by $g_i(d_i) = B_i[d_i]$, $d_i \in D_i$.*

Then
$$\bigcap_{i \in I} A_i \neq \phi.$$

PROOF : First, we prove that for each $i \in I$, S_i^{-1} is transfer open-valued. In fact, for each $d_i \in D_i$ and $x \in S_i^{-1}(d_i)$, we have $d_i \in S_i(x) = B_i[\hat{x}_i] \cap D_i$ and so we have

$$\hat{x}_i \in B_i[d_i] = g_i(d_i) \quad \text{and} \quad d_i \in D_i.$$

By the condition (2), g_i is transfer open-valued and hence there exists $z_i \in D_i$ such that $\hat{x}_i \in \text{int}\, g_i(z_i)$. Thus there exists a neighbourhood $N(\hat{x}_i)$ of $\hat{x}_i$ such that $N(\hat{x}_i) \subset g_i(z_i)$. Denote

$$U(x) = X_i \times N(\hat{x}_i).$$

Then $U(x)$ is also a neighbourhood of x in X. For each $u \in U(x)$, we have $\hat{u}_i \in N(\hat{x}_i) \subset g_i(z_i)$. Therefore, we have

$$\begin{aligned} z_i \in g_i^{-1}(\hat{u}_i) \cap D_i &= \{x_i \in D_i : \hat{u}_i \in g_i(x_i)\} \cap D_i \\ &= \{x_i \in D_i : \hat{u}_i \in B_i[x_i]\} \cap D_i \\ &= \{x_i \in D_i : x_i \in B_i[\hat{u}_i]\} \cap D_i \\ &= B_i[\hat{u}_i] \cap D_i = S_i(u). \end{aligned}$$

Hence, $u \in S_i^{-1}(z_i)$ for all $u \in U(x)$, which implies that $U(x) \subset S_i^{-1}(z_i)$ and so $x \in U(x) \subset \text{int}\, S_i^{-1}(z_i)$, i.e., S_i^{-1} is transfer open-valued.

Summing up the above arguments, we know that for each $i \in I, S_i$ and T_i satisfy all the conditions in Theorem 2.2. Hence, there exists a point $\bar{x} \in D$ such that

$$\bar{x}_i \in T_i(\bar{x}) = A_i[\hat{\bar{x}}_i] \cap D_i \quad \text{for all} \quad i \in I,$$

i.e., $(\bar{x}_i, \hat{\bar{x}}_i) = \bar{x} \in A_i$ for all $i \in I$. Therefore, we have $\bar{x} \in \bigcap_{i \in I} A_i$. This completes the proof.

Remark 5.1 : Theorem 5.1 contains Theorem 3 in [8] as its special case. In addition, a similar result is proved in [19].

Now, we are in a position to study the existence problem of equilibrium for abstract economies of Shafer-Sonnenschein's version (see [2] or [15]).

In the sequel, let I be a set of agents. For each $i \in I, X_i$ be a convex subset of a locally convex Hausdorff topological vector space E_i and D_i be a nonempty compact subset of X_i. Denote

$$X = \prod_{i \in I} X_i \quad \text{and} \quad D = \prod_{i \in I} D_i.$$

Definition 5.1 — An abstract economy is a family of ordered quadruples $\Gamma = (X_i, A_i, B_i,$

$P_i)_{i \in I}$, where $A_i, B_i : X = \prod_{i \in I} X_i \to 2^{X_i}$ are constraint correspondences and $P_i : X \to 2^{X_i}$ is a preference correspondence. An equilibrium for Γ is a point $\bar{x} \in X$ such that for each $i \in I, \bar{x}_i \in \overline{B_i(\bar{x})}$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = 0$.

Theorem 5.2 — *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy such that for each $i \in I$,*

(1) *for each $x \in X$, $B_i(x)$ is nonemtpy convex and $A_i(x) \subset B_i(x) \subset D_i$;*

(2) *the mapping $\overline{B}_i : X \to 2^{X_i}$ defined by $\overline{B}_i(x) = \overline{B_i(x)}, x \in X$, is u.s.c.;*

(3) *$T_i^{-1} : D_i \to 2^X$ is transfer open-valued, where $T_i : X \to 2^{D_i}$ is defined by*

$$T_i(x) = A_i(x) \cap P_i(x), x \in X;$$

(4) *for each $x \in X, x_i \notin coT_i(x)$, and*

(5) *the set $K_i = \{x \in X : T_i(x) \neq 0$ is a paracompact subset in X.*

Then there exists an equilibrium of Γ in X, i.e., there exists a point $\bar{x} \in X$ such that $\bar{x}_i \in \overline{B_i(\bar{x})}$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = 0$ for all $i \in I$.

