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THE POINT SPECTRA AND REGULARITY FIELDS OF PRODUCTS OF QUASI-DIFFERENTIAL OPERATORS

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In this paper, the general ordinary quasi-differential expression τ of nth order with complex coefficients and its formal adjoint τ^+ are considered, and a number of results concerning the location of the point spectra and regularity fields of products of the operators which are generated by such expressions are obtained. Some of these are extensions or generalizations of those proved for symmetric case in [1, 12, 13, 14, 15], whilst others for general case in [9, 10, 11].

Key Words : Quasi-Differential Expressions; Essential Spectra; Joint Field of Regularity; Regularly Solvable Operators; Product Operators

1. INTRODUCTION

The minimal operators T_0 and T_0^+ generated by a general quasi-differential expressions τ and its formal adjoint τ^+ respectively, form an adjoint pair of closed, densely-defined operators in the underlying L_w^2-space, that is $T_0 \subset (T_0^+)^*$. The operators which fulfill the role that the self-adjoint and maximal symmetric operators play in the case of a formally symmetric expression τ are those which are regularly solvable with respect to T_0 and T_0^+. Such an operator S satisfies $T_0 \subset S \subset (T_0^+)^*$, and for $\lambda \in \mathbb{C}$, $(S - \lambda I)$ is a Fredholm property of zero index; this means that S has the desirable Fredholm property that the equation $(S - \lambda I)u = f$ has a solution if and only if f is orthogonal to the solutions of $(S^* - \bar{\lambda} I)v = 0$, and furthermore the solution spaces of $(S - \lambda I)u = 0$ and $(S^* - \bar{\lambda} I)v = 0$ have the same finite dimension. This notion was originally due to Visik[16].

The main objectives of this paper are to investigate the location of the point spectra and regularity fields of product of ordinary quasi-differential operators. Also, the results concerning the product of differential operators generalize all of those given in [13, 14, 15] for symmetric case and in [10, 11] for non-symmetric case, by removing the condition on the regularity fields.

We deal throughout with a quasi-differential expression τ of arbitrary order n defined by a general Shin-Zettl matrix given in [8] and [9], and the minimal operator T_0 is generated by $w^{-1}\tau[\cdot]$ in $L_w^2(I)$, where w is a positive weight function on the underlying interval I. The left-hand end point of I is assumed to be regular but the right-hand end point may be either regular or singular.

2. PRELIMINARIES

In this section, we give some of the definitions and results which will be needed later; see [1], [3], [4] and [5].

The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$ respectively and $N(T)$ will denote its null space. The nullity of T, written nul (T), is the dimension of $N(T)$ and the deficiency of T, written def(T), is the co-dimension of $R(T)$ in H; thus if T is densely-defined and $R(T)$ is closed, then def(T) = nul(T^*). The Fredholm domain of T is (in the notation of [3]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values $\lambda \in \mathbb{C}$ which are such that $(T-\lambda I)$ is a Fredholm operator, where I is the identity operator on H. Thus $\lambda \in \Delta_3(T)$ if and only if $(T-\lambda I)$ has closed range and finite nullity and deficiency. The index of $(T-\lambda I)$ is the number ind $(T-\lambda I)$ = nul $(T-\lambda I)$ − def $(T-\lambda I)$, this being defined for $\lambda \in \Delta_3(T)$.

Two closed densely-defined operators A and B acting in H are said to form an adjoint pair if $A \subset B^*$ and consequently $B \subset A^*$; equivalently, $(Ax, y) = (x, By)$, for all $x \in D(A)$ and $y \in D(B)$, where $(\cdot, \cdot)$ denotes the inner-product on H.

The field of regularity $\Pi(A)$ of A is the set of all $\lambda \in \mathbb{C}$ for which there exists a positive constant $k(\lambda)$ such that,

$$\| (A-\lambda I)x \| \geq k(\lambda) \| x \| \text{ for all } x \in D(A),$$

or equivalently, on using the Closed-Graph Theorem nul $(A-\lambda I) = 0$ and $R(A-\lambda I)$ is closed.

The joint field of regularity $\Pi(A, B)$ of A and B is the set of $\lambda \in \mathbb{C}$ which are such that $\lambda \in \Pi(A)$, $\bar{\lambda} \in \Pi(B)$ and both def $(A-\lambda I)$ and def $(B-\bar{\lambda} I)$ are finite. An adjoint pair A and B is said to be Compatible if $\Pi(A, B) \neq \phi$.

Definition 2.1 — A closed operator S in H is said to be regularly solvable with respect to the compatible adjoint pair A and B if $A \subset S \subset B^*$ and $\Pi(A, B) \cap \Delta_4(s) \neq \phi$, where $\Delta_4(S)$ = $\{\lambda : \lambda \in \Delta_3(S), \text{ index } (S-\lambda I) = 0\}$. The terminology "regularly solvable" comes from Visik's paper[16].

