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NEARLY FUZZY HAUSDORFF SPACES

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The notion of fuzzy set was introduced by Zadeh. Fuzzy topological spaces were introduced by Chang[2] and studied by many eminent authors like Lowen[6&7], Wong[9-11]. A notion of fuzzy Hausdorff space was introduced in [8]. A different notion of fuzzy Hausdorff space was introduced in [4]. In this paper, we introduce the notion of nearly fuzzy Hausdorff spaces (n.f.T_2 spaces) which generalise both the definitions of fuzzy Hausdorffness mentioned above and study the properties of such spaces in detail.

Key Words : Nearly Fuzzy T_2; Nearly Fuzzy T_1; Converge Fuzzily; Fuzzy T_2; Quotient Fuzzy Topology; Fuzzy Continuous; Fuzzy Open; Product Fuzzy Topology; Induced Topology; Topologically Generated.

INTRODUCTION

Fuzzy topological spaces were studied by C.L. Chang, R. Lowen, C. K. Wong and others[1-11]. In this paper, we introduce the notion of Nearly Fuzzy Hausdorff spaces and study their properties. In most cases we are able to generalise and find analogous results available for Hausdorff spaces and where analogous results fail, we give counter examples. Different versions of fuzzy Hausdorff spaces were introduced by Rekha Srivastava[8] and Ghanim *et al.*[4] We give their definitions and their relationship with our definition. We give example to show that the class of nearly fuzzy hausdorff spaces introduced by us is strictly larger than their class.

Definition 1 — Two fuzzy sets μ and ν of X are said to intersect at x if $\mu(x) + \nu(x) > 1$.

Definition 2 — Two fuzzy sets μ and ν of X are said to be disjoint if they do not intersect at any point of X i.e.,

$$\mu(x) + \nu(x) \leq 1, \ \forall \ x \in X.$$

Definition 3 — A point $x \in X$ is to lie in μ (i.e., $x \in \mu$) if $\mu(x) > \frac{1}{2}$.

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Definition 4 — A Fuzzy Topological Space (X, δ) is said to be a nearly fuzzy Hausdorff space $(n \cdot f \cdot T_2)$ if for every pair of elements $x \neq y$ of X, there exists nonzero disjoint fuzzy open sets $\mu, \nu \in \delta$ such that $\mu(x) > \frac{1}{2}$ and $\nu(y) > \frac{1}{2}$.

i.e., there exist fuzzy open sets $\mu \neq 0$ and $\nu \neq 0$ in δ such that $\mu(x) > \frac{1}{2}$ and $\nu(y) > \frac{1}{2}$ and $\mu(z) + \nu(z) \leq 1 \;\; \forall \; z \in X$.

Definition 5 — Let (X, δ) be a fuzzy topological space. A singleton $\{x\} \subset X$ is said to be fuzzy closed if there exists a fuzzy closed set μ with support $\{x\}$ and $\mu(x) > \frac{1}{2}$.

Note 1 — Let $X = \{x_1, x_2, \ldots, x_n\}$ and let $a_1, a_2, \ldots, a_n \in [0, 1]$, then $\mu_{x_1, x_2, \ldots, x_n}^{a_1, a_2, \ldots, a_n}$ denotes the fuzzy set $\mu \in I^X$ such that $\mu(x_1) = a_1, \mu(x_2) = a_2, \ldots, \mu(x_n) = a_n$.

Note 2 — In a nearly fuzzy T_2 space singletons need not be fuzzy closed. This can be seen from the following example.

Example 1 — Let $X = \{a, b, c\}$.

$$\delta = \Big\{ 0, 1, \mu_{a,b,c}^{z_0, 0, 0}, \mu_{a,b,c}^{0, x_0, 0}, \mu_{a,b,c}^{0, 0, y_0}, \mu_{a,b,c}^{x, x_0, 0}, \mu_{a,b,c}^{0, y, y_0}, \mu_{a,b,c}^{z_0, 0, z}$$

$$\mu_{a,b,c}^{z_0, x_0, z}, \mu_{a,b,c}^{z_0, y, y_0} \mu_{a,b,c}^{x, x_0, y_0}, \mu_{a,b,c}^{z_0, x_0, y_0}, \mu_{a,b,c}^{x, x_0, z}$$

$$\mu_{a,b,c}^{x, y, y_0}, \mu_{a,b,c}^{z_0, y, z}, \mu_{a,b,c}^{x, y, z} \Big\}$$

where

$$x, y, z > \frac{1}{2} \text{ are fixed; } x_0, y_0, z_0 < \frac{1}{2} \text{ are also fixed; and } x_0 + y < 1, y_0 + z < 1, z_0 + x < 1.$$

$$\delta^c = \Big\{ 0, 1, \mu_{a,b,c}^{1-x, 1-x_0, 1}, \mu_{a,b,c}^{1, 1-y, 1-y_0}, \mu_{a,b,c}^{1-z_0, 1, 1-z}, \mu_{a,b,c}^{1-z_0, 1, 1},$$

$$\mu_{a,b,c}^{1, 1-x_0, 1}, \mu_{a,b,c}^{1, 1, 1-y_0}, \mu_{a,b,c}^{1-z_0, 1-x_0, 1-z}, \mu_{a,b,c}^{1-z_0, 1-y, 1-y_0}$$

$$\mu_{a,b,c}^{1-x, 1-x_0, 1-y_0}, \mu_{a,b,c}^{1-z_0, 1-x_0, 1-y_0}, \mu_{a,b,c}^{1-x, 1-x_0, 1-z},$$

$$\mu_{a,b,c}^{1-x, 1-y, 1-y_0}, \mu_{a,b,c}^{1-z_0, 1-y, 1-z}, \mu_{a,b,c}^{1-x, 1-y, 1-z} \Big\}.$$

Here singletons are not fuzzy closed (since for $a \in X$, for $\mu_{a,b,c}^{r,0,0} \;\; \forall \; r > \frac{1}{2}$ is not fuzzy closed).

Remark 1 : There exists $n \cdot f \cdot T_2$ spaces in which every singleton is fuzzy closed.

Example 2 — Let $X = \{a, b, c\}$

$$\delta = \Big\{ 0, 1, \mu_{a,b,c}^{1-x, 1, 1}, \mu_{a,b,c}^{1, 1-y, 1}, \mu_{a,b,c}^{1, 1, 1-z}, \mu_{a,b,c}^{1-x, 1-y, 1} \mu_{a,b,c}^{1-x, 1, 1-z}$$

$$\mu_{a,b,c}^{1,1-y,1-z}, \mu_{a,b,c}^{1-x,1-y,1-z}\Big\} \quad \text{where } x, y, z > \frac{1}{2} \text{ are fixed.}$$

$$\mathcal{S} = \left\{0, 1, \mu_{a,b,c}^{x,0,0}, \mu_{a,b,c}^{0,y,0}, \mu_{a,b,c}^{0,0,z}, \mu_{a,b,c}^{x,y,0}, \mu_{a,b,c}^{x,0,z}, \mu_{a,b,c}^{0,y,z} \mu_{a,b,c}^{x,y,z}\right\} .$$

Let $\sigma = \langle \delta \cup \mathcal{S} \rangle$. Clearly, (X, σ) is a $n \cdot f \cdot T_2$ space and every singleton is fuzzy closed.

Example — The example (2) in which $x, y, z \leq \frac{1}{2}$ is sufficient.

Definition 6 — (R. Lowen[7]) — The initial fuzzy topology on E for the family of fuzzy topological spaces $(F_j, \gamma_j)_{j \in J}$ and the family of functions $f_j : E \to (F_j, \gamma_j)$ is the smallest fuzzy topology on E making each function f_j fuzzy continuous.

