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ON SOME GENERALIZED MULTI-VALUED VARIATIONAL INEQUALITIES

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In this paper, we have introduced a particular kind of generalized quasi-variational inequality and have proved an existence theorem for its solution in reflexive Banach spaces under assumption of h-pseudo-monotonicity and h-hemi-continuity. A fixed-point theorem of h-pseudo-monotone mapping is also proved, which is a further generalization of such result obtained earlier by Singh, Tarafdar, and Watson[22].

Key Words : Variational Inequality; h-Pseudo-monotonicity; h-Hemi-continuity Hausdorff Spaces and Minimization Problems

1. INTRODUCTION

Some of the recent treatment of existence of solution of variational-inequalities are formulated in the framework of KKM-map principle or alternatively under fixed point consideration. In the present article, we try to obtain some results which deal with generalized quasi-variational inequalities of particular type which also take care of multivalued analogue of variational-inequalities under the similar framework of KKM-mapping principle along with the appropriately formulated fixed point theorems.

Some of the earlier works in this direction were done by authors like Shih and Tan[18 & 19], Ding and Tan[8], Fan[9], Siddiqi, Ansari and Khaliq[20], Siddiqi, Ahmad & Khan[21], and Singh, Tarafdar & Watson[22]. Giannessi has dealt vector-variational inequalities in a finite dimensional Euclidean space while Chen and Chen[5], Chen and Yang[6], Siddiqi, Ansari and Khaliq[20] and Lee, *et al.*[14] have studied vector variational-inequalities in abstract space. The existence theory of solutions to the generalized quasi-variational-inequalities problem has been recently established by Chan and Pang[4], Shih and Tan[18 & 19], Ding and Tan[8] and others in different setting. Bhattacharya and Vetrivel[2] have obtained a general existence theorem of solutions to the generalized quasi-variational inequalities (GQVI) problem which contains the existence theorem of Ky Fan's best approximation problem. In 1968, Browder [3, theorem 1] established his famous fixed-point theorem which is equivalent to celebrated Fan's lemma[9]. Fan's lemma[9] is an infinite-dimensional generalization of the classical KKM theorem.

The paper is organized as follows. In the next section, we will give some preliminaries and notations. Subsequently, we state and prove some results on existence of solution to the generalized multivalued quasi-variational inequalities (GMQVI) which extend the results of Siddiqi, Ahmad, and Khan[21]. Further, in later section, an appropriate fixed point theorem is used to obtain an existence result of solution for GMQVI problems. This last stated result is a generalized form of earlier results of similar type of Singh, Tarafdar and Watson[22].

2. Preliminaries and Notations

Let X and Y be two normed spaces and K a nonempty closed and convex subset of X. Let $A : L(X, Y) \to L(X, Y)$ be a mapping, $T : K \to 2^{L(X, Y)}$ be a multivalued mapping, where $L(X, Y)$ is the space of all continuous mapping from X into Y. Let $\{C(x) : x \in K\}$ be a family of closed pointed convex cones in Y with int $C(x) \neq \phi$ $\forall\, x \in K$, where int $C(x)$ is the interior of the set $C(X)$.

Siddiqi *et al.*[21] considered the following (GMVVI) problem:

Find $x_0 \in K$ such that for each $x \in K$, $\exists$ $s_0 \in T(x_0)$ s.t.

$$\langle As_0, x - x_0 \rangle \notin \text{-int } C(x_0),$$

where $\langle As_0, y \rangle$ denotes the evaluation of the linear mapping As_0 at y.

We will be interested in the following problem :

Find $x_0 \in K\, s.\, t.$

$$\langle As_0, x - x_0 \rangle + h(x) - h(x_0) \notin - \mathit{int}\ C(x_0),$$

where $h : X \to Y$ is a lower semi-continuous and convex functional.

We will call it generalized multivalued quasi-variational-inequality problem (GMQVI).

Definitions — Let X, Y be two normed spaces, $A : L(X, Y) \to L(X, Y)$ be a mapping and $T : X \to 2^{L(X, Y)}$ be a multivalued mapping and let $C = \bigcap_{x \in K \subset X} C(x)$ be non empty.