Definition 2.2 — The resolvent set $\rho(S)$ of a closed operator S in H consists of the complex numbers λ for which $(S-\lambda I)^{-1}$ exists, is defined on H and is bounded. The complement of $\rho(S)$ in $\mathbb{C}$ is called the spectrum of S and written $\sigma(S)$. The point spectrum $\sigma_p(S)$, continuous spectrum $\sigma_c(S)$ and residual spectrum $\sigma_r(S)$ are the following subsets of $\sigma(S)$ (see [2] and [3]):

$$\sigma_p(S) = \{\lambda \in \sigma(S) : (S-\lambda I) \text{ is not injective}\},$$

i.e., the set of eigenvalues of S;

$$\sigma_c(S) = \{\lambda \in \sigma(S) : (S-\lambda I) \text{ is injective, } R(S-\lambda I) \subsetneq \overline{R(S-\lambda I)} = H\};$$

$$\sigma_r(S) = \{\lambda \in \sigma(S) : (S-\lambda I) \text{ is injective, } \overline{R(S-\lambda I)} \neq H\}.$$

For a closed operator S we have

$$\sigma(S) = \sigma_p(S) \cup \sigma_c(S) \cup \sigma_r(S).$$

An important subset of the spectrum of a closed densely-defined T in H is the so-called *essential spectrum*. The various essential spectra of T are defined as in [3, chapter IX] to be the sets

$$\sigma_{ek}(T) = \mathbb{C} \setminus \Delta_k(T), \ (k = 1, 2, 3, 4, 5); \qquad \dots (2.1)$$

$\Delta_3(T)$ and $\Delta_4(T)$ have been defined earlier.

The sets $\sigma_{ek}(T)$ are closed and $\sigma_{ek}(T) \subset \sigma_{ej}(T)$ if $k < j$. The inclusion being strict in general. We refer the reader to [1], [2] and [3, Chapter IX] for further information about the sets $\sigma_{ek}(T)$.

Given two operators A and B, both acting in a Hilbert space H, we wish to consider the product operator AB. This is defined as follows:

$D(AB) = \{x \in D(B) \mid Bx \in D(A)\}$ and $(AB)x = A(Bx)$ for all $x \in D(AB)$.

It may happen in general that $D(AB)$ contains only the null element of H. However, is the case of many differential operators the domains of the product will be dense in H; see [17, Section 3].

The next result gives conditions under which the deficiency of a product is the sum of the deficiencies of the factors. It is a generalization of Theorem A (Glazman) in [7, 17, Section 3].

Lemma 2.3 — Let A and B be closed operators with dense domains in a Hilbert space H and suppose that $\lambda \in \Pi(A, B)$. Then AB is a closed operator with dense domain and

$$\text{def}\,(AB - \lambda I) = \text{def}(A - \lambda I) + def(B - \lambda I). \qquad \dots (2.2)$$

Evidently, Lemma 2.3 extends to the product of any finite number of operators $A_1, A_2, \dots, A_n$.

3. Quasi-Differential Expressions

The quasi-differential expressions are defined in terms of a Shin-Zettl matrix F on an interval I. The set $Z_n(I)$ of Shin-Zettl matrices on I consists of $(n \times n)$-matrices $F = \{f_{rs}\}$, $1 \le r, s \le n$ whose entries are complex-valued functions on I which satisfy the following conditions :

$$\left.\begin{array}{lll} f_{rs} \in L^1_{loc}(I) & & (1 \le r, s \le n, n \ge 2), \\ f_{r, r+1} & \text{a.e. on } I & (1 \le r \le n-1), \\ f_{rs} = 0 & \text{a.e. on } I & (2 \le r + 1 < s \le n). \end{array}\right] \qquad \dots (3.1)$$

For $F \in Z_n(I)$, the quasi-derivatives associated with F are defined by :

$$\left.\begin{array}{l} y^{[0]} := y, \\ y^{[r]} := f^{-1}_{r, r+1}\left\{ (y^{[r-1]})' - \sum\limits_{s=1}^{r} f_{rs}\, y^{[s-1]} \right\} (1 \le r \le n-1), \\ \text{and } y^{[n]} := (y^{[n-1]})' - \sum\limits_{s=1}^{n} f_{ns}\, y^{[s-1]}, \end{array}\right], \qquad \dots (3.2)$$

where the prime $'$ denotes differentiation.

The quasi-differential expression τ associated with F is given by:

$$\tau[y] := i^n y^{[n]}, \ (n \geq 2), \qquad \ldots (3.3)$$

this being defined on the set,

$$V(\tau) := \left\{ y : y^{[r-1]} \in AC_{loc}(I), \ r = 1, 2, \ldots, n \right\}, \qquad \ldots (3.4)$$

where $AC_{loc}(I)$ denotes the set of functions which are absolutely continuous on every compact subinterval of I.

The formal adjoint τ^+ of τ is defined by the matrix $F^+ \in z_n(I)$ given by :

$$F^+ := -L^{-1} F^* L, \qquad \ldots (3.5)$$

where F^* is the conjugate transpose of F and L is the non-singular $n \times n$-matrix

$$L = \{(-1)^r \delta_{r, n+1-s}\}, \ (1 \leq r, s \leq n) \qquad \ldots (3.6)$$

δ being the kronecker delta. If $F^+ = \{f^+_{rs}\}$, then it follows that

$$f_{rs}{}^+ = (-1)^{r+s+1} \bar{f}_{n-s+1, n-r+1}, \text{ for each } r \text{ and } s. \qquad \ldots (3.7)$$

The quasi-derivatives associated with F^+ are therefore:

$$\left.\begin{array}{l} y_+^{[0]} := y, \\ y_+^{[r]} := \bar{f}^{-1}_{n-r, n-r+1} \left\{ (y_+^{[r-1]})' - \displaystyle\sum_{s=1}^{r} (-1)^{r+s+1} \bar{f}_{n-s+1, n-r+1} \, y_+^{[s-1]} \right\} \\ \text{and} \quad y_+^{[n]} := (y_+^{[n-1]})' - \displaystyle\sum_{s=1}^{n} (-1)^{n+s+1} \bar{f}_{n-s+1, 1} \, y_+^{[s-1]}. \end{array}\right]$$

Note that, $(F^+)^+ = F$ and so $(\tau^+)^+ = \tau$. We refer to [4], [5], [8], [9] and [18] for a full account of the above and subsequent results on quasi-differential expressions.