Definition 7 — Let $f : X \to Y$ be a function. Let $\mu \in I^X$ be a fuzzy subset of X, then $f(\mu)$ is a fuzzy subset of Y defined by $f(\mu)(y) = \sup_{x \in f^{-1}(y)} \mu(x)$. Let $\nu \in I^Y$ be a fuzzy subset of Y, then $f^{-1}(\nu)$ is a fuzzy subset of X defined by $f^{-1}(\nu)(x) = \sup_{x \in f^{-1}(y)} \nu(f(x))$.

Theorem 1 — *a) A fuzzy subspace of a $n \cdot f \cdot T_2$ space is $n \cdot f \cdot T_2$ space. b) Arbitrary product of $n \cdot f \cdot T_2$ spaces is $n \cdot f \cdot T_2$ space.*

PROOF : *a*) Let (X, δ) be $n \cdot f \cdot T_2$ and $Y \subseteq X$. Consider the fuzzy subspace on Y given by $\delta \mid Y$ the collection of all members of δ restricted to Y. Let $x_1, x_2 \in Y$ such that $\cdot\, x_1 \neq x_2$. Since $Y \subseteq X$ and (X, δ) is $n \cdot f \cdot T_2$ space. Hence, there exist $\mu \neq 0, \nu \neq 0 \in \delta s \cdot t\, \mu(x_1) > \frac{1}{2}$ and $\nu(x_2) > \frac{1}{2}$ and $\mu(z) + \nu(z) \leq 1, \ \forall\ z \in X$.

Now $\mu \mid Y, \nu \mid Y \in \delta \mid Y$ such that $(\mu \mid Y)(x_1) > \frac{1}{2}$ and $\nu \mid Y(x_2) > \frac{1}{2}$ and $(\mu \mid Y)(z) + (\nu \mid Y)(z) \leq 1$ $\forall\ z \in Y$ then $(Y, \delta \mid Y)$ is $n \cdot f \cdot T_2$ space.

b) Let I be any indexed set and $(X_i, \sigma_i)_{i \in I}$ be a family of $n \cdot f \cdot T_2$ space.

The product fuzzy topology $\Pi\, \sigma_i$ on $\Pi\, X_i$ is the initial fuzzy topology on $\Pi\, X_i$ for the family of fuzzy topological spaces (X_i, σ_i) and functions $f_j : \Pi\, X_i \to X_j$ where f_j, the projection on the jth coordinate.

To prove that $(\Pi\, X_i, \Pi\, \sigma_i)$ is $n \cdot f \cdot T_2$ space, let $x, y \in \Pi\, X_i$ such that $x \neq y$. $x \neq y$. then there exists $j \in I$ such that $x_j \neq y_j$ and $x_j, y_j \in X_j$. Hence, there exist $\mu_j \neq 0, \nu_j \neq 0 \in \sigma_j$ such that $\cdot$

$\mu_j(x_j) > \frac{1}{2}$, $\nu_j(y_j) > \frac{1}{2}$ and $\mu_j(z_j) + \nu_j(z_j) \leq 1 \ \forall \ z_i \in X_j$.

Consider $f_j^{-1}(\mu_j)$ and $f_j^{-1}(\nu_j)$. $f_j^{-1}(\mu_j)(x) = \mu_j(f_j(x)) = \mu_j(x_j) > \frac{1}{2}$ and similarly $f_1^{-1}(\nu_j)(y) > \frac{1}{2}$.

To prove that $f_j^{-1}(\mu_j)(2) + f_j^{1}(\nu_j)(2) \leq 1 \ \forall \ 2 \in \Pi X_i$, Suppose not $f_j^{-1}(\mu_j)(z) + f_j^{-1}(\nu_j)(z) > 1$, for some $z \in \Pi x_i$. This implies $\mu_j(f_j(z)) + \nu_j(z)) > 1$ i.e., $\mu_j(z_j) + \nu_j(z_j) > 1$, where $z_j \in X_j$, but to $\mu_j(z_j) + \nu_j(z_j) \leq 1 \ \forall \ z_j \in X_j$.. By the definition of product topology $\Pi \sigma_i$ on $\Pi X_i, f_j^{-1}(\mu_j)$ and $f_j^{-1}(\nu_j)$ are fuzzy open (since f_j is fuzzy continuous in $\Pi \sigma_j$). Hence, $(\Pi X_i, \Pi \sigma_i)$ is $n \cdot f \cdot T_2$ space.

Definition 8 — A sequence of points x_n of X is said to converge fuzzily to $x \in X$ in (X, δ), denoted as $x_n \xrightarrow{f} x$ if, $\forall \ \mu \in \delta$ such that $\mu(x) > \frac{1}{2}$ there exists N such that $\mu(x_n) > \frac{1}{2} \ \forall$ $n \geq N$.

Remark 2 : We can extend this definition to net convergence fuzzily.

***Theorem* 2** — *Let (X, δ) be $n \cdot f \cdot T_2$ space. If any sequence of points of X converges fuzzily, then it converges uniquely.*

PROOF : Suppose $x_n \in X$ converges fuzzily to x and y such that $x \neq y$ since (X, δ) is $n \cdot f \cdot T_2$ space there exist open sets $\mu \neq 0, \nu \neq 0$ such that $\mu(x) > \frac{1}{2}$, $\nu(y) > \frac{1}{2}$ and $\mu(z) + \nu(z) \leq 1$, $\forall$ $z \in X$. From $x_n \xrightarrow{f} x$ and $\mu(x) > \frac{1}{2}$, there exists m_1 such that $\mu(x_n) > \frac{1}{2} \ \forall \ n \geq m_1$. Similarly, there exists m_2 such that $\nu(x_n) > \frac{1}{2} \ \forall \ \nu(x_n) > \frac{1}{2} \ \forall \ n \geq m_2$.

Let $m = \max \{m_1, m_2\}$, $\mu(x_n) > \frac{1}{2}$ and $\nu(x_n) > \frac{1}{2} \ \forall \ n \geq m$, then $\mu(x_n) + \nu(x_n) > 1 \ \forall \ n \geq m$; this contradicts $\mu(z) + \nu(z) \leq 1 \ \forall \ z \in X$. Therefore, $x_n \xrightarrow{f} x$ uniquely.

Remark 3 : Converse of the above theorem need not be true.

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Example 3 — Let X be an uncountable set and $\delta = \{\mu \in I^X \,/\, \mu^c$-has countable support or $\mu^c = 1\}$. Clearly (X, δ) is fuzzy topological space. Every sequence $\{x_n\}$ of points of X converges fuzzily uniquely if it converges. (In fact, here every sequence does not converge. For, suppose $x_n \xrightarrow{f} x$, consider $\mu \in \delta$ such that $\mu(z) = \begin{cases} 1 & \text{if } z \neq x_n, x \text{ or } z = y \\ \frac{1}{4} & \text{if } z = x_n, \\ 3/4 & \text{if } z = x \end{cases}$ $\mu(x) > \frac{1}{2}$, but $\mu(x_n) < \frac{1}{2}$ $\forall\, n \in Z_+$. Hence, x_n does not converge to x fuzzily). (X, δ) is not $n \cdot f \cdot T_2$. For, $x \neq y$, suppose there exist $\mu, \nu \in \delta$ s . t $\mu(x) > \frac{1}{2}$ $\nu(y) > \frac{1}{2}$ and $\mu(z) + \nu(z) \leq 1$, $\forall\ z \in X_j$. μ^c, ν^c has countable support $\{x_n\}_{n \in Z_+}$ and $\{y_m\}_{m \in Z_+}$ respectively. Hence, μ and ν has value 1 on X- $\{x_n\}_{n \in Z_+}$ and X-$\{y_m\}_{m \in Z_+}$ respectively. Therefore, μ and ν has value 1 on X-$\left\{x_n, y_m\right\}_{n, m \in Z_+}$. X is uncountable, there exists $z \in X - \left\{x_n, y_m\right\}_{n, m \in Z_+}$ such that $\mu(z) = 1$, $\nu(z) = 1$, but $\mu(z) + \nu(z) \leq 1$, $\forall\ z \in X$..

