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ELASTIC-PLASTIC TRANSITION IN A NON-HOMOGENEOUS DISC WITH VARIABLE THICKNESS SUBJECTED TO INTERNAL PRESSURE

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Elastic-Plastic transitional stresses in a non-homogeneous disc with variable thickness, subjected to internal pressure have been derived, using transition approach. Non-homogeneity in the disc is considered due to variation of Poisson's ratio of the material. Results obtained have been discussed numerically and depicted graphically. It is found that the presence of non-homogeneity and thickness variation influence significantly the stresses and pressure required for initial yielding. The thickness variation reduces the magnitude of the stresses and pressure needed for fully plastic state.

Key Words : Elastic-plastic Transition; Disc; Internal Pressure; Generalized Strain Measure

INTRODUCTION

A literature survey indicates that circular discs with constant material properties have been analysed by several workers under various conditions. Durban[2] found an exact solution for the internally pressurized elastico-plastic, strain-hardening, annular plate, and Chaudhuri[1] obtained stresses in a non-homogeneous rotating annulus by varying Poisson's ratio of the material while keeping Young's modulus constant. The problem of a rotating disc with varying thickness and inhomogeneity subjected to a steady inhomogeneous temperature field has been solved by Yeh and Han[15], and Tutuncu[14] investigated effect of anisotropy on stresses in rotating discs. Guven[5&6] studied the plane state of stress in elastic-plastic annular discs with variable thickness subjected to external and internal pressures, assuming Tresca's yield condition, its associated flow rule and strain hardening. These investigations are based on the assumption of infinitesimal strains. Following transition approach, we worked on the problem of a thin rotating annular disc with variable thickness, with varying rigidity modulus and of a thick cylinder with variable compressibility under internal pressure[3, 4, 10, 11]. The theory utilizes the concept of generalized strain measure and asymptotic transition through the critical points of differential system defining the deformed field. Seth[8] has defined the generalized strain measure as,

$$e_{ii} = \frac{1}{m}\left[1 - \left(1 - 2\,\overset{A}{e}_{ii}\right)^{m/2}\right], \qquad \ldots (1)$$

where m is the measure and $\overset{A}{e}_{ii}$ are the principal Almansi finite strain components.

In this paper an attempt has been made to study the elastic-plastic transition in a non-homogeneous disc with variable thickness subjected to internal pressure. Non-homogeneity in the disc is taken due to variation of Poisson's ratio of the material. The thickness 'h' and Poisson's ratio 'ν' are assumed to vary in the radial direction as,

$$h = h_0 \, [r/b]^{-k}, \; \nu = \nu_0 \, [r/b]^{n}, \qquad \text{... (2)}$$

where h_0, ν_0, k and n are real constants and $0 < \nu \leq 1/2$.

GOVERNING EQUATIONS

Consider a thin disc of non-constant thickness with inner radius 'a' and outer radius 'b' subjected to internal pressure p. The disc is made of the material having varying Poisson's ratio and thickness is taken sufficiently small so that it is effectively in a state of plane stress. Let r, θ, z refer to the radial, tangential and axial directions of the disc; $J_{rr}, J_{\theta\theta}, J_{zz}$ and $e_{rr}, e_{\theta\theta}, e_{zz}$ are the stress and strain components in the respective directions. The displacement components are given by[2]

$$u = r(1 - \beta), \; v = 0, \; w = d_1 z, \qquad \text{... (3)}$$

where β is a function of r only and d_1 is a constant. The generalized components of strain are obtained as[8&9]

and

$$\left. \begin{aligned} C_{rr} &= \frac{1}{m}[1 - (r\beta' + \beta)^{m}], \; e_{\theta\theta} = \frac{1}{m}[1 - \beta^{m}], \\ e_{zz} &= \frac{1}{m}[1 - (1 - d_1)^{m}], \; e_{r\theta} = e_{\theta z} = e_{zr} = 0, \end{aligned} \right| \qquad \text{... (4)}$$

where

$$\beta' = d\beta/dr.$$

The stress-strain relations are[12]

$$\left. \begin{aligned} J_{rr} &= \lambda c \, [e_{rr} + e_{\theta\theta}] + 2\mu e_{rr}, \\ T_{\theta\theta} &= \lambda c \, [e_{rr} + e_{\theta\theta}] + 2\mu \, e_{\theta\theta} \\ \text{and} \quad J_{zz} &= J_{zr} = J_{r\theta} = J_{\theta z} = 0, \end{aligned} \right| \qquad \text{... (5)}$$

where λ is a constant, $\mu = \mu(r)$ and $c = 2\mu/(\lambda + 2\mu)$.