Let the interval I have end-points $a, b \ (-\infty \leq a < b \leq \infty)$, and let $w : I \to I\!R$ be a non-negative weight function with $w \in L^1_{loc}(I)$ and $w(x) > 0$ (for almost all $x \in I$). Then $H = L^1_w(I)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions such that $\int_I w|f|^2 < \infty$ the inner-product is defined by $(f, g) := \int_I wf\bar{g}, \ (f, g \in L^2_w(I))$. The equation

$$\tau[u] - \lambda wu = 0 \ (\lambda \in \mathbb{C}) \text{ on } I, \qquad \ldots (3.9)$$

is said to be regular at the left end-point $a \in I\!R$, if for all $x \in (a, b)$,

$$a \in I\!R; \ w, a_{rs} \in L^1[a, x], \ (r, s = 1, 2, \ldots, n). \qquad \ldots (3.10)$$

Otherwise (3.9) is said to be singular at a. Similarly, we define the terms regular and singular at b. If (3.9) is regular at both end-points, then it is said to be regular; in this case we have

$$a, b \in I\!R;\ w, a_{rs} \in L^1 (a, b), (r, s = 1, 2, ..., n). \qquad ... (3.11)$$

We shall be concerned with the case when a is a regular end-point of (3.9), the end-point b being allowed to be either regular or singular.

Note that, in view of (3.7), an end-point of I is regular for (3.9), if and only if it is regular for the equation,

$$\tau^+ [v] - \bar{\lambda} w v = 0 \ (\lambda \in \mathbb{C}) \text{ on } I. \qquad ... (3.12)$$

Note that, at a regular end-point a, say, $u^{[r-1]}(a)\,(v_+^{[r-1]}(a))$ is defined for all $u \in V(\tau)\,(v \in V(\tau^+))$, $r = 1, ..., n$. Set,

$$D \ : = \ \{u : u \in V(\tau), u \text{ and } w^{-1}\ \tau[u] \in L_w^2 (a, b)\}, \qquad ... (3.13)$$

$$D^+ \ : = \ \{v : v \in V(\tau^+), v \text{ and } w^{-1}\ \tau^+ [v] \in L_w^2 (a, b)\}. \qquad ... (3.14)$$

The subspaces D and D^+ of $L_w^2 (a, b)$ are domains of the so-called maximal operators T and T^+ respectively, defined by

$$Tu : = w^{-1}\ \tau[u]\ (u \in D) \text{ and } T^+ v : = w^{-1}\ \tau^+ [v]\ (v \in D^+).$$

For the regular problem the minimal operators T_0 and T_0^+ are the restrictions of $w^{-1}\ \tau[u]$ and $w^{-1} \tau^+ [v]$ to the subspaces:

$$\left.\begin{aligned} D_0 &: = \left\{ u : u \in D, u^{[r-1]}(a) = u^{[r-1]}(b) = 0, r = 1, 2, ..., n \right\} \\ D_0^+ &: = \left\{ v : v \in D^+, v_+^{[r-1]}(a) = v_+^{[r-1]}(b) = 0, r = 1, 2, ..., n \right\} \end{aligned}\right\} \qquad ... (3.15)$$

respectively. The subspaces D_0 and D_0^+ are dense in $L_w^2 (a, b)$ and T_0 and T_0^+ are closed operators (see [3], [4], [5] and [18, Section 3]).

In the singular problem we first introduce operators T_0' and (T_0^+); T_0' being the restriction of $w^{-1}\ \tau[\cdot]$ to

$$D_0' = \{u : u \in D, \text{ supp } u \subset (a, b)\},$$

and with (T_0^+) defined similarly. These operators are densely-defined and closable in $L_w^2 (a, b)$; and we defined the minimal operators T_0 and T_0^+ to be their respective closures (see [3], [5] and [18, Section 5]). We denote the domains of T_0 and T_0^+ by D_0 and D_0^+ respectively.

It can be shown that

$$u \in D_0 \Rightarrow u^{[r-1]}(a) = 0, (r = 1, 2, ..., n), \qquad ... (3.16)$$

and

$$v \in D_0^+ \Rightarrow v_+^{[r-1]}(a) = 0, (r = 1, 2, ..., n) \qquad ... (3.17)$$

because we are assuming that a is a regular end-point. Moreover, in both the regular and singular problems, we have

$$T_0^* = T^+, \; T_0^+ = T^*; \qquad ... (3.18)$$

see [18, Section 5] in the case when $\tau = \tau^+$ and compare with the treatment in [3, Section III.10.3] and [5] in general case.