PROOF : For each $i \in I$, define a mapping $S_i : X \to 2^{D_i}$ by

$$S_i(x) = \mathrm{co}T_i(x), x \in X.$$

By the condition (3) and Lemma 1.2, $(\mathrm{co}T_i)^{-1} = S_i^{-1}$ is also transfer open-valued. Hence, it follows from the condition (5) that $S_i|_{K_i} : K_i \to 2^{D_i}$ is nonempty convex valued and $(S_i|_{K_i})^{-1}$ is transfer open-valued. By Theorem 2.1, $S_i|_{K_i}$ has a continuous selection $f_i : K_i \to D_i$ such that $f_i(x) \in S_i(x)$ for all $x \in K_i$. Now, we define a mapping $G_i : X \to 2^{D_i}$ by

$$G_i(x) = \begin{cases} (f_i(x)) \text{ if } x \in K_i, \\ \overline{B_i(x)} \text{ if } x \in X \setminus K_i. \end{cases}$$

By the condition (1), G_i is a mapping with nonempty closed convex values.

Next, we prove that K_i is an set in X. In fact, for each $x \in K_i$, since f_i is a continuous selection of $S_i|_{K_i}$, we have $f_i(x) \in S_i(x)$. Letting $y_i = f_i(x) \in D_i$, then we have $x \in S_i^{-1}(y_i)$. Since S_i^{-1} is transfer open-valued, there exists a point $z_i \in D_i$ such that $x \in \mathrm{int}\, S_i^{-1}(z_i)$. Hence, there exists a neighbourhood $N(x)$ of x in X such that $N(x) \subset S_i^{-1}(z_i)$ and so, for each $b \in N(x), z_i \in S_i(b) = \mathrm{co}\, T_i(b)$. Therefore, for each $b \in N(x), T_i(b) \neq 0$. This implies that $N(x) \subset K_i$, i.e., K_i is an open set in X and so $X \setminus K_i$ is a closed set in X. By the condition (2), G_i is u.s.c.

Summing up the above arguments, we know that $G_i : X \to 2^{D_i}$ is a u.s.c. mapping with nonempty closed convex values. By Lemma 1.4, there exists a point $\bar{x} \in D$ such that $\bar{x}_i \in G_i(\bar{x})$ for all $i \in I$. If $\bar{x} \in K_i$, then we have $\bar{x}_i = f_i(\bar{x}) \in S_i(\bar{x}) = \text{co}T_i(\bar{x})$, which contradicts the condition (4). Therefore, $\bar{x} \in X \setminus K_i$ and we have

$$\bar{x}_i \in \overline{B_i(\bar{x})} \text{ and } A_i(\bar{x}) \cap P_i(\bar{x}) = 0.$$

This completes the proof.

Remark 5.2 : Theorem 5.2 extends and improves the corresponding results of [7, Theorem 3.1], [12, Theorem 1], [20, Theorem 6.1] and [17].

Theorem 5.3 — *Let $\Gamma = (X_i, \Lambda_i, B_i, P_i)_{i \in I}$ be an abstract economy such that, for each $i \in I$,*

(1) *for each $x \in X, A_i(x) \subset B_i(x) \subset D_i$ and $B_i(x)$ is nonempty convex;*

(2) *B_i^{-1} is transfer open-valued;*

(3) *$T_i^{-1} : D_i \to 2^X$ is transfer open-valued, where $T_i : X \to 2^{D_i}$ is defined by*

$$T_i(x) = A_i(x) \cap coP_i(x), x \in X;$$

(4) *for each $x \in X, x_i \notin coP_i(x)$; and*

(5) *the set $M_i = \{x \in X : A_i(x) \cap P_i(x) \neq 0\}$ is closed in X.*

Then there exists a point $\bar{x} \in D$ such that

$$\bar{x}_i \in B_i(\bar{x}) \text{ and } A_i(\bar{x}) \cap P_i(\bar{x}) = 0 \text{ for all } i \in I.$$