Substituting eq. (4) in eq. (5), the stress components can be obtained as,

$$\left. \begin{aligned} J_{rr} &= \frac{2\mu}{m}[3 - 2c - \beta^{m}[1 - c + (2 - c)(p + 1)^{m}]] \\ \text{and} \quad J_{\theta\theta} &= \frac{2\mu}{m}[3 - 2c - \beta^{m}[2 - c + (1 - c)(p + 1)^{m}]], \end{aligned} \right| \qquad \text{... (6)}$$

where

$$r\beta'/\beta p.$$

The equation of equilibrium is[6&13]

$$\frac{d}{dr}(h_r, J_{rr}) - h J_{\theta\theta} = 0. \quad \text{... (7)}$$

Using eq. (6) in eq. (7), we get a nonlinear differential equation in β as,

$$m(2-c)p\,\beta^{m+1}(p+1)^{m-1}\frac{dp}{d\beta}$$

$$= \left[\begin{array}{l} 2\left(\frac{\mu'}{\mu}+\frac{h'}{h}\right)[3-2c-\beta^m[1-c+(2-c)(p+1)^m]]+ \\ +\beta^m[1-(p+1)^m]+rc'\beta^m[1+(p+1)^m] \\ -m\beta^m p[1-c+(2-c)(p+1)^m-2rc', \end{array}\right] \quad \text{... (8)}$$

where

$$h' = \frac{dh}{dr}, \mu' = \frac{d\mu}{dr}, c' = \frac{dc}{dr}.$$

The critical points of β in eq. (8) are $\rho = -1$ and $\pm\infty$.

The boundary conditions are

$$J_{rr} = -p \text{ at } r = a$$

and

$$J_{rr} = 0 \text{ at } r = b. \quad \text{... (9)}$$

SOLUTION

It is seen[3, 4, 7, 11] that the asymptotic solution through the principal stress leads from elastic to plastic state at the transition point $\rho \to \pm\infty$.

Therefore, we take the transition function R_1 as,

$$R_1 \equiv J_{\theta\theta} = \frac{2\mu}{m}[3-2c-\beta^m[2-c+(1-c)(\rho+1)^m]]. \quad \text{... (10)}$$

Taking the logarithmic differentiation of eq. (10) with respect to r, substituting the value of $dp/d\beta$ from eq. (8) and taking the asymptotic value as $\rho \to \pm\infty$, we get after integration

$$J_{\theta\theta} = \frac{A\gamma}{h}\exp f_1(r) \quad \text{... (11)}$$

where A_1 is a constant of integration and

$$f_1(r) = -\int \frac{dr}{r(2-c)}.$$

Substituting eq. (11) in eq. (7), we get after integration

$$rhJ_{rr} = A_2 + A_1\, I_1(r), \qquad \text{... (12)}$$

where A_2 is a constant of integration and

$$I_1(r) = \int v \exp f_1(r)\, dr.$$

Using boundary conditions (9) in eq. (12), the stresses (11) and (12) can be obtained as

$$\left.\begin{aligned} J_{rr} &= \frac{aph(a)}{h}\left[\frac{I_1(r) - I_1(b)}{I_1(b) - I_1(a)}\right] \\ \text{and} \qquad J_{\theta\theta} &= \frac{apv\, h(a)}{h[I_1(b) - I_1(a)]} \exp f_1(r). \end{aligned}\right| \qquad \text{... (13)}$$