1. A is said to be h-monotone in C regarding to T if for any $x, y \in X, s \in T(x)$ and $t \in T(y)$, such that

$$\langle As - At, x - y \rangle + h(x) - h(y) \in C,$$

where $h : X \to Y$ is l.s.c. and convex functional

2. A is said to be h-pseudo-monotone in C regarding to T if for any $x, y \in X$ $\exists$ $s \in T(x)\, s.\, t.$

$\langle As, y - x \rangle + h(y) - h(x) \notin - \mathit{int}\ C(x)$, where $h : X \to R$ is l.s.c and convex functional, implies that $\exists$ $t \in T(y)\, s.\, t.$

$$\langle At, y - x \rangle + h(y) - h(x) \notin -\mathit{int}\, C(x).$$

3. A is said to be h-hemi-continuous if for any $x, y \in X, \alpha > 0$ and $t_\alpha \in T(x + \alpha y)$, $\exists$ $t_0 \in T(y)$ s.t. for any $z \in X$,

$$\langle At_\alpha, z \rangle + h(z) \to \langle At_0, z \rangle + h(z) \text{ as } \alpha \to 0^+.$$

Existence Theorem : *Let X be a reflexive Banach space and Y is a Banach space. Let K be a nonempty closed bounded convex subset of X.*

Let us assume that

1. Let $C : K \to 2^Y$ be a multivalued mapping such that for each $x \in K$, $C(x)$ is closed pointed convex cone with int $C(x) \neq \phi$ and let $C = \bigcap_{x \in K} C(x)$ with a nonempty interior in $C \neq \phi$;

2. The mapping $W(x) = Y \{\text{-int } C(x)\}$ is upper semi-continuous (u.s.c.) and concave;

3. $T : X \to 2^{L(X, Y)}$ is a compact valued, multivalued mapping;

4. $A : L(X, Y) \to L(X, Y)$ is h-pseudo-monotone and h-hemi-continuous. Then (GMQVI) problem is solvable.

PROOF : Let $F_1 = \{x \in K : \exists\ s \in T(x)\ s.t. \langle As, y - x \rangle + h(y) - h(x) \notin -\ int\ C(x)\}$, for any $y \in K$, where $h : K \to Y^+$ is a lower semi-continuous (l.s.c.) and convex functional.

We have to prove that F_1 is KKM-mapping on K.

Suppose $\{x_1, x_2, ..., x_n\} \subset K$, $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geq 0$, $i = 1, 2, ..., n$, and

$$x = \sum_{i=1}^{n} \alpha_i x_i \notin \bigcup_{i=1}^{n} F_i(x_i).$$

Then for any $s \in T(x)$,

$$\langle As, x_i - x \rangle + h(x_i) - h(x) \in -\ int\ C(x), i = 1, 2, ..., n;$$

Then we have

$$\langle As, x \rangle + h(x) = \left\langle As, \sum_{i=1}^{n} \alpha_i x_i \right\rangle + h\left(\sum_{i=1}^{n} \alpha_i x_i \right)$$

$$= \sum_{i=1}^{n} \alpha_i (\langle As, x_i \rangle + h(x_i))$$

$$\in \sum_{i=1}^{n} \alpha_i (\langle As, x \rangle + h(x) - int\ C(x))$$

$$= \langle As, x \rangle + h(x) - int\ C(x).$$

Hence, $0 \in$ int $C(x)$, which contradicts the pointedness of $C(x)$.

Therefore, we have derived that F_1 is a KKM-mapping on K.

Now, let us define a set-valued mapping

$F_2 : K \to 2^K$ by following, i.e., for any $y \in K$,

$$F_2(y) = \{x \in K : \exists\ t \in T(y)\ s.t. \langle At, y - x \rangle + h(y) - h(x) \notin -\ int\ C(x)\}.$$

Let $x \in F_1(y)$, then $\exists\, s \in T(x)$ $s.\,t.$

$$\langle As, y - x \rangle + h(y) - h(x) \notin -\, int\ C(x)$$

Since A is h-pseudo-monotone, therefore $\exists\, t \in T(y)$ $s.\ t.$

$$\langle At, y - x \rangle + h(y) - h(x) \notin -\, int\ C(x), \text{ for any } y \in K.$$

That is, $x \in F_2(y)$. Hence, $F_1(y) \subseteq F_2(y)$, for any $y \in K$.

Therefore, F_2 is a KKM-mapping on K.

Again, for any $y \in K$, we shall show that $F_2(y)$ is closed. Indeed let $\{x_n\}$ be a sequence in $F_2(y) s.\,t.$,

$$x_n \to x_0 \in K.$$

Since $x_n \in F_2(y)$ for all n, $\exists\, t \in T(y)$ $s.t.$,

$$\langle At_n, y - x_n \rangle + h(y) - h(x_n) \notin -\, int\ C(x_n)$$

or,

$$\langle At_n, y - x_n \rangle + h(y) - h(x) \in W(x_n)$$

Since $T(y)$ is compact, we can assume that $\exists\, t_0 \in T(y)$ $s.\,t.$ $t_n \to t_0$.