PROOF : For each $i \in I$, define a mapping $G_i : X \to 2^{D_i}$ by

$$G_i(x) = \begin{cases} coT_i(x) \text{ if } x \in M_i, \\ B_i(x) \quad \text{if } x \in X \setminus M_i. \end{cases}$$

By the conditions (1) and (5), for each $x \in X, G_i(x)$ is nonempty convex. Besides, for each $d_i \in D_i$, we have

$$\begin{aligned} G_i^{-1}(d_i) &= \{x \in X : d_i \in G_i(x)\} \\ &= \{x \in M_i : d_i \in \text{co}T_i(x)\} \cup \{x \in X \setminus M_i : d_i B_i(x)\} \\ &= (M_i \cap (\text{co}T_i)^{-1}(d_i)) \cup ((X \setminus M_i) \cap B_i^{-1}(d_i)). \end{aligned}$$

Therefore, for each $y \in G_i^{-1}(d_i)$, we have $y \in M_i \cap (\text{co}T_i)^{-1}(d_i)$ or $y \in (X \setminus M_i) \cap B_i^{-1}(d_i)$.

If $y \in M_i \cap (\mathrm{co}T_i)^{-1}(d_i)$, then $y \in M_i$ and $y \in (\mathrm{co}T_i)^{-1}(d_i)$. By the condition (3) and Lemma 1.2, the mapping $(\mathrm{co}T_i)^{-1}$ is transfer open-valued. Thus there exists a point $z_i \in D_i$ such that $y \in (\mathrm{int}\,(\mathrm{co}T_i)^{-1}(z_i)) \cap M_i$. Since $(\mathrm{int}\,(\mathrm{co}T_i)^{-1}(z_i)) \cap M_i$ is an open set in M_i, we have

$$(\mathrm{int}\;(\mathrm{co}T_i)^{-1}(z_i)) \cap M_i = \mathrm{int}\;\{(\mathrm{int}\;(\mathrm{co}T_i)^{-1}(z_i)) \cap M_i\}$$

and so

$$y \in \mathrm{int}\;\{(\mathrm{int}\;(\mathrm{co}T_i)^{-1}(z_i)) \cap M_i\} \subset \mathrm{int}\;\{(\mathrm{co}T_i)^{-1}(z_i) \cap M_i\}$$

$$= \mathrm{int}G_i^{-1}(z_i). \qquad \ldots (5.1)$$

If $y \in (X \setminus M_i) \cap B_i^{-1}(d_i)$, then $y \in X \setminus M_i$ and $y \in B_i^{-1}(d_i)$. By the condition (2), there exists a point $u_i \in D_i$ such that $y \in (\mathrm{int}\,B_i^{-1}(u_i)) \cap (X \setminus M_i)$. Since $(\mathrm{int}B_i^{-1}(u_i)) \cap (X \setminus M_i)$ is an open set in $X \setminus M_i$, we have

$$\mathrm{int}\;\{\mathrm{int}\;B_i^{-1}(u_i)) \cap (X \setminus M_i)\} = (\mathrm{int}\;B_i^{-1}(u_i)) \cap (X \setminus M_i).$$

Therefore, we have

$$y \in \mathrm{int}\;\{(\mathrm{int}\;B_i^{-1}(u_i)) \cap (X \setminus M_i)\} \subset \mathrm{int}\;\{B_i^{-1}(u_i)) \cap (X \setminus M_i)\}$$

$$= \mathrm{int}\;G_i^{-1}(u_i). \qquad \ldots (5.2)$$

Combining (5.1) and (5.2), it follows that G_i^{-1} is transfer open-valued. By Theorem 2.2, there exists a point $\bar{x} \in D$ such that $\bar{x}_i \in G_i(\bar{x})$ for all $i \in I$. In view of the condition (4), we know that $\bar{x} \notin M_i$ and so $\bar{x} \in X \setminus M_i$. Hence, we have $\bar{x}_i \in B_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = 0$ for all $i \in I$. This completes the proof.

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