Definition 9 — A function $f: (X, \delta) \to (Y, \sigma)$ is said to be fuzzy open if $f(\mu) \in \sigma$ $\forall\ \mu \in \delta$.

Theorem 3 — *Let $f: (X, \delta) \to (Y, \sigma)$ be a bisective function from X to Y and f be fuzzy open, then (Y, σ) is $n \cdot f \cdot T_2$ space if (X, δ) is $n \cdot f \cdot T_2$ space.*

PROOF : Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since f is bijective, there exist unique $x_1 \neq x_2$ such that $f(x_1) = y_1, f(x_2) = y_2$. (X, δ) is $n \cdot f \cdot T_2$ space and $x_1 \neq x_2$ hence there exist $\mu, \nu \in \delta$ such that $\mu(x_1) > \frac{1}{2}$, $\nu(x_2) > \frac{1}{2}$ and $\mu(z) + \nu(z) \leq 1$ $\forall\ z \in X$. Variable 'f' is fuzzy open which implies $f(\mu)$ and $f(\nu)$ are fuzzy open in (Y, σ), (i.e.,) $f(\mu), f(\nu) \in \sigma$.

$$f(\mu)\,(y_1) = \sup_{x \in f^{-1}(y_1)} \mu(x) = \mu(x_1) > \frac{1}{2}. \text{ (since } x_1 \in f^{-1}(y_1)\text{)}. \text{ Similarly, } f(\nu)\,(y_2) > \frac{1}{2}.$$

Claim — $f(\mu)$ and $f(\nu)$ are disjoint, i.e., to prove that $f(\mu)\,(z) + f(\nu)\,(z) \leq 1$, $\forall\ z \in Y$. On the contrary, assume there exists

$$z \in Y \text{ such that } f(\mu)\,(z) + f(\nu)\,(z) > 1. \qquad \text{... (1)}$$

Since f is surjective there exists bijective $x \in X$ such that $f(x) = z$ unique. Hence $f(\mu)\,(z) = \mu(x)$ and $f(\nu)\,(z) = \nu(x)$. From (1) we have $\mu(x) + \nu(x) > 1$ but to $\mu(z) + \nu(z) \leq 1$, $\forall\ z \in X$. Therefore, $\forall$ $y_1, y_2 \in Y$ such that $y_1 \neq y_2$, there exist $f(\mu), f(\nu) \in \sigma$ such that $f(\mu)\,(y_1) > \frac{1}{2}, f(\nu)\,(\mu) > \frac{1}{2}$ and $f(\mu)\,(z) + f(\nu)\,(z) \leq 1$, $\forall\ z \in Y$. Therefore, (Y, σ) is $n \cdot f \cdot T_2$ space.

Note 4 — The requirement that f is fuzzy open cannot be dropped.

Example 4 — Let $X = \{a, b, c\}$ and $Y = \{r, s, t\}$.

$$\text{Let } \delta = \left\{ 0, 1, \mu_{a,b,c}^{x,0,0}, \mu_{a,b,c}^{0,y,0}, \mu_{a,b,c}^{0,0,z}, \mu_{a,b,c}^{x,y,0}, \mu_{a,b,c}^{0,y,z}, \mu_{a,b,c}^{x,0,z}, \mu_{a,b,c}^{x,y,z} \right\},$$

$$1 > x \geq y \geq z > \frac{1}{2}.$$

$$\sigma = \left\{ 0, 1, v_{r,s,t}^{y,0,0} \right\} .$$

let $f : X \to Y$ by $a \to r, b \to s, c \to t$.

$$f(\delta) = \left\{ 0, 1, v_{r,s,t}^{x,0,0}, v_{r,s,t}^{0,y,0}, v_{r,s,t}^{0,0,z}, v_{r,s,t}^{x,y,0}, v_{r,s,t}^{0,y,z}, v_{r,s,t}^{x,0,z}, v_{r,s,t}^{x,y,z} \right\}$$

Clearly f is one to one and onto. But f is not fuzzy open and (X, δ) is $n \cdot f \cdot T_2$ space, but (Y, σ) is not $n \cdot f \cdot T_2$ space.

Note 5 — The requirement that f is onto cannot be dropped.

Example 5 — Let $X = \{a, b\}$ and $Y = \{r, s, t\}$

$$\delta = \left\{ 0, 1, \mu_{a,b}^{x,0}, \mu_{a,b}^{0,y}, \mu_{a,b}^{x,y} \right\}, x, y > \frac{1}{2}$$

and

$$\sigma = \left\{ 0, 1, v_{r,s,t}^{x,0,0}, v_{r,s,t}^{0,y,0}, v_{r,s,t}^{x,y,0}, v_{r,s,t}^{x,1,0}, v_{r,s,t}^{1,y,0}, v_{r,s,t}^{1,1,0} \right\}$$

Let $f : X \to Y$ by $a \to Y, b \to s$. Hence, $f(\delta) = \left\{ 0, v_{r,s,t}^{1,1,0}, v_{r,s,t}^{x,0,0}, v_{r,s,t}^{0,y,0}, v_{r,s,t}^{x,y,0} \right\}$.

Clearly f is 1-1 and fuzzy open, but f is not onto. And (X, δ) is $n \cdot f \cdot T_2$ space but (Y, σ) is not $n \cdot f \cdot T_2$ space.

Note 6 — The requirement that f is one-to-one cannot be dropped.

Example 6 — Let $X = \{x_n / n \in Z_+\}$ and $Y = \{x, y\}$; then

$$\delta = \langle \{ 0, 1, \mu_i \mid i \in Z_+ \} \rangle, \mu_i \in I^X \text{ is given by } \mu_i(x_j) = \begin{cases} \frac{1}{2}\left(1 + \frac{1}{i}\right) & j = i \\ \frac{1}{2}\left(1 - \frac{1}{j}\right) & j \neq i \end{cases},$$

$$\sigma = \langle \{ 0, 1, v_i \mid i \in Z_+ \} \rangle, v_i \in I^Y$$

is given by

$$v_{2i}(x) = \frac{1}{2}\left(1 + \frac{1}{2i}\right) \text{ and } v_{2i}(y) = \frac{1}{2} \text{ and } v_{2i-1}(x) = \frac{1}{2}, v_{2i-1}(y) = \frac{1}{2}\left(1 + \frac{1}{2i-1}\right)$$

Let $f : X \to Y$ be defined by $f(x_{2i}) = x$ and $f(y_{2i-3}) = y$. Clearly, f is onto, but it is not one-one.

$$f(\delta) = \langle \{0, 1, f(\mu_i)\}\rangle,$$

where

$$f(\mu_{2i})(x) = \frac{1}{2}\left(1 + \frac{1}{2i}\right), f(\mu_{2i})(y) = \frac{1}{2};$$

and

$$f(\mu_{2i-1})(x) = \frac{1}{2}, f(v_{2i-1})(y) = \frac{1}{2}\left(1 + \frac{1}{2i-1}\right)$$

Clearly, $f(\delta \subseteq \sigma$. Hence, f is fuzzy open (X, δ) is $n \cdot f \cdot T_2$ space, For $x_i \neq x_j$, there exist $\mu_i, \mu_j \in \delta$ such that $\mu_i(x_i) > \frac{1}{2}$ and $\mu_j(x_j) > \frac{1}{2}$ and $\mu_i(z) + \mu_j(z) \leq 1$. But (Y, σ) is not $n \cdot f \cdot T_2$ space.

