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REMARKS ON HERMAN RINGS OF TRANSCENDENTAL MEROMORPHIC FUNCTIONS

ZHENG JIAN-HUA

Department of Mathematical Sciences, Tsing Hua University, Beijing, P.R. China

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It is proved that a meromorphic function of finite type only has a finite number of Herman rings, and we construct a meromorphic function which has an infinite number of Herman rings.

Key Words : Herman Rings; Transcendental Meromorphic Functions

1. INTRODUCTION AND RESULTS

Let $f: C \mapsto \hat{C}$ be a transcendental meromorphic function, and $f^n, n \in N$, denote the nth iterate of f. Then $f^n(z)$ is defined for all $z \in C$ except possibly for a countable set of the poles of $f, f^2, \ldots, f^{n-1}$. Define Fatou set of f by

$$F(f) = \{z \in C;\ \{f^n\} \text{ is defined and normal in some neighbourhood of } z\}$$

and Julia set of f by $J(f) = \hat{C} \setminus F(f)$. It is well known that $F(f)$ is open and completly invariant under f, i.e., $z \in F(f)$ if and only if $f(z) \in F(f)$. Let U be a component of $F(f)$, then $f^n(U) \subseteq U_n$, where U_n is a component of $F(f)$. If for a smallest integer $p > 0$, $U_p = U$, then U is said to be a periodic component of period p, and if, in addition, U is an annulus, then U is said to be a Herman ring and $\{U_0, U_1, \ldots, U_{p-1}\}$ $(U_0 = U)$ a cycle of Herman rings.

A point z_0 is called periodic if for some $n > 0$, $f^n(z_0) = z_0$. In this case, the smallest n with this property is called the period of z_0. A periodic point z_0 of period n is called attracting, indifferent, or repelling according as $|(f^n)'(z_0)|$ is less than, equal to, or greater than 1. For an indifferent periodic point z_0 of period n, we have $(f^n)'(z_0) = e^{2\pi\alpha i}, 0 \le \alpha < 1$. When α is irrational, z_0 is irrationally indifferent and furthermore, z_0 is a Siegel point if $z_0 \in F(f)$ or a Cremer point if $z_0 \in J(f)$. And when α satisfies the diophantine condition of Siegel type, z_0 must be a Siegel point and f has a cycle of Siegel disks one of which contains z_0.

Denote by sing (f^{-1}) the set of singularities of the inverse function of f, that is, the set of critical and asymptotic values and limit points of these values. A meromorphic function is said to be of finite type if using $f^{-1} < \infty$.

It is well known that an entire function has no Herman rings and a rational function may has Herman rings, but at most a finite number. In this note, we discuss the existence and number of Herman rings of a transcendental meromorphic function. The work was stimulated from the discussion in Zheng[5] on uniform perfectness of Julia set of a transcendental meromorphic function of finite type. By the methods of quasiconformal deformation of Sullivan (cf. [1]) and of Eremenko and Lyubich[3], we prove the following.

Theorem 1 — *Let f be a meromorphic function of finite type. Then f has only a finite number of Herman rings.*

Theorem 1 may not be true for a meromorphic function not being of finite type, and a transcendental meromorphic function may have Herman rings. By the method of quasiconformal surgery of Shishikura[4], we prove the following.

Theorem 2 — *There exists a transcendental meromorphic function which has an infinite number of Herman rings.*

2. Proof of Theorems

Proof of Theorem 1 — We take m annuli $U_1, U_2, \ldots, U_m$ from distinct cycles of Herman rings of f. For each j, U_j is conformally equivalent to the round annulus $\{1 < |\zeta| < r_j\}$. Define the function μ_{t_j} on U_j which in ζ-coordinate is given by $\upsilon_j := t_j \zeta^2/|\zeta|^2$, $0 < t_j < 1$. Since the ellipse field corresponding to υ_j is invariant under the rotation, we have

$$\mu_{t_j}(z) = \mu_{t_j}(f(z))\frac{\overline{f'(z)}}{f'(z)}, \text{ on } U_j. \quad \ldots (1)$$

The solution of Beltrami equation to coefficient μ_{t_j} increases the modulus of the annulus U_j, for by calculation, the solution to v_j does so.

For $t \in T := \{t = (t_1 t_2, \ldots, t_m)\ t \in \boldsymbol{R}^m : 0 < t_j < 1, 1 \le j \le m\}$, define the Beltrami coefficient $\mu_t = \mu_{t_j}$ on U_j. Extend μ_t to the inverse iterates of U_j's under f and $\mu_t = 0$ elsewhere, so that it satisfies (1). It is obvious that $\|\mu_t\|_\infty < 1$.