Since $\langle \cdot, \cdot \rangle$ is continuous, $W(x)$ is u.s.c. and $t_n \to 0, x_n \to x_0$, we have

$$\langle At_n, y - x_n \rangle + h(y) - h(x_n) \to \langle At_0, y - x_0 \rangle + h(y) - h(x_0) \in W(x_0).$$

Hence, $\langle At_0, y - x_0 \rangle + h(y) - h(x_0) \notin -\, int\ C(x_0)$. Therefore, $x_0 \in F_2(y)$ and so $F_2(y)$ is closed.

Now we will show that $F_2(y)$ is convex.

Let $x_1, x_2 \in F_2(y)$ and $\alpha, \beta \geq 0$ $s.t.$ $\alpha + \beta = 1$.

Then $\exists\, t \in T(y)$ $s.\,t.$

$$\langle At, y - x_1 \rangle + h(y) - h(x_1) \notin -\, int\ C(x_1); \qquad \ldots (1)$$

and

$$\langle At, y - x_2 \rangle + h(y) - h(x_2) \notin -\, int\ C(x_2) \qquad \ldots (2)$$

Multiplying (1) by α and (2) by β and adding (1) & (2), we have

$$\alpha \langle At, y - x \rangle + \beta \langle At, y - x_2 \rangle + \alpha h(y) - \alpha h(x_1) + \beta h(y) - \beta h(x_2).$$

$$\in \alpha W(x_1) + \beta W(x_2).$$

Since W is concave, we have

$$\langle At, y-(\alpha x_1+\beta x_2)\rangle + h(y) - h(\alpha x_1+\beta x_2) \in W(\alpha x_1+\beta x_2).$$

Hence, $\alpha x_1+\beta x_2 \in F_2(y)$, therefore $F_2(y)$ is convex.

Since K is closed, bounded and convex subset of a reflexive Banach space X, K is weakly compact.

Since $F_2(y) \subset K$, $F_2(y)$ is weakly compact.

By KKM-Fan Lemma 2.1[9]

$$\bigcap_{x\in K} F_2(y) \neq \phi.$$

Let $x \in \bigcap_{x\in K} F_2(y) \neq \phi$. Then for any $y \in K, \exists\, t_y \in T(y)$

s. t. $\langle At_y, y-x\rangle + h(y) - h(x) \notin -\,int\, C(x)$. By convexity of K, for any

$$\alpha \in (0,1), \exists\, t_\alpha \in T(\alpha y+(1-\alpha)x)\ s.\,t.$$

$$\langle At_\alpha, \alpha(y-x) + \alpha(h(y)-(h(x))\rangle \notin -\,int\, C(x)$$

Dividing by α, we get $\langle At_\alpha, y-x\rangle + h(y) - h(x) \notin -\,int\, C(x)$

By h-hemi-continuity of A, $\exists\, t_0 \in T(x)\ s.\,t.$

$$\langle At_0, y-x\rangle + h(y) - h(x) \notin -\,int\, C(x)$$

Hence,

$$x \in \bigcap_{y\in K} F_1(y)$$

Then

$$\bigcap F_1(y) \neq \phi$$

Consequently, there exists x_0 s.t. for each $x \in K, \exists\, s_0 \in T(x_0)\ s.\,t.$

$$\langle As_0, x-x_0\rangle + h(x) - h(x_0) \notin -\,int\, C(x_0).$$

Hence the theorem.

3. Fixed Point Theorem

The following lemma is needed in the sequel :

Lemma 3.1 — If $T: K \to 2^{L(X, Y)}$ is a compact valued, multivalued mapping and $A : L(X, Y) \to L(X, Y)$ is h-pseudo-monotone and h-hemi continuous then $x_0 \in K$ is a solution of

$$\langle As_0, x - x_0 \rangle + h(x) - h(x_0) \notin -int\, C(x_0) \text{ for all } x \in K, \text{ where } s_0 \in T(x_0) \qquad \text{... (3.1)}$$

iff $x_0 \in K$ is a solution of

$$\langle As, x - x_0 \rangle + h(x) - h(x_0) \notin -int\, C(x_0) \text{ for all } x \in K \text{ and } s \in T(x). \qquad \text{... (3.2)}$$

PROOF : Let $x_0 \in K$ be a solution of (3.1). Then by h-pseudo-monotone of A, we have

$$\langle As, x - x_0 \rangle + h(x) - h(x_0) \notin -int\, C(x_0) \text{ for all } x \in K, \text{ where } s \in T(x).$$