Note 7 — Let $f: (X, \delta) \to (Y, \sigma)$ be one to one, onto and fuzzy continuous map. If (X, δ) is $n \cdot f \cdot T_2$ space, but (Y, σ) need not be $n \cdot f \cdot T_2$ space.

Example 7 — Let $X = \{a, b, c\}$ and $Y = \{r, s, t\}$. $f: X \to Y$ is defined by $a \to r, b \to s, c \to t$.

$$\text{Let } \delta = \left\{0, 1, \mu_{a,b,c}^{x,0,0}, \mu_{a,b,c}^{0,y,0}, \mu_{a,b,c}^{0,0,z}, \mu_{a,b,c}^{x,y,0}\right.$$

$$\left.\mu_{a,b,c}^{x,0,z}, \mu_{a,b,c}^{0,y,z}, \mu_{a,b,c}^{x,y,z}\right\}, x, y, z \geq \frac{1}{2}$$

$$\sigma = \left\{0, 1, v_{r,s,t}^{x,0,0}\right\} \quad \text{and hence, } f^{-1}(\sigma) = \left\{0, 1, \mu_{a,b,c}^{x,0,0}\right\} \subseteq \delta.$$

Clearly, f is one-to-one, onto, fuzzy continuous and (X, δ) is $n \cdot f \cdot T_2$ space, but (Y, σ) is not $n \cdot f \cdot T_2$ space.

Theorem 4 — *Let $f: (X, \delta) \to (Y, \sigma)$ be one-to-one, onto, fuzzy closed, then (Y, σ) is $n \cdot f \cdot T_2$ space if (X, δ) is $n \cdot f \cdot T_2$ space.*

PROOF : To prove (Y, σ) is $n \cdot f \cdot T_2$ space, let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since f is bijective, there exist unique $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$ and $x_1 \neq x_2$. Since (X, δ) is $n \cdot f \cdot T_2$ space, there exist $\mu, v \in \delta$ such that $\mu(x_1) > \frac{1}{2}$, $v(x_2) > \frac{1}{2}$ and $\mu(z) + v(z) \leq 1 \; \forall z \in X$. '$f$' is fuzzy closed then $f(\mu^c) \in \sigma^c, f(v^c) \in \sigma^c$. Consider $\mu' = [f(\mu^c)]^c$ and $v' = [f(v^c)]^c$. Clearly, $\mu', v' \in \sigma, \mu'(y_1) = 1 - f(\mu^c)(y_1) = 1 - \mu^c(x_1) = \mu(x_1) > \frac{1}{2}$.

Similarly $v'(y_2) > \frac{1}{2}$. Clearly $\mu'(z) + v'(z) \leq 1, \; \forall z \in Y$. Hence, (Y, σ) is $n \cdot f \cdot T_2$ space.

Note 8 — We can give examples to show that the above theorem is not true if any one of the hypothesis is dropped.

Definition 10 — Let (X, δ) be a fuzzy topological space and let $f: (X, \delta) \to Y$ be surjective map, then the Quotient fuzzy topology on Y induced by the map f is the largest fuzzy topology in which f is fuzzy continuous.

Remark 4 : Let $f: (X, \delta) \to Y$ be a bijective map. Let $Q(\delta)$ be the quotient fuzzy topology on Y induced by f. Then $(Y, Q(\delta))$ is $n \cdot f \cdot T_2$ space if (X, δ) is $n \cdot f \cdot T_2$ space.

PROOF : 'f' is one-to-one, then $f^{-1}(f(\mu)) = \mu$. Therefore, $\mu \in \delta$ implies $f^{-1}(f(\mu)) \in \delta$. Hence $f(\mu) \in Q(\delta)$ if $n \in S$ and hence f is fuzzy open. By theorem 3, $(Y, Q(\delta))$ is $n \cdot f \cdot T_2$ space.

Corollary 1 — $n \cdot f \cdot T_2$ is a fuzzy topological property. Proof is obvious from theorem 3.

Definition 11 — (Nearly fuzzy T_1). A fuzzy topological space (X, δ) is to be nearly fuzzy $T_1(n \cdot f \cdot T_1)$ if for every pair $x, y \in X$ such that $x \neq y$, there exist $\mu, \nu \in \delta$ such that $\mu(x) > \frac{1}{2}$, $\mu(y) \leq \frac{1}{2}$ and $\nu(y) > \frac{1}{2}$, $\nu(x) \leq \frac{1}{2}$.

Proposition 1 — $n \cdot f \cdot T_2 \Rightarrow n \cdot f \cdot T_1$

PROOF : Let (X, δ) be $n \cdot f \cdot T_2$. To prove (X, δ) is $n \cdot f \cdot T_1$, consider $x, y \in X$ such that $x \neq y$. (X, δ) is $n \cdot f \cdot T_2$, there exist $\mu, \nu \in \delta$ such that $\mu(x) > \frac{1}{2}$, $\nu(y) > \frac{1}{2}$ and $\mu(z) + \nu(z) \leq 1$ $z \in X$. $\mu(x) > \frac{1}{2}$ and $\mu(x) + \nu(x) \leq 1 \Rightarrow \nu(x) \leq \frac{1}{2}$. Similarly, $\mu(y) \leq \frac{1}{2}$. Hence, (X, δ) is $n \cdot f \cdot T_1$.

Remark 5 : The converse of the above proposition need not be true.

Example 8 — Let X be an Infinite set and $\sigma = \{\mu \in I^X$ such that either support μ^c is finite or $\mu^c = 1\}$. Clearly (X, σ) is fuzzy topological space and $n \cdot f \cdot T_1$. For $x \neq y$, define μ, ν such that

$$\mu(z) = \begin{cases} 0 & \text{if } z = x \\ 1 & \text{otherwise} \end{cases}$$

and

$$\nu(z) = \begin{cases} 0 & \text{if } z = y \\ 1 & \text{otherwise} \end{cases}.$$

Clearly, supports of μ^c and ν^c are $\{x\}$ and $\{y\}$ respectively. Therefore, $\mu, \nu \in \sigma$ such that $\nu(x) > \frac{1}{2}$, $\nu(y) \leq \frac{1}{2}$ and $\mu(y) > \frac{1}{2}$, $\mu(x) \leq \frac{1}{2}$. But (X, σ) is not $n \cdot f \cdot T_2$. For, let $x \neq y \in X$. Suppose there exist $\mu, \nu \in \delta$ such that $\mu(x) > \frac{1}{2}$, $\nu(y) > \frac{1}{2}$ and $\mu(z) + \nu(z) \leq 1$, $\forall\ z \in X$. $\mu, \nu \in \delta \Rightarrow \mu^c, \nu^c$ has finite supports $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ respectively. [neither μ^c, nor ν^c equals 1, suppose $\mu^c = 1, \mu = 0$, but to $\mu(x) > \frac{1}{2}$]. Hence, μ has value 1 on X-$\{x_1, \ldots, x_n\}$ and ν has value 1 on

$X\text{-}\{y_1, \ldots, y_m\}$. Since X is infinite, there exists $z \in X\text{-}\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ on which $\mu(z) = 1$, $\nu(z) = 1$ contradicts $\mu(z) + \nu(z) \leq 1$, $\forall\ z \in X$.