For a disc whose material and thickness vary according to eq. (2), the transitional stresses (13) become

$$J_{\theta\theta} = \frac{v_0 pt_2 f_2(R)}{d_0 - t_0} R^{t_1},\ J_{rr} = pt_2\left[\frac{f_2(R) - d_0}{d_0 - t_0}\right] R^{k-1}, \qquad \text{... (14)}$$

where

$$R = r/b,\ R_0 = a/b,\ d_0 = \exp(v_0/n),\ t_0 = \exp(v_0 R_0^n/n),$$

and

$$f_2(R) = \exp(v_0 R^n/n),\ t_1 = n + k - 1,\ t_2 = R_0^{1-k}.$$

It can be seen from eq. (14) that $|J_{\theta\theta} - J_{rr}|$ has the greatest value at $R = R_0$ for

$$1 > v_0\left[\frac{t_2 d_0 - t_0 R_0^n}{d_0 - t_0}\right]. \qquad \text{... (15)}$$

Therefore, yielding in this case starts at the inner surface of the disc (otherwise at the outer surface) and we have

$$|J_{\theta\theta} - J_{rr}|_{R=R_0} = \left|\frac{\gamma_0 pt_0 R_0^n}{d_0 - t_0} + p\right| \equiv Y. \qquad \text{... (16)}$$

The pressure necessary for initial yielding is given by

$$\frac{p}{Y}=\frac{1}{\left[1+\dfrac{v_0 t_0 R_0^n}{d_0-t_0}\right]}\equiv p_1 \qquad \text{... (17)}$$

and the stresses (14) can be expressed as

$$\sigma_\theta=\left(\frac{1-p_1}{t_0 t_3}\right)f_2(R)\,R^{t_1},\ \sigma_r=\left(\frac{1-p_1}{v_0 t_0 t_3}\right)[f_2(R)-d_0]\,R^{k-1}, \qquad \text{... (18)}$$

where

$$t_3=R_0^{t_1},\ \sigma_\theta=\frac{J_{\theta\theta}}{Y},\ \sigma_r=\frac{J_{rr}}{Y}.$$

As a result, plasticity in the disc spreads from inner to outer surface and for fully plastic state, that is, $\gamma=1/2$ (or $\lambda\to\infty$)[11, 12] eq. (14) gives

$$|J_{\theta\theta}-J_{rr}|_{R=1}=\left|\frac{pt_2}{2(1-\sqrt{R_0})}\right|\equiv Y^*. \qquad \text{... (19)}$$

The presence required by the disc to become fully plastic is given by

$$p_1^*=\frac{2(1-\sqrt{R_0})}{t_2}, \qquad \text{... (20)}$$

and the plastic stresses can be obtained from eq. (14) as

$$\sigma_\theta^*=R^{t_4},\ \sigma_r^*=2(\sqrt{R}-1)\,R^{k-1}, \qquad \text{... (21)}$$

where

$$t_4=k-\frac{1}{2},\ p_1^*=\frac{p}{Y^*},\ \sigma_\theta^*=\frac{J_{\theta\theta}}{Y^*},\ \sigma_r^*=\frac{J_{rr}}{Y^*}.$$

Homogeneous Case

For a disc of homogeneous material (i.e., $n = 0$) eq. (2) gives

$$v=v_0 \text{ (a constant).} \qquad \text{... (22)}$$

Consequently, the transitional stresses (13) become

$$J_{\theta\theta}=\frac{v_0 p t_2 R^{t_5}}{1-t_6},\ J_{rr}=\frac{pt_2 f_3}{1-t_6}(R)\,R^{k-1}, \qquad \text{... (23)}$$

where

$$t_5 = k + \nu_0 - 1, t_6 = R_0^{\nu_0}, f_3(R) = R^{\gamma} - 1.$$

Yielding in the disc starts at the inner surface for

$$1 > \nu_0 \left(\frac{t_2 - t_6}{1 - t_6} \right) \quad \text{... (24)}$$

and we have

$$\left. \begin{array}{l} Y = \left| \left(\dfrac{\nu_0 p t_6}{1 - t_6} \right) + p \right|, p_1 = \dfrac{1}{\left[1 = \dfrac{\nu_0 t_6}{1 - t_6} \right]}, \\ \text{and} \\ \sigma_\theta = \left(\dfrac{1 - p_1}{t_7} \right) R^{t_5}, \sigma_r = \left(\dfrac{1 - p_1}{\nu_0 t_7} \right) f_3(R) R^{k-1} \end{array} \right| \quad \text{... (25)}$$