We see from (3.18) that $T_0 \subset T = (T_0^+)^*$ and hence T_0 and T_0^+ form an adjoint pair of closed, densely-defined operators in $L_w^2(a, b)$. By [3, Corollary III.3.2], def $(T_0 - \lambda I)+$ def $(T_0^+ - \bar{\lambda} I)$ is constant on the joint field of regularity $\Pi(T_0, T_0^+)$ and we have shown in [5] that,

$$n \leq \text{def } (T_0 - \lambda I) + \text{def } (T_0^+ - \bar{\lambda} I) \leq 2n \text{ for all } \lambda \in \Pi(T_0, T_0^+). \qquad ... (3.19)$$

In the regular problem,

$$\text{def } (T_0 - \lambda I) + \text{def } (T_0^+ - \bar{\lambda} I) = 2n, \text{ for all } \lambda \in \Pi(T_0, T_0^+). \qquad ... (3.20)$$

Theorem 3.1 — *Suppose $f \in L_{loc}^1(I)$ and suppose that the conditions (3.1) are satisfied. Then given any complex numbers $c_j \in \mathbb{C}, j = 0, 1, ..., n-1$ and $x_0 \in (a, b)$ there exists a unique solution of $\tau[\phi] = wf$ in (a, b) which satisfies*

$\phi^{[j]}(x_0) = c_j$, j = 0, 1, ..., n − 1.

PROOF : See [1], [3] and [12, Part II, Theorem 16.2.2].

Theorem 3.2 — (*cf.* [9, Theorem II.2.5]) — *Let τ be a regular quasi-differential expression of order n on [a, b]. For $f \in L_w^2(a, b)$, the equation $w^{-1}\tau[u] = f$ has a solution $\phi \in V(\tau)$ satisfying,*

$$\phi^{[r]}(a) = \phi^{[r]}(b) = 0, \; (r = 0, 1, ..., n-1)$$

if and only if f is orthogonal in $L_w^2(a, b)$ to the solution space of $\tau^+[\psi] = 0$, i.e.,

$$R[T_0(\tau) - \lambda I)] = N[T(\tau^+) - \bar{\lambda} I)^{\perp}. \qquad ... (3.21)$$

Corollary 3.3 — (*cf.* Corollary II.2.6]

As a result from Theorem 3.2, we have that

$$R[T_0(\tau) - \lambda I]^{\perp} = N(T(\tau^+) - \bar{\lambda} I].$$

4. SPECTRA OF PRODUCTS OF OPERATORS

We start by listing some properties and results of general quasi-differential expressions $\tau_1, \tau_2, \ldots, \tau_n$ each of order n. For proofs the reader is referred to [7] and [17].

$$(\tau_1 + \tau_2)^+ = \tau_1^+ + \tau_2^+ \qquad \ldots (4.1)$$

$$(\tau_1\, \tau_2)^+ = \tau_2^+\, \tau_1^+ \,, \; (\lambda \tau)^+ = \bar{\lambda} \tau^+ \text{ for } \lambda \text{ a complex number.} \qquad \ldots (4.2)$$

A consequence of Properties (4.1) and (4.2) is that if $\tau = \tau^+$ then $p(\tau)^+ = p(\tau)^+$ for p any polynomial with complex coefficients. Also we note that the leading coefficients of a product is the product of the leading coefficients. Hence, the product of regular differential expressions is regular.

Lemma 4.1 — Suppose τ_j is a regular differential expressions on the interval $[a, b)$ and $\lambda \in \prod_{j=1} [T_0(\tau_1\, \tau_2 \ldots \tau_n), T_0(\tau_1, \tau_2 \ldots \tau_n)^+]$, then we have

(*i*) the product operator $\prod_{j=1}^{n} [T_0(\tau_j)]$ is closed, densely-defined and

$$\text{def}\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right] = \sum_{j=1}^{n} \text{def}\, [T_0(\tau_j) - \lambda I] \text{ and,}$$

$$\text{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right] = \sum_{j=1}^{n} \text{def}\, [T_0(\tau_j^+) - \bar{\lambda} I].$$

(*ii*) $[T_0(\tau_1\, \tau_2 \ldots \tau_n)] \subseteq \prod_{j=1}^{n} [T_0(\tau_j)]$ and $[T_0(\tau_1\, \tau_2 \ldots \tau_n)^+] \subseteq \prod_{j=1}^{n} [T_0(\tau_j^+)]$.

Lemma 4.2 — Let $\tau_1, \tau_2, \ldots, \tau_n$ be regular differential expressions on $[a, b)$ suppose that $\lambda \in \Pi\, [T_0(\tau_1\, \tau_2 \ldots \tau_n), T_0(\tau_1\, \tau_2 \ldots \tau_n)^+]$, then

$$[T_0(\tau_1\, \tau_2 \ldots \tau_n)] = \sum_{j=1}^{n} [T_0(\tau_j)], \qquad \ldots (4.3)$$

if and only if the following partial separation condition is satisfied:

$f \in L_w^2(a, b), f^{[s-1]} \in AC_{loc}(a, b)$, where s is the order of the product expression $\tau_1\, \tau_2 \ldots \tau_n$ and $(\tau_1\, \tau_2 \ldots \tau_n)^+ f \in L_w^2(a, b)$ together imply that $\left(\prod_{j=1}^{n} \tau_j^+\right) f \in L_w^2(a, b)$ for $k = 1, \ldots, n-1$.