Put $W := \text{sing} f^{-1}\ V\{\infty\} = \{a_1, a_2, \ldots, a_q, \infty\}$, $q := \text{sing} f^{-1}$. Then

$$f : C \setminus f^{-1}(W) \to C \setminus W$$

is an unbranched covering map. Choose two distinct $b_1, b_2 \notin f^{-1}(W)$, and $a_{q+1} = f(b_1), a_{q+2} = f(b_2)$. Let ϕ_t be the quasiconformal mapping corresponding to μ_t which fixes b_1, b_2 and ∞. Then $f_t := \phi_t \circ f \circ \phi_t^{-1}$ is a transcendental meromorphic function and ϕ_t is continuous in t. $\phi_t(U_j)$ is for f_t a Herman ring whose modulus is a strictly increasing function of t_j, and hence for $t \ne \tau, t, \tau \in T, f_t \ne f_\tau$

Suppose that $m > 2q + 4$. Then there exists a non-constant arc $\gamma : t = t(\sigma), \sigma \in I = [0, 1]$, in T on which $\phi_{t(\sigma)}(a_j) = \phi_{t(0)}(a_j)$ on I. To simplify the notation, below we write $\phi_\sigma = \phi_t(\sigma)$,

$f_\sigma = f_{t(\sigma)}$. Therefore, $\phi_\sigma(a_j) = \phi_0(a_j)$, for $0 \le \sigma \le 1$, $1 \le j \le q+2$. By the Covering Homotopy Theorem there exists a continuous family of homeomorphisms $h_\sigma : C f^{-1}(W) \to C\ \phi_1 \circ f^{-1}(W)$ such that $h_1 = \phi_1$ and we have the commuting diagram

$$\begin{array}{ccc} C \setminus f^{-1}(W) & \xrightarrow{\phi_\sigma \circ f} & C \setminus \phi_\sigma(W) \\ h_\sigma \downarrow & & \downarrow \text{id} \\ C \setminus \phi_1 \circ f^{-1}(W) & \xrightarrow{f_1} & C \setminus \phi_1(W) \end{array}$$

that is, $\phi_\sigma \circ f = f_1 \circ h_\sigma$, $0 \le \sigma \le 1$. Extend h_σ a homeomorphism from $\hat{C}$ onto $\hat{C}$. The functions $h_\sigma(b_i)$, $i = 1, 2$, are continuous in σ. Since

$$f_1 \circ h_\sigma(b_i) = \phi_\sigma \circ f(b_i) = \phi_\sigma(a_{q+i}) = \phi_0(a_{q+i}), \ (i = 1, 2),$$

$h_\sigma(b_i)$ take a discrete set of values. Hence noting $h_1 = \phi_1$, $h_\sigma(b_i) = b_i$, $(i = 1, 2)$. Putting $\sigma = 0$ we obtain $f_0 \circ \phi_0 = \phi_0 \circ f = f_1 \circ h_0$, thus $f_0 = f_1 \circ (h_0 \circ \phi_0^{-1})$. The homeomorphism $h_0 \circ \phi_0^{-1} : C \to C$ is conformal outside a discrete set and has two fixed points b_1, b_2. Therefore, $h_0 \circ \phi_0^{-1} = \text{id}$ and $f_0 = f_1$. Thus we have derived a contradiction, and so $m \le 2q+4$.

Theorem 1 follows.

In order to prove Theorem 2, we need the following result, which is a special version of Main Lemma in [2].

Lemma — Let $\{a_j\}$ be a sequence of complex numbers and such that $a_j \to \infty$, as $j \to \infty$ and let λ be a constant. Then there exists an entire function f such that

$$f(a_j) = a_j, f'(a_j) = \lambda, j \in N.$$

From Lemma, we can immediately deduce the following

Theorem 3 — *Given a sequence $\{e^{2\pi i\alpha_j}\}$ of complex numbers such that for j, α_j satisfies the diophantine condition of Siegel type and a sequence $\{p_j\}$ of positive integers, there exists an entire function f which has cycles of Siegel disks of period p_j with rotation numbers $e^{2\pi i\alpha_j}$, j = 1, 2, ...*

PROOF OF THEOREM 2 — Put

$$P(z) := e^{-2\pi\alpha i} z + z^2, 0 < \alpha < 1,$$

where α is given and satisfies the diophantine condition of Siegel type. Therefore $P(z)$ has a Siegel disk containing 0. We take a sequence $\{a_j\}$ of complex numbers such that $a_j \to \infty$, as $j \to \infty$. Set $P_j(z) := P(z - a_j) + a_j$, and then for each j, $P_j(z)$ has a Siegel disk U_j containing a_j and with the rotation number $e^{-2\pi\alpha i}$. We can take a_j, $j = 1, 2, \ldots$ such that for $i \ne j$, $U_i \cap U_j = 0$. For each j, U_j is conformally equivalent to the disk $\{|\zeta| < 2\}$ by conformal map h_j, and

$$h_j(a_j) = 0,\ h_j \circ P_j \circ h_j^{-1}(\zeta) = e^{-2\pi\alpha i}\zeta, |\zeta| < 2.$$

Set

$$\varphi_j := 1/h_j.$$

Then

$$\varphi_j(a_j) = \infty,\ \varphi_j \circ P_j \circ \varphi_j^{-1}(\zeta) = e^{2\pi\alpha i}\zeta, \frac{1}{2} < |\zeta|. \qquad \ldots (2)$$

Extend φ_j such that it maps conformally $C \setminus U_j$ onto $\left\{|\zeta| < \frac{1}{2}\right\}$ and $\varphi_j(\infty) = 0$.

It follows from Lemma that there exists an entire function f such that

$$f(a_j) = a_j, f'(a_j) = \lambda = e^{2\pi\alpha i}.$$

Then each a_j is a Siegel fixed point of f, and f has the Siegel disks V_j containing a_j and with the rotation number λ. It is clear that for $i \neq j$, $V_i \cap V_j = 0$. Since any polynomial has at most finitely many fixed points, f is transcendental. For each j, V_j is conformally equivalent to the disk $\{|\zeta| < 2\}$ by conformal map ψ_j, and

$$\psi_j \circ f \circ \psi_j^{-1}(\zeta) = e^