Definition 12 — (C.L. Chang[2]) : Let (X, δ) be fuzzy topological space. A fuzzy subset $\mu \in I^X$ is said to be fuzzy compact if every fuzzy open covering admits afinite subcovering. i.e., "If for every collection $\sigma \subseteq \delta$ such that $\bigvee_{\nu \in \sigma} \nu = \mu$, then there exists a finite $\gamma_1, \gamma_2, \ldots, \gamma_n$ in σ such that $\bigvee_{i=1}^{n} \gamma_i = \mu$."

Note 9 — Fuzzy compact subsets of $n \cdot f \cdot T_2$ space need not be fuzzy closed. This follows from the following example.

Example 9 — Let X be $\{a, b, c\}$ and σ be

$$\left\{0, 1, \mu^{x,0,0}_{a,b,c}, \mu^{0,y,0}_{a,b,c}, \mu^{0,0,z}_{a,b,c}, \mu^{x,y,0}_{a,b,c}, \mu^{0,y,z}_{a,b,c}, \mu^{x,0,z}_{a,b,c}\ \mu^{x,y,z}_{a,b,c}\right\} \quad x,\ y,\ z > \frac{1}{2}.$$

Here (X, σ) is $n \cdot f \cdot T_2$ space and every fuzzy subset $\mu \in I^X$ is fuzzy compact (since the whole fuzzy topology σ is finite). But $\mu^{1/2,1/2,1/2}_{a,b,c}$ is not fuzzy closed.

Theorem 5 — *Let (X, σ) be f.t.s. If 1_Δ the characteristic function of diagonal Δ of $X \times X$, is fuzzy closed in the fuzzy product topology, $(X \times X, \sigma \times \sigma)$, then (X, σ) is $n \cdot f \cdot T_2$ space, but not conversely.*

PROOF : First we prove a claim.

Claim — For every pair $x, y \in X$ such that $x \neq y$ there exists a fuzzy open set $\mu \times \nu \in \sigma \times \sigma$ such that $\mu \times \nu(x, y) > \frac{1}{2}$ and $\mu \times \nu(z_1, z_2) + 1_\Delta(z_1, z_2) \leq 1$ for all $(z_1, z_2) \in X \times X$.

PROOF OF THE CLAIM — $x \neq y \Rightarrow 1_\Delta(x, y) = 0\ \overline{1_\Delta(x, y)} = 1_\Delta(x, y) = 0$ (since 1_Δ is fuzzy closed, i.e. $\overline{1}_\Delta = 1_\Delta$). Since $\overline{1}_\Delta(x, y) = 0 < \frac{1}{2}$, there exists $\mu \times \nu \in \sigma \times \sigma$ such that $\mu \times \nu(x, y) > \frac{1}{2}$ and $\mu \times \nu(z_1, z_2) + 1_\Delta(z_1, z_2) \leq 1\ \ \forall\ (z_1, z_2) \in X \times X$, with

$$\mu \times \nu(z_1, z_2) = \begin{cases} 0 & \text{if } z_1 = z_2 \\ \leq 1 & \text{if } z_1 \neq z_2 \end{cases}$$

and

$$1_\Delta(z_1, z_2) = \begin{cases} 1 & \text{if } z_1 = z_2 \\ 0 & \text{if } z_1 \neq z_2 \end{cases}.$$

Hence the claim.

Now to prove the theorem, we have to prove (X, σ) is $n \cdot f \cdot T_2$ space. Let $x, y \in X$ such that

$x \neq y$, by the claim there exists $\mu \times \nu \in \sigma \times \sigma$ such that $\mu \times \nu(x, y) > \frac{1}{2}$ and

$$\mu \times \nu(z_1, z_2) + 1_\Delta(z_1, z_2) \leq 1, \ \forall \ (z_1, z_2) \in X \times X. \qquad (*)$$

From $\mu \times \nu(x, y) > \frac{1}{2}$ we have $\min(\mu(x), \nu(y)) > \frac{1}{2}$; hence $\mu(x) > \frac{1}{2}$ and $\nu(y) > \frac{1}{2}$,

It is enough to prove $\mu(z) + \nu(z) \leq 1 \ \forall \ z \in X$..

Suppose not, $\mu(z) + \nu(z) > 1$ for some $z \in X$ then $\mu \times \nu(z, z) > 0$ and $1_\Delta(z, z) = 1$. Hence, $\mu \times \nu(z, z) + 1_\Delta(z, z) > 1$, but from (*) :

$\mu(z) + \nu(z) \leq 1 \ \forall \ z \in X$. Therefore, (X, σ) is $n \cdot f \cdot T_2$ space. The converse of the theorem need not be true.

Example 10 — Let $X = \{a, b, c\}$

$$\delta = \Big\{ 0, 1, \mu_{a,b,c}^{z_0, 0, 0}, \mu_{a,b,c}^{0, x_0, 0}, \mu_{a,b,c}^{0, 0, y_0}, \mu_{a,b,c}^{x, x_0, 0}, \mu_{a,b,c}^{0, y, y_0}, \mu_{a,b,c}^{z_0, 0, z}$$

$$\mu_{a,b,c}^{z_0, x_0, z}, \mu_{a,b,c}^{z_0, y, y_0} \mu_{a,b,}^{x, x_0 y_0}, \mu_{a,b,c}^{z_0, x_0, y_0}, \mu_{a,b,c}^{x, x_0, z}$$

$$\Big\{ \mu_{a,b,c}^{x, y, y_0}, \mu_{a,b,c}^{z_0, y, z}, \mu_{a,b,c}^{x, y, z} \Big\}_{\frac{3}{4} > x, y, z > \frac{1}{2} \ \& \ x_0, y_0, z_0 < \frac{1}{2} \ \& \ x_0 + y < 1, \, y_0 + z < 1, \, z_0 + x < 1}$$

Clearly (X, δ) is $n \cdot f \cdot T_2$ space. $\Delta = \{(a, a), (b, b), (c, c)\}$ and

$$1_\Delta = \mu_{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9}^{1, 0, 0, 0, 1, 0, 0, 1}.$$

1_Δ is not fuzzy closed in $(X \times X, \delta \times \delta)$.

Theorem 6 — *Let (X, δ) be a f.t.s and R an equivalence on X. The quotient fuzzy topology $Q(\delta)$ on $\frac{X}{R}$ is $n \cdot f \cdot T_2$ space if (X, δ) is $n \cdot f \cdot T_2$ space, 1_R is fuzzy closed in $(X \times X, \delta \times \delta)$ and the identification map P is fuzzy open.*

We start with the following claim.

Claim — Let (Y, σ) be a fuzzy topological space and $\mu \in I^Y$.

a) $x \in \bar{\mu}$ (i.e., $\bar{\mu}(x) > \frac{1}{2}$) implies every fuzzy open set $\nu \in \sigma$ such that $\nu(x) > \frac{1}{2}$ intersects μ i.e., $\mu(z) + \nu(z) > 1$, for some $z \in Y$.

b) If $x \in X$ such that $\bar{\mu}(x) < \frac{1}{2}$, then there exists $\nu \in \delta$ such that $\nu(x) > \frac{1}{2}$ and $\nu(z) + \mu(z) \leq 1 \ \forall \ z \in X$.