$$\ldots (4.4)$$

Furthermore, $[T_0(\tau_1 \tau_2 \dots \tau_n)] = \prod_{j=1}^{n} [T_0(\tau_j)]$ and $[T_0(\tau_1 \tau_2 \dots \tau_n)] = \prod_{j=1}^{n} [T_0(\tau_j^+)]$ if and only if,

$$\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right] = \sum_{j=1}^{n} \operatorname{def}\,[T_0(\tau_j) - \lambda I] \text{ and,}$$

$$\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right] = \sum_{j=1}^{n} \operatorname{def}\,[T_0(\tau_j^+) - \bar{\lambda} I].$$

We shall say that the product $\tau_1 \tau_2 \dots \tau_n$ is partially separated expressions in $L^2_w(a, b)$ whenever property (4.4) holds.

Lemma 4.3 — $\Pi[T_0(\tau_1 \tau_2 \dots \tau_n), T_0(\tau_1 \tau_2 \dots \tau_n)^+] = \Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} [T_0(\tau_j^+)\right]$.

PROOF : Let $\lambda \in \Pi\,[T_0(\tau_1 \tau_2 \dots \tau_n), T_0(\tau_1 \tau_2 \dots \tau_n)^+]$, then from definition of the field of regularity we have $\lambda \in \Pi\,[T_0(\tau_1 \tau_2 \dots \tau_n)]$ and $\bar{\lambda} \in \Pi\,[T_0(\tau_1 \tau_2 \dots \tau_n)^+]$, i.e., each of the operators $[T_0(\tau_1 \tau_2 \dots \tau_n) - \lambda I]$ and $[T_0(\tau_1 \tau_2 \dots \tau_n)^+ - \lambda I]$ has closed range and densely-defined on H with finite deficiency indices.

Consequently, by Lemma 4.2 each of the operators $\prod_{j=1}^{n} [T_0(\tau_j) - \lambda I]$ and $\prod_{j=1}^{n} [T_0(\tau_j^+) - \bar{\lambda} I]$ has closed range and their deficiency indices are finite, i.e., $\lambda \in \Pi\left[\prod_{j\ 1}^{n} [T_0(\tau_j)], \prod_{j=1}^{n} [T_0(\tau_j^+)]\right]$. The rest of the proof follows from definition and Lemma 4.2.

Corollary 4.4 — Let τ_j be a regular differential expressions on $[a, b)$ for $j = 1, \dots, n$. If all solutions of the differential equations $(\tau_j - \lambda I)y = 0$ and $(\tau_j^+ - \bar{\lambda} I)y = 0$ on $[a, b)$ are in $L^2_w(a, b)$ for $j = 1, \dots, n$ and $\lambda \in \mathbb{C}$; then all solutions of $\left[\prod_{j=1}^{n} (\tau_j) - \lambda I\right] y = 0$ and $\left[\prod_{j=1}^{n} (\tau_j^+) - \bar{\lambda} I\right] y = 0$ are in $L^2_w(a, b)$ for all $\lambda \in \mathbb{C}$; see [6] and [7, Corollary 1].

Corollary 4.4 — Let τ_j be a regular differential expressions on $[a, b)$ for $j = 1, \ldots, n$. If all solutions of the differential equations $(\tau_j - \lambda I)y = 0$ and $(\tau_j^+ - \bar{\lambda} I)y = 0$ on $[a, b)$ are in $L_w^2(a, b)$ for $j = 1, \ldots, n$ and $\lambda \in \mathbb{C}$; then all solutions of $\left[\prod_{j=1}^{n} (\tau_j) - \lambda I\right] y = 0$ and $\left[\prod_{j=1}^{n} (\tau_j^+) - \bar{\lambda} I\right] y = 0$ are in $L_w^2(a, b)$ for all $\lambda \in \mathbb{C}$; see [6] and [7, Corollary 1].

Corollary 4.5 — Suppose τ_1, τ_2 and $\tau_1 \tau_2$ are all regular differential expressions on $[a, b)$. Then the product is in the maximal deficiency case at b if and only if both τ_1 and τ_2 are in the maximal deficiency case at b; see [7, Corollary 2].

Theorem 4.6 — *The point spectra* $\sigma_p\left[\prod_{j=1}^{n} T_0(\tau_j)\right]$ *and* $\sigma_p\left[\prod_{j=1}^{n} T_0(\tau_j^+)\right]$ *of* $\prod_{j=1}^{n} T_0(\tau_j)$ *and* $\prod_{j=1}^{n} T_0(\tau_j^+)$ *are empty.*

PROOF : Let $\lambda \in \sigma_p\left(\prod_{j=1}^{n} [T_0(\tau_j)]\right)$. Then there exists a non-zero element

$\phi \in D_0\left[\prod_{j=1}^{n} (\tau_j)\right]$, such that

$$\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]\phi = 0.$$

In particular, this gives that

$$\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]\phi = \lambda w \phi,$$

$$\phi^{[r]}(a) = \phi^{[r]}(b) = 0, \; (r = 0, 1, \ldots, n^2 - 1).$$

From Theorem 3.1, it follows that $\phi \equiv 0$ and hence $\sigma_p\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \varnothing$.