Now assume that x_0 satisfies (3.2) and let $x \in K$ be arbitrary. Then

$$x_1 = (1 - t)\, x_0 + tx \in K \text{ for all } t \in (1, 0). \text{ Since } K \text{ is convex, } \exists\, s_1 \in T(x_1).$$

$$\langle As_1, x_1 - x_0 \rangle + h(x_1) - h(x_0) \notin -int\, C(x_0)$$

So,

$$\langle As_1, (1 - t), x_0 + tx - x_0 \rangle + h(1 - t)\, x_0 + tx) - h(x_0) \notin -int\, C(x_0)$$

$$\langle As_1, t(x - x_0) \rangle + t(h(x) - h(x_0)) \notin -int\, C(x_0)$$

If $0 < t < 1$, then

$$\langle As_1, x - x_0 \rangle + h(x) - h(x_0) \notin -int\, C(x_0)$$

As

$$t \to 0, x_1 \to x_0,$$

We get

$$\langle As_0, x - x_0 \rangle + h(x) - h(x_0) \notin -int\, C(x_0).$$

Theorem 3.1 — *Let L be a nonempty closed and convex subset of reflexive Banach space X. Let* $A : L(X, Y) \to L(X, Y)$ *be mapping of h-pseudo-monotone & h-hemi-continuous type,* $T : K \to 2^{L(X, Y)}$ *be multivalued mapping and* $h : K \to Y$ *be a l.s.c. and convex function. Further assume that there exists an x zero nonempty set* K_0 *contained in a compact, convex subset* K_1 *of K s. t. the set*

$$D = \bigcap_{x_0 \in K_0} \bigcap_{s_0 \in T(x_0)} \{x \in K : \langle As_0, x_0 - x \rangle + h(x_0) - h(x) \notin -\,int\, C(x)\}$$

is either empty or compact.

Then there exists a $x_0 \in K$ such that

$$\langle As_0, x - x_0 \rangle + h(x) - h(x_0) \notin -\,int\, C(x_0)$$

PROOF : Let us suppose that for $x \in K$ there exists $x_0 \in K$ $s.\,t.$ $s_0 \in T(x_0)$ and

$$\langle As_0, x - x_0 \rangle + h(x_0) - h(x) \in -\,int\, C(x) \qquad \text{... (3.3)}$$

Then our supposition (3.3) may or may not hold. In either case, we will prove the existence of a $x_0 \in K$, where $s_0 \in T(x_1)$.

First suppose that (3.3) does not hold. This means that there exists at least one $\bar{x} \in K$, $s.\,t$,

$$\langle As_0, x_0 - \bar{x} \rangle + h(x_0) - h(\bar{x}) \notin -\,int\, C(\bar{x}) \text{ for all } x_0 \in K,$$

i.e., $\bar{x} \in K$ is solution of (3.1), by lemma 3.1.

Next suppose that (3.3) holds. If possible, suppose that there is no solution under assumption (3.3).

Then $\quad x_0 \in K$, the set

$$F(x0) = \{x \in K : \langle As0, x - x0 \rangle + h(x) - h(x0) \in -\,int\, C(x0),$$

where

$$s0 \in T(x0)\}$$

must be nonempty.

It also follows from the convexity of h that the set $F(x_0)$ is convex for each $x_0 \in K$. Then $F : K \to 2^{L(X,\,Y)}$ is set-valued map with $F(x_0)$ nonempty and convex for each $x_0 \in K$.

Now for each $x_0 \in K$,

$$F^{-1}(x_0) = \{x \in K, x_0 \in F(x)\}$$

$$= \{\ x \in K : \exists\, s \in T(x), \langle As, x_0 - x \rangle + h(x_0) - h(x) \in -\,int\, C(x)\}$$

Hence, for each $x_0 \in K$,

$$(F^{-1}(x0))^c = \{x \in K : \exists\, s \in T(x), \langle As, x0 - x \rangle + h(x0) - h(x) \notin -\,int\, C(x)\}$$

since A is h-pseudo-monotone, therefore

$$\subset \{x \in K : \exists\, s_0 \in T(x_0), \langle As_0, x_0 - x \rangle + h(x_0) - h(x) \notin -\,int\, C(x)\}$$

$= G(x_0)$ which is a relatively closed subsets of K since h is l.s.c., Therefore, for all $x_0 \in K, \exists\, s_0 \in T(x_0)$ $s.\ t.$

$$F^{-1}(x_0) \subset (G(x_0))^c = Ox_0 \text{ (say), which is relatively open subset of } K.$$

Now by condition (3.3)

$$\bigcup_{x_0 \in K} Ox_0 = K$$

$$D = \bigcap_{x_0 \in K_0} \bigcap_{s_0 \in T(x_0)} G(x_0)$$

$$= \bigcap_{x_0 \in K_0} \bigcap_{s_0 \in T(x_0)} (Ox_0)^c, \text{ from given condition.}$$

Hence, by theorem A as given in Tarafdar[23] there exists a x $x_0 \in K$, $s.\ t.$ $x_0 \in F(x_0)$ i.e.,

$$\langle As_0, x_0 - x_0 \rangle + h(x_0) - h(x_0) \in -\operatorname{int} C(x_0),$$

which is impossible. Hence, there is a solution in this case.

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