where

$$t_7 = R^{t_5}.$$

Therefore, for fully plastic state (i.e., $\nu_0 = 1/2$), eq. (23) gives

$$\left. \begin{array}{l} Y^* = \left| \dfrac{p t_2}{2(1 - \sqrt{R_0})} \right|, p_1^* = \dfrac{2(1 - \sqrt{R_0})}{t_2}, \\ \text{and} \\ \sigma_\theta^* = R^{t_4}, \sigma_r^* = 2(\sqrt{R} - 1) R^{k-1} \end{array} \right| . \quad \text{... (26)}$$

It can be seen that eqs. (21) and (26) for the non-homogeneous and homogeneous material are idential for fully plastic state; a result in agreement with the earlier obtained[3]. For vanishing k, the solution describes the behaviour of an elastic-plastic annular disc with uniform thickness.

Numerical Results and Discussion

To study the combined effects of non-homogeneity and thickness variation as stipulated in eq. (2), a disc with radii ratio $R_0 = 0.5$ and $\nu_0 = 0.3$ has been considered. The values for the pressure and stresses have been calculated from eqs. (17)-(21), (25) and (26) for different values of k and n. It is found that a non-homogeneous disc requires higher pressure for initial yielding as compared to the homogeneous case, that is, $p_1 = 0.43518$, 0.47525 and 0.51898 for $n = 0$, 0.5 and 1 respectively, and the pressure is independent of the thickness variation. However, a disc with variable thickness needs lesser pressure to become fully plastic as compared to the flat disc, that is, $p_1^* = 1.17157$, 0.82843 and 0.58579 for $k = 0$, 0.5 and 1 respectively.