Similarly $\sigma_p\left[\prod_{j=1}^{n} T_0(\tau_j^+)\right] = \varnothing$.

Theorem 4.7 — *(i)* $\rho\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \emptyset$; *(ii)* $\sigma_p\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \sigma_c\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \emptyset$; *and* *(iii)* $\sigma\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \sigma_r\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \mathbb{C}$.

PROOF : (*i*) Since $R\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]$ is a proper closed subspace of $L_w^2(a, b)$, then the resolvent set $\rho\left[\prod_{j=1}^{n} T_0(\tau_j)\right]$ is empty.

(*ii*) Since $R\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]$ is closed, then the continuous spectrum of $\prod_{j=1}^{n} T_0(\tau_j)$ is the empty set, i.e., $\sigma_c\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \emptyset$.

(*iii*) From (i), (ii) and Theorem 4.6, it follows that

$$\sigma\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \sigma_r\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \mathbb{C}.$$

Corollary 4.8 — (*i*) $\sigma_c\left[\prod_{j=1}^{n} T(\tau_j)\right] = \sigma_r\left[\prod_{j=1}^{n} T(\tau_j)\right] = \emptyset$,

(*ii*) $\sigma\left[\prod_{j=1}^{n} T(\tau_j)\right] = \sigma_p\left[\prod_{j=1}^{n} T(\tau_j)\right] = \mathbb{C}$ and (iii) $\rho\left[\prod_{j=1}^{n} T(\tau_j)\right] = \emptyset$.

PROOF : From Theorem 3.2 and since $\prod_{j=1}^{n} T(\tau_j) = \prod_{j=1}^{n} [T_0(\tau_j^+)]^*$ it follows that $R\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right]$ is closed for every $\lambda \in \mathbb{C}$; see [3, Theorem I.3.7]. Also, we have

$$\text{nul}\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right] = \text{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right] = \sum_{j=1}^{n} \text{def}\,[T_0(\tau_j^+) - \bar{\lambda} I] = n^2,$$

and

$$\text{def}\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right] = \text{nul}\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right] = \sum_{j=1}^{n} \text{nul}\,[T_0(\tau_j^+) - \bar{\lambda} I] = 0.$$

(*i*) Since, $R\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right]$ is closed and $\text{def}\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right] = 0$, then

$R\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right] = H$ and this yields that,

$$\sigma_c\left[\prod_{j=1}^{n} T(\tau_j)\right] = \sigma_r\left[\prod_{j=1}^{n} T(\tau_j)\right] = \varnothing,$$

(*ii*) Since, $\text{nul}\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right] = n^2$ for every $\lambda \in \mathbb{C}$, then we have that,

$\sigma_p\left[\prod_{j=1}^{n} T(\tau_j)\right] = \mathbb{C}$. It also follows that $\sigma\left[\prod_{j=1}^{n} T(\tau_j)\right] = \mathbb{C}$ and hence

$$\rho\left[\prod_{j=1}^{n} T(\tau_j)\right] = \varnothing.$$

Lemma 4.9 — (cf. [3, Lemma IX.9.1]) — If $I = [a, b]$, with $-\infty < a < b < \infty$, then for any $\lambda \in \mathbb{C}$, the operator $\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]$ has closed range, zero nullity and deficiency n^2. Hence,

$$\sigma_{ek}\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \begin{cases} \varnothing & (k = 1, 2, 3), \\ \mathbb{C} & (k = 4,5). \end{cases}$$

5. Field of Regularity of Product of Operators

We now obtain some results which in fact are a natural consequence of those in Section 4.

Theorem 5.1 — *(i)* $\Pi\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \Pi\left[\prod_{j=1}^{n} T_0(\tau_j^+)\right] = \mathbb{C}$, *and for every* $\lambda \in \mathbb{C}$,

$$\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right] = \operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right] = n^2.$$

(*ii*) $\Pi\left[\prod_{j=1}^{n} T(\tau_j)\right] = \Pi\left[\prod_{j=1}^{n} T(\tau_j^+)\right] = \varnothing$, and for every $\lambda \in \mathbb{C}$,

$$\operatorname{nul}\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right] = \operatorname{nul}\left[\prod_{j=1}^{n} T(\tau_j^+) - \bar{\lambda} I\right] = n^2.$$

Proof : (*i*) We have from Theorem 3.2 and Lemma 4.9 that, for every $\lambda \in \mathbb{C}$, $\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]^{-1}$ exists and its domain $R\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]$ is a closed subspace of $L_w^2(a, b)$. Hence, since $\left[\prod_{j=1}^{n} T_0(\tau_j)\right]$ is a closed operator, then $\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]^{-1}$ is also closed and so, it follows from the Closed Graph Theorem that, $\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]^{-1}$ is bounded and hence $\Pi\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \mathbb{C}$. From Theorem 3.2, $R\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]^{\perp}$ is the n^2-dimensional subspace of $L_w^2(a, b)$.