PROOF OF THE CLAIM : *a*) Let $x \in \overline{\mu}$. To prove that $\forall\ \nu \in \delta$ such that $\nu(x) > \frac{1}{2}$ intersects μ. Sppose not, there exists $\nu \in \delta$ such that $\nu(x) > \frac{1}{2}$ and $\nu(z) + \mu(z) \leq 1$, $\forall\ z \in Y$. Hence, $\nu^c(z) \geq \mu(z)$, $\forall\ z \in Y$. $\therefore$ ν^c is a fuzzy closed set containing μ. Hence, $\nu^c \geq \overline{\mu}$ and hence $\nu^c(x) \geq \overline{\mu}(x)$. But $\nu^c(x) = 1 - \nu(x) < \frac{1}{2}$. $\therefore$ $\overline{\mu}(x) < \frac{1}{2}$.

This contradicts $x \in \overline{\mu}$.

b) If $x \in X$ such that $\overline{\mu}(x) < \frac{1}{2}$, it is sufficient to consider the open set.

Now we prove the theorem. To prove that X/R is a $n \cdot f \cdot T_2$ space in quotient fuzzy topology, assume $p(x), p(y) \in X/R$ s.t $p(x) \neq p(y)$ and remark that x and y are not related (i.e., $(x, y) \notin R$), hence $1_R(x, y) = 0$. By the hypothesis, 1_R is fuzzy closed then $\overline{1}_R = \overline{1}_R$. $\therefore$ $\overline{1}_R(x, y) = 0 < \frac{1}{2}$. By the claim (*b*), there exists a fuzzy open set $\nu = \mu \times \mu_2 \in I^{X \times X}$, in product fuzzy topology such that $\mu_1 \times \mu_2\,(x, y) > \frac{1}{2}$ and $\mu_1 \times \mu_2\,(x, y) + 1_R\,(x, y) \leq 1$, $\forall\ (x, y) \in X \times X$. $\mu_1 \times \mu_2\,(x, y) = \min\,(\mu_1(x), \mu_2(y)) > \frac{1}{2}$, where $\mu_1, \mu_2 \in \delta$. Hence, $\mu_1\,(x) > \frac{1}{2}$ and $\mu_2\,(y) > \frac{1}{2}$. The map p is fuzzy open (by our hypothesis) then $p(\mu_1)$ and $p(\mu_2)$ are fuzzy open. From $\mu(x_1) > \frac{1}{2}$ we have $p(\mu_1)\,(p(x)) = \sup_{z \in p^{-1}(p(x))} \mu_1(z) \geq \mu_1(x) > \frac{1}{2}$. Similarly, $p(\mu_2)\,(p(y)) > \frac{1}{2}$. Hence, we have two fuzzy open sets $p(\mu_1)$ and $p(\mu_2)$ such that $p(\mu_1)\,(p(x)) > \frac{1}{2}$ and $p(\mu_2)\,(p(y)) > \frac{1}{2}$. Hence, it is enough to prove that $p(\mu_1)$ and $p(\mu_2)$ are disjoint, i.e., to prove that $p(\mu_1)\,(p(z)) + p(\mu_2)\,(p(z)) \leq 1$ $\forall\ z \in X$ (p is onto). Suppose not, $p(\mu_1)\,(p(z)) + p(\mu_2)\,(p(z)) > 1$ for some $z \in X$. Hence, $p(\mu_1)\,(p(z)) > 0$ and $p(\mu_2)\,(p(z)) > 0$. By definition $p(\mu_1)\,(p(z)) = \sup_{x \in p^{-1}(p(z))} \mu_1(x) > 0$, we have $\mu_1(t) > 0$ for some $t \in p^{-1}\,(p(z))$. Similarly, $\mu_2(s) > 0$ for some $s \in p^{-1}\,(p(z))$.

Hence, $\mu_1 \times \mu_2(t, s) > 0$, where $t, s \in p^{-1}\,(p(z))$. Hence, we have $\mu_1 \times \mu_2\,(t, s) > 0$, where tRs (i.e., $(t,\ s) \in R$, since $t, s \in [z]$) and $\mu_1 \times \mu_2\,(t, s) + 1_R\,(t, s) > 1$ (Since $(t, s) \in R$, $1_R\,(t, s) = 1$) which is a contradiction. Hence the theorem.

Remark 6 : The converse of the above theorem need not be true.

Example 11 — Let $X = \left\{ x_i, y_j \right\}_{i, j \in Z}$.

$$\delta = \left\{ 0, 1, \mu_{x_i, y_j}^{a, b}, \mu_{x_i, y_j}^{a, 0}, \mu_{x_i, y_j}^{0, b} \right\} a, b > \frac{1}{2},$$

where

$$\mu^{a,b}_{x_i, y_i} \text{ means } \mu(x_i) = av_i\, \mu(y_i) = b - v_j$$

and

$$R = \{(x_i x_j)_\forall\, i, j\} \cup \{(y_i, y_j) \,/\, \forall\, i, j\}$$

Clearly (X, δ) is a fuzzy topology.

$$X/R = \left\{ (x_i)_{i \in Z_+}, (y_i)_{j \in Z_+} \right\} = \{x, y\}, \text{ where } x = (x_i)_{i \in Z_+} \;\&\; y = (y_i)_{i \in Z_+}.$$

$$p : X \to X/R \text{ by } p(x_i) = x \;\forall\, i, \text{ and } p(y_j) = y \;\forall\, j$$

Quotient Fuzzy Topology on $X/R = Q(\delta) = \left\{ \mu \in I^{X/R} / p^{-1}(\mu) \in \delta \right\}$

$$= \left\{ \mu^{a,b}_{x,y} / p^{y} \left(\mu^{a,b}_{x,y} \right) = \mu^{a,b}_{x_i, y_j} \in \delta \right\}$$

Therefore, $Q(\delta) = \left\{ 0, 1, \mu^{a,D}_{x,y}\, \mu^{0,b}_{x,y}, \mu^{a,b}_{x,y} \right\} a, b > \frac{1}{2}$.

Hence, clearly $(X/R, Q(\delta)$ is $n \cdot f \cdot T_2$ but (X, δ) is not $n \cdot f \cdot T_2$

Fuzzy T_2 Space

From example 2 we find that there are $n \cdot f \cdot T_2$ spaces, in which singletons are fuzzy closed in the sence that $\forall\, x \in X$ there exists a fuzzy closed set with singleton support $\{x\}$ and fuzzy value $> \frac{1}{2}$.

Definition 13 — A fuzzy topological space (X, δ) is said to be fuzzy T_2 space if (X, δ) is $n \cdot f \cdot T_2$ and singletons are fuzzy closed.

Theorem 7 — *a) A subspace of $F \cdot T_2$ space is $F \cdot T_2$. b) Arbitrary product of $F \cdot T_2$ spaces is again $F \cdot T_2$ space in product fuzzy topology.*

PROOF : *a)* Let (X, δ) be a $F \cdot T_2$ space. Let $Y \subseteq X$. A subspace fuzzy topology on $Y = \delta \mid Y = \{\mu \mid Y \;\forall\, \mu \in \delta\}$.

To prove that $\delta \mid Y$ is $F \cdot T_2$.

By theorem 1, we have $\delta \mid Y$ is $n \cdot f \cdot T_2$.

Now we prove singletons of Y is fuzzy closed.

Let $y \in Y \subseteq X$, then $\{y\}$ is fuzzy closed in (X, δ) (since (X, δ) is $F \cdot T_2$).

Hence, there exists a fuzzy closed set μ such that $\mu(y) > \frac{1}{2}$ and $\mu(z) = 0 \;\forall\, z \neq y, z \in X$.