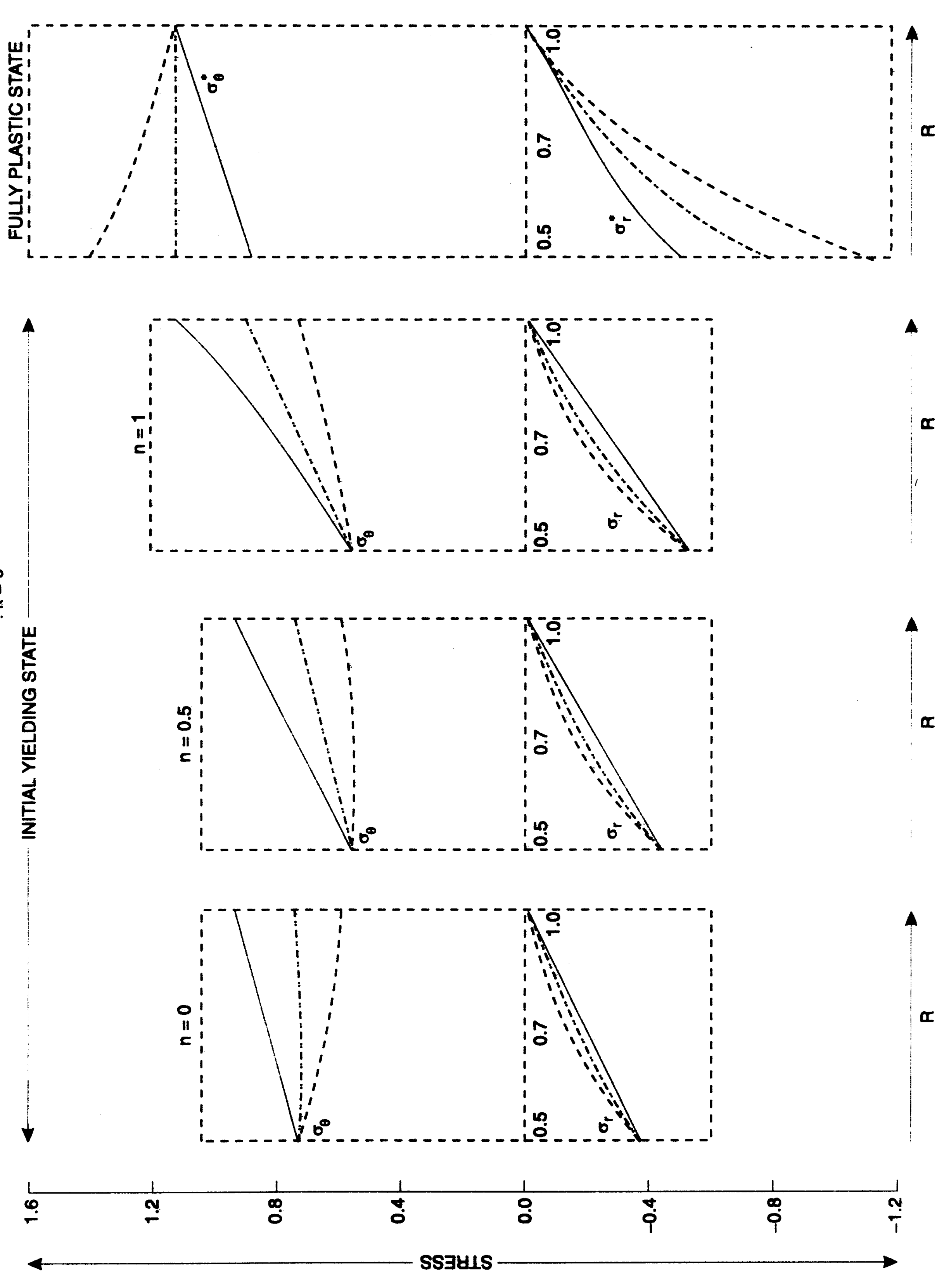

FIG. 1. Stress distribution in a non-homogeneous disc ($\nu = \nu_0 R^n$) with variable thickness ($h = h_0 R^{-k}$) subjected to internal pressure.

Using these values, the curves for radial and circumferential stresses have been drawn in Fig. 1. It can be observed that the presence of non-homogeneity and thickness variation in the analysis of a thin disc subjected to internal pressure influence the elastic-plastic transitional stresses significantly. For a disc made of non-homogeneous material, the magnitude of the circumferential stress increases at the outer surface for initial yielding. The thickness variation, $0 < k \leq 1$, decreases the magnitude of the plastic stresses at the inner surface of the disc. Thus, the use of material possessing non-homogeneity and thickness variation in accordance with eq. (2) may be beneficial for the manufacture of disc components.

REFERENCES

1. P. K. Chaudhuri, *Bull. Calcutta mat. Soc.* **71** (1979), 103.
2. D. Durban, *Acta Mechanica* **66** (1987), 111.
3. S. K. Gupta and R. K. Shukla, *Indian J. pure appl. Math.* **25** (1994), 1089.
4. S. K. Gupta and R. K. Shukla, *Proc. natn. Acad. Sci. (India)* **64** (1994), 479.
5. U. Guven, *Acta Mechanica* **92** (1992), 29.
6. U. Guven, *ZAMM* **75** (1995), 245.
7. S. Hulsurkar, *Indian J. pure appl. Math.* **10** (1979), 597.
8. B. R. Seth, *Int. J. nonlin. Mech.* **1** (1966), 35.
9. B. R. Seth, *ZAMM* **43** (1963), 345.
10. R. K. Shukla, *Indian J. pure appl. Math.* **27** (1996), 487.
11. R. K. Shukla, *Indian J. pure appl. Math.* **28** (1997), 277.
12. I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, McGraw Hill New York, 1950.
13. S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw Hill, New York, 1951.
14. N. Tutuncu, *Int. J. mech. Sci.* **37** (1995), 873.
15. K. Y. Yeh and R.P.S. Han, *J. appl. Mech. Trans ASME,* **61** (1994), 186.