Thus,

$$\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right] = \dim R\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]^{\perp} = n^2, \text{ for every } \lambda \in \mathbb{C}.$$

Similarly,

$$\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right] = \dim R\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right]^{\perp} = n^2, \text{ for every } \lambda \in \mathbb{C}.$$

(*ii*) As $\Pi\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \mathbb{C}$, we have, for every $\lambda \in \mathbb{C}$ that $\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right]$ has closed range and so, since $\left[\prod_{j=1}^{n} T(\tau_j)\right] = \left[\prod_{j=1}^{n} T_0^*(\tau_j^+)^*\right]$, then $\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right]$ has closed range, see [3, Theorem I.3.7]. Furthermore, from (i),

$$\operatorname{nul}\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right] = \operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right] = n^2.$$

Hence, $\lambda \notin \Pi\left[\prod_{j=1} T(\tau_j)\right]$ and so part (ii) of the theorem follows.

Corollary 5.2 — The operators $\prod_{j=1}^{n} T_0(\tau_j)$, $\prod_{j=1}^{n} T_0(\tau_j^+)$ form a compatible adjoint pair with

$$\Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^+)\right] = \mathbb{C}.$$

PROOF : From part (*i*) of Theorem 5.1, it follows that,

$\Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^+)\right] = \mathbb{C}$. Using (3.18), the Corollary follows.

Theorem 5.3 — *If for some* $\lambda_0 \in \mathbb{C}$, *there are* n^2 *linearly independent solutions of the equations*

$$\left[\prod_{j=1}^{n} (\tau_j)\right]\phi = \lambda_0 w\phi \text{ and } \left[\prod_{j=1}^{n} (\tau_j^+)\right]\psi = \bar{\lambda} w\psi,$$

are in $L_w^2(a, b)$ for all $\lambda \in \mathbb{C}$.

PROOF : The proof follows from {9, Proposition III.3.24] and Cororllary 4.4.

From Corollary 5.2 and Theorem 5.3 we have the following Lemma.

Lemma 5.4 — If, for some $\lambda_0 \in \mathbb{C}$, there are n^2 linearly independent solutions of the equations

$$\left[\prod_{j=1}^{n}(\tau_j)\right]\phi=\lambda_0 w\phi \text{ and } \left[\prod_{j=1}^{n}(\tau_j^{+})\right]\psi=\bar{\lambda}_0 w\psi,$$

in $L^2_w(a,b)$, then $\lambda_0 \in \Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^{+})\right]$;

see also [13, Theorem 2.1] and [15, Lemma 5.1].

Theorem 5.5 — *Let* $\prod_{j=1}^{n} T_0(\tau_j)$ *and* $\prod_{j=1}^{n} T_0(\tau_j^{+})$ *be the product of the minimal operators* $T_0(\tau_j)$ *and* $T_0(\tau_j^{+})$ *respectively, defined on the interval* $[a, b)$. *If* $\Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^{+})\right]$ *is empty, then*

$$\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j)-\lambda I\right]+\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j^{+})-\bar{\lambda} I\right]\neq 2n^2.$$

In particular, if $\Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^{+})\right]$ is empty and $n = 1$, then

$$\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j)-\lambda I\right]+\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j^{+})-\bar{\lambda} I\right]=1.$$

PROOF : If $\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j)-\lambda I\right]=\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j^{+})-\bar{\lambda} I\right]=n^2$ for some $\lambda_0 \in \mathbb{C}$, then,

$$\left[\prod_{j=1}^{n}(\tau_j)\right]\phi=\lambda_0 w\phi \text{ and } \left[\prod_{j=1}^{n}(\tau_j^{+})\right]\psi=\bar{\lambda}_0 w\psi,$$

each have $n^2 - L^2_w(a,b)$ solutions (see [6]). Hence by Theorem 5.3, we have that all solutions of

$$\left[\prod_{j=1}^{n}(\tau_j)\right]\phi=\lambda w\phi \text{ and } \left[\prod_{j=1}^{n}(\tau_j^{+})\right]\psi=\bar{\lambda} w\psi,$$

are in $L^2_w(a,b)$ for all $\lambda \in \mathbb{C}$, and hence by Corollary 5.2, we have that

$$\lambda \in \Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^+)\right].$$

Thus if

$$\Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^+)\right]$$

is empty, we cannot have

$$\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right] + \operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right] = 2n^2.$$

In particular, if $n = 1$, then the relation (3.21) gives that,

$$1 \leq \operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right] + \operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right] \leq 2,$$

so if $\Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^+)\right]$ is empty we have

$$\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right] + \operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right] = 1.$$

For a regularly solvable operator, we have the following general theorem:

Theorem 5.6 — *Suppose for a regularly solvable extension of the product of minimal operator* $\prod_{j=1}^{n} T_0(\tau_j)$ *that*

$$\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right] + \operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right] = N, \; n^2 \leq N \leq 2n^2,$$

for all $\lambda \in \Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^+)\right].$

Then
$$\text{nul}\left[\prod_{j=1}^{n} T(\tau_j)-\lambda I\right]+\text{nul}\left[\prod_{j=1}^{n} T(\tau_j^+)-\bar{\lambda} I\right]\leq N \text{ for all } \lambda\in \mathbb{C}.$$

If $\Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^*)\right]$ is empty, then

$$\text{nul}\left[\prod_{j=1}^{n} T(\tau_j)-\lambda I\right]+\text{nul}\left[\prod_{j=1}^{n} T(\tau_j^+)-\bar{\lambda} I\right]< N.$$

PROOF : Let $\text{def}\left[\prod_{j=1}^{n} T_0(\tau_j)-\lambda I\right]=r$, $\text{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+)-\bar{\lambda} I\right]=s$ such that

$$\text{def}\left[\prod_{j=1}^{n} T_0(\tau_j)-\lambda I\right]+\text{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+)-\bar{\lambda} I\right]=r+s,\ n^2\leq r+s\leq 2n^2,$$

for all $\lambda\in \Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^+)\right]$.