Therefore, we have $\mu^c(z) = \begin{cases} 1 & \text{if } z \neq y \\ < \frac{1}{2} & z = y \end{cases}$ and $\mu^c \in \delta$. Therefore, $\mu^c \mid \delta \in \delta \mid Y$.

$$\mu^c \mid \delta(z) = \begin{cases} 1 & \text{if } z \neq y,\ z \in Y \\ < \frac{1}{2} & z = y \end{cases}$$

and

$$(\mu^c \mid \delta)^c(z) = \begin{cases} 0 & \text{if } z \neq y,\ z \in Y \\ > \frac{1}{2} & z = y \end{cases}$$

and $(\mu^c \mid \delta^c)$ is fuzzy closed and have support singleton $\{y\}$ and value $> \frac{1}{2}$. Then $\delta \mid Y$ is $f \cdot T_2$.

b) Let I be any indexed set and $(X_i, \delta_i)_{i \in I}$ be any arbitrary indexed family of $F \cdot T_2$ spaces.

By theorem 1, clearly $\left(\prod_{i \in I} x_i, \prod_{i \in I} \delta_i \right)$ is $n \cdot f \cdot T_2$

Now we prove in product fuzzy topolgy $(\Pi X_i, \Pi \delta_i)$ singletons are fuzzy closed. Let $x \in \prod_{i \in I} X_i$, such that $x = (x_i)_{i \in I}$ where $x_i \in X_i$.

If $x_i \in X_i$ and (X_i, δ_i) is $f \cdot T_2$, then there exists $\mu_i^c \in \delta_i$ such that that $\mu_i(x_i) > \frac{1}{2}$ and μ_i has support $= \{x_i\}$. Now take $p_i^{-1}(\mu_i^c)$, where p_i is projection on $\Pi X_i \to X_i$ and p_i is fuzzy continuous in product fuzzy topology (by definition of product fuzzy topology).

$$\therefore \quad p_i^{-1}(\mu_i^c) \in \prod_{i \in I} \delta_i (p_i^{-1}(\mu_i^c))^c$$

is fuzzy closed and

$$\left(p_i^{-1}(\mu_i^c) \right)^c (z) = \begin{cases} 0 & z \in \Pi X_i \text{ such that } p(z) \neq x_i \\ > \frac{1}{2} & z \in \Pi X_i \text{ such that } p_i(z) = x_i \end{cases}$$

i.e.,

$$\left(p_i^{-1}(\mu_i^c) \right)^c (z) = \begin{cases} > \frac{1}{2} & \text{if } z \in \Pi X_i \text{ such that } z = (z_i) \text{ and } z_i = x_i \\ 0 & \text{otherwise} \end{cases}$$

$$\bigwedge_{i \in I} \left(p_i^{-1} (\mu_i^c) \right)^c (z) = \inf_{i \in I} \mu_i (z_i) = \begin{cases} > \frac{1}{2} \text{ if } z_i = x_i \; i \in I \\ 0 \text{ otherwise.} \end{cases}$$

(since for any $z_i \neq x_i$, $\mu_i(z_i) = 0$ and inf $= 0$)

and $\bigwedge_{i \in I} \left(p_i^{-1} (\mu_i^c) \right)^c$ is fuzzy closed (since arbitrary intersection of fuzzy closed sets is fuzzy closed) and support $= \{x\}$ and value $> \frac{1}{2}$ and hence $(\Pi X_i, \Pi \delta_i)$ is $F \cdot T_2$.

Definition 14 (R. Lowen[6]) — Let (X, δ) be a fuzzy topological space. The induced topology $i(\delta)$ on X is the smallest topology on X in which every $\mu \in \delta$ is continuous from $X \to (I, \tau_r)$, τ_r is a subspace topology on I induced by the topology on π given by $\tau = R\ \{\phi, R, (\alpha, \infty] \mid \alpha \in R\}$

Theorem 8 — *If (X, δ) is $F \cdot T_2$ then $(X, i(\delta))$ is T_2. (where $i(\delta)$ is the induced topology on X by fuzzy topology δ).*

PROOF : Let (X, δ) be fuzzy T_2, consider $x, y \in X$ such that $x \neq y$, then there exist $\mu, \nu \in \delta$ such that $\mu(x) > \frac{1}{2}$, $\nu(y) > \frac{1}{2}$ and $\mu(z) + \nu(z) \leq 1$, for every $z \in X$. Since $i(\delta)$ is the smallest topology on X in which every $\mu \in \delta$ is continuous from $X \to (I\ \tau_r)$, where $\tau_r = \left\{ \mu^{-1} (\alpha, 1] \mid \alpha \in R, \phi, I \right\}$. Then $\mu^{-1}\left[\frac{1}{2}, 1\right] \in i(\delta)$ and $x \in \mu^{-1}\left[\frac{1}{2}, 1\right]$. Similarly, $\nu^{-1}\left[\frac{1}{2}, 1\right] \in i(\delta)$ and $y \in \nu^{-1}\left[\frac{1}{2}, 1\right]$

We prove $\mu^{-1}\left[\frac{1}{2}, 1\right] \cap \nu^{-1}\left[\frac{1}{2}, 1\right] = \phi$. Suppose not, then there exists $z \in \mu^{-1}\left[\frac{1}{2}, 1\right] \cap \nu^{-1}\left[\frac{1}{2}, 1\right]$ and hence $\mu(z) > \frac{1}{2}$, $\nu(z) > \frac{1}{2}$ $\therefore$ $\mu(z) + \nu(z) > 1$ which contradicts $\mu(z) + \mu(z) \leq 1$ $\forall$ $z \in X$. Hence, $(X, i(\delta))$ is Hausdorff space.

Note 10 — The converse of the above theorem need not be true.

Example 12 — Let $X = \{a, b, c\}$

and

$$\delta = \left\{ 0, 1, \mu_{a,b,c}^{x,0,0}, \mu_{a,b,c}^{0,y,0}, \quad \mu_{a,b,c}^{0,0,z}, \mu_{a,b,c}^{x,y,0}, \mu_{a,b,c}^{x,0,z}, \mu_{a,b,c}^{0,y,z}, \mu_{a,b,c}^{x,y,z} \right\} \frac{1}{2} \geq x > y > z$$

(X, δ) is not $F \cdot T_2$.

But $i(\delta) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ is clearly T_2.

Definition 15 (R. Lowen[6]) — Let (X, τ) be a topological space, then $w(\tau) = c[(X, \tau), (I, \tau_r)]$ is a fuzzy topology induced by τ.

Theorem 9 — *$(X, w(\tau))$ is fuzzy $T_2 \Leftrightarrow (X, \tau)$ is Hausdorff space.*

PROOF : Let $(X, w(\tau))$ is fuzzy T_2. To prove τ is T_2 let $x, y \in X$ such that $x \neq y$. Since $(X, w(\tau))$ is fuzzy T_2 there exist $\mu, v \in w(\tau)$ such that $\mu(x) > \frac{1}{2}$, $v(y) > \frac{1}{2}$ and $\mu(z) + v(z) \leq 1$ $\forall$ $z \in X$. By definition of $w(\tau), \mu, v : X \to I_r$ are fuzzy continuous. Hence, $\mu^{-1}\left[\frac{1}{2}, 1\right], v^{-1}\left[\frac{1}{2} \cdot 1\right] \in \tau$. Let $A = \mu^{-1}\left[\frac{1}{2}, 1\right]$ and $B = v^{-1}\left[\frac{1}{2} \cdot 1\right]$. Clearly, $A, B \in \tau$ and $x \in A$ and $y \in B$ and $A \cap B = \phi$ (suppose not, it contradicts $\mu(z) + v(z) \leq 1$ $\forall$ $z \in X$). Hence, (X, τ) is Hausdorff if $(X, w(\tau))$ is fuzzy T_2.

Let (X, τ) be T_2. To prove $(X, w(\tau))$ is fuzzy T_2. Let $1\{x\}$ be the characteristic function on $\{x\}$. $\left(1^c_{(x)}\right)^{-1}(\alpha, 1] = X\text{-}\{x\} \in \tau$ (since (X, τ) is T_2 and hence singletons are closed). This is true for every $\alpha \in I$ and $\left(1^c_{(x)}\right)^{-1}(I) = X \in \tau$. This implies $1^c_{(x)} \in w(\tau)$. Hence, singletons are fuzzy closed. Now we prove $(X, w(\tau))$ is $n \cdot f \cdot T_2$. Let $x, y \in X$ such that $x \neq y$. (X, τ) is T_2, then there exist $A, B \in \tau$ such that $x \in A, y \in B$ and $A \cap B = \phi$. Now consider 1_A, 1_B the characteristic function on A and B respectively. $1_A^{-1}(\alpha, 1] = A$ $\forall$ $\alpha \in I$ and $1_A^{-1}(I) = X$ implies $1_A \in w(\tau)$. Similarly, $1_B \in w(\tau)$. Clearly, $1_A(z) + 1_B(z) \leq 1$ $\forall$ $z \in X$, then $(X, w(\tau))$ is $n \cdot f \cdot T_2$. Hence, $(X, w(\tau))$ is $f \cdot T_2$ if (X, τ) is T_2.

Definition 16 (R. Lowen[6]) — A fuzzy topological space (X, δ) is said to be topologically generated if there exists a topology τ on X such that $w(\tau) = \delta$.

Note 11 — Let (X, δ) be a topologically generated fuzzy topological space. If $(X, i(\delta))$ is Hausdorff then (X, δ) is fuzzy T_2. The proof is obvious from theorem 9.

Definition A (Rekha Srivastava[8]) — A fuzzy topological space (X, δ) is said to be fuzzy Hausdorff iff for any two distinct fuzzy points $p, q \in X$, there exist $U, V \in \tau$ with $p \in U$ and $q \in V$. Here, distinct fuzzy points means their singleton supports are distinct.

Definition B (Ghanim *et al*[4]) — A Fuzzy topological space (X, δ) is said to be fuzzy Hausdorff iff for every pair of fuzzy singletons p_1 and p_2 with different supports, there exist open fuzzy sets O_1 and O_2 such that $p_1 \subseteq O_1 \subseteq co\, P_2, p_2 \subseteq O_2 \subseteq co\, p_1$ and $O_1 \subseteq co\, O_2$.

(B') Equivalently, for every pair $x, y \in X, x \neq y$, there exist $U, V \in \delta$ such that $U(x) = 1 = V(y)$, $U(y) = 0 = V(x)$ and $U \subseteq Co\ V$. This equivalence is proved in [1] by Arun K. Srivastava and Dewan Muslim Ali. We remember $n \cdot f \cdot T_2$ and T_2 space definitions.

Definition C_0 — A Fuzzy Topological Space (X, δ) is said to be a nearly fuzzy Hausdorff space $(n \cdot f \cdot T_2)$ if for every pair of elements $x \neq y$ of X, there exist nonzero disjoint fuzzy open sets $\mu, \nu \in \delta$ such that $\mu(x) > \frac{1}{2}$ and $\nu(y) > \frac{1}{2}$. i.e., there exists $\mu \neq 0$ and $\nu \neq 0$ in δ such that $\mu(x) > \frac{1}{2}$ and $\nu(y) > \frac{1}{2}$ and $\mu(z) + \nu(z) \leq 1 \ \forall\ z \in X$.

Definition C — A fuzzy topological space (X, δ) is said to be fuzzy T_2 space if (X, δ) is $n \cdot f \cdot T_2$ and singletons are fuzzy closed.

The following implications are true :

$$(A) \Rightarrow (C_0) \text{ and } (B) \Rightarrow (C) \Rightarrow (C_0).$$

PROOF : Consider a fuzzy topological space (X, δ) in which A holds. To prove (C_0) also holds. Let $x, y \in X$ such that $x \neq y$. Define fuzzy points p and q such that $p(x) = \frac{3}{4}$ and $p(z) = 0$, for every $z \neq x$ and $q(y) = \frac{3}{5}$ and $q(z) = 0$, for every $z \neq y$. Hence, by (A), there exist $\mu, \nu \in \delta$ such that $p \in \mu, q \in \nu$ and $\mu \wedge \nu = 0$. $p \in \mu \Rightarrow p(x) < \mu(x)$. Hence, $\mu(x) > \frac{1}{2}$. Similarly, $\nu(y) > \frac{1}{2}$ and $\mu \wedge \nu = 0 \Rightarrow \mu(z) + \nu(z) \leq 1$, for every $z \in X$ and Hence, (C_0) holds.

Now consider a fuzzy topological space in which (B) holds. Hence (B') holds. Clearly (C_0) holds. By [10], every fuzzy singleton with value 1 is fuzzy closed and hence (C) holds.

But (C_0) does not imply (A) and (C_0) does not imply (B) are seen from the following example.

Let

$$X = \{a, b, c\}$$

and

$$\delta = \left\{ 0, 1, \mu_{a,b,c}^{x,0,0}, \mu_{a,b,c}^{0,y,0}, \mu_{a,b,c}^{0,0,z}, \mu_{a,b,c}^{x,y,0}, \quad \left\{ \mu_{a,b,c}^{x,0,z}, \mu_{a,b,c}^{0,y,z}, \mu_{a,b,c}^{x,y,z} \right\}, \frac{1}{2} < x, y, z < 1$$

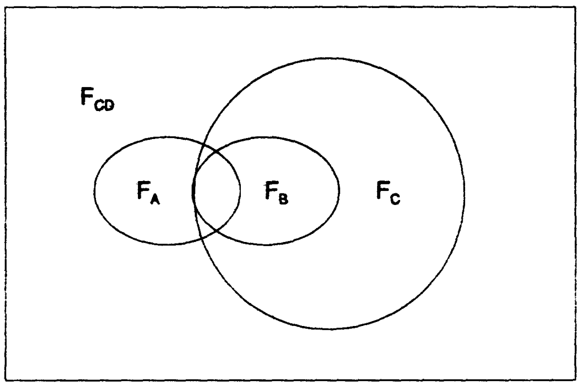
are fixed.

Clearly (X, δ) is a fuzzy topological space in which (C_0) holds. But (A) does not hold. For, consider fuzzy points p, q such that $p(a) = r$, $p(t) = 0$, $\forall\ t \neq a$ and $q(b) = s$ and $q(t) = 0$, $\forall$ $t \neq b$, where r, s satisfies $\max\ \{x, y, z\} < r, s < 1$.

Then p and q are distinct and the only fuzzy open set containing p is 1 and similarly for q also. Hence, there are no fuzzy open sets satisfies the requirements of (A). Similarly there are no fuzzy open sets satisfies the requirements of (B). Hence, (C_0) does not imply (A) and (B). Similarly, (C) does not imply (B) is verified from the example (2).

Let F_A, F_B and F_{C_0} be the fuzzy topological spaces with (A), (B) and (C_0) respectively. Then $F_A \subsetneqq F_{C_0}$ and $F_B \subsetneqq F_C \subsetneqq F_{C_0}$. Hence, the class of fuzzy topological spaces introduced by us is strictly bigger than class of Rekha Srivastava[8] and Ghanim[4].

Remark 7 : But according to [11] there is no relation between (A) and (B). But F_A and F_B must have common elements like I^X. So we have



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