Then for any closed extension S of $\prod_{j=1}^{n} T_0(\tau_j)$ which is regularly solvable with respect to $\prod_{j=1}^{n} T_0(\tau_j)$ and $\prod_{j=1}^{n} T_0(\tau_j^+)$, we have from [3, Theorem III.3.5] that,

$$\dim\left\{D(S)/D_0\left[\prod_{j=1}^{n}(\tau_j)\right]\right\}=\text{def}\left[\prod_{j=1}^{n} T_0(\tau_j)-\lambda I\right]=r$$

and

$$\dim\left\{D(S^*)/D_0\left[\prod_{j=1}^{n}(\tau_j^+)\right]\right\}=\text{def}\left[\prod_{j=1}^{n} T_0(\tau_j^+)-\bar{\lambda} I\right]=s.$$

Hence, S and S^* are finite dimensional extensions of $\prod_{j\ 1}^{n} T_0(\tau_j)$ and $\prod_{j=1}^{n} T_0(\tau_j^+)$ respectively. Thus from [3, Corollary IX.4.2], we get

$$\sigma_{ek}\left[\prod_{j\ 1}^{n} T_0(\tau_j)\right] = \sigma_{ek}(S),\ (k = 1, 2, 3). \qquad \dots (5.1)$$

Since $\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]$ has closed range, zero nullity and deficiency r (see Lemma 4.9), then for any $\lambda \in \mathbb{C}$, we have that

$$\Pi\left[\prod_{j=1}^{n} T_0(\tau_j)\right] \cap \sigma_{ek}\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \varnothing,\ (k = 1, 2, 3).$$

By (5.1), we have that

$$\Pi\left[\prod_{j=1}^{n} T_0(\tau_j)\right] \cap \sigma_{ek}(S) = \varnothing,\ (k = 1, 2, 3).$$

Therefore,

$$\Delta_k\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \Delta_k(S) = \mathbb{C},\ (k = 1, 2, 3).$$

Similarly,

$$\Delta_k\left[\prod_{j=1}^{n} T_0(\tau_j^{+})\right] = \Delta_k(S^*) = \mathbb{C},\ (k = 1, 2, 3).$$

Furthermore, the equations

$$\left[\prod_{j=1}^{n} (\tau_j)\right]\phi = \lambda w \phi \text{ and } \left[\prod_{j=1}^{n} (\tau_j^{+})\right]\psi = \bar{\lambda} w \psi,$$

has at most r and s linearly independent solutions for $\lambda \in \mathbb{C}$ respectively.

Hence,

$$\text{nul}\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right] + \text{nul}\left[\prod_{j=1}^{n} T(\tau_j^{+}) - \bar{\lambda} I\right] \le N \text{ for all } \lambda \in \mathbb{C}.$$

But if, for any $\lambda_0 \notin \Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^+)\right]$, then either $\lambda_0 \notin \Pi\left[\prod_{j=1}^{n} T_0(\tau_j)\right]$ or $\lambda_0 \notin \Pi\left[\prod_{j=1}^{n} T_0(\tau_j^+)\right]$.

If $\lambda_0 \notin \Pi\left[\prod_{j=1}^{n} T_0(\tau_j)\right]$, then either λ_0 is an eigenvalue of $\prod_{j=1}^{n} T_0(\tau_j)$ or $R\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right]$ is not closed. Similarly for $\bar{\lambda}_0 \notin \Pi\left[\prod_{j=1}^{n} T_0(\tau_j^+)\right]$. But $\prod_{j\ 1}^{n} T_0(\tau_j)$ and $\prod_{j=1}^{n} T_0(\tau_j^+)$ have no eigenvalues, then if $\lambda_0 \notin \Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^+)\right]$, we have $R\left[\prod_{j=1}^{n} T(\tau_j) - \lambda_0 I\right]$ and $R\left[\prod_{j=1}^{n} T(\tau_j^+) - \bar{\lambda}_0 I\right]$ are both not closed and so we cannot have

$$\text{nul}\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right] + \text{nul}\left[\prod_{j=1}^{n} T(\tau_j^+) - \bar{\lambda} I\right] = N.$$

Hence,

$$\text{nul}\left[\prod_{j=1}^{n} T(\tau_j) - \lambda I\right] + \text{nul}\left[\prod_{j=1}^{n} T(\tau_j^+) - \bar{\lambda} I\right] = N,$$

for any $\lambda \notin \Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^+)\right]$.

Remark : It remains an open question as to how many of the solutions of

$$\left[\prod_{j=1}^{n} (\tau_j)\right] \phi = \lambda w \phi \text{ and } \left[\prod_{j=1}^{n} (\tau_j^+)\right] \psi = \lambda w \psi,$$

may be in $L_w^2(a, b)$ for any $\lambda \in \mathbb{C}$ when $\Pi\left[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n} T_0(\tau_j^+)\right]$ is empty, except that we know from above that not all of them are in $L_w^2(a, b)$. We refere to [2, 7, 13, 14, 15] for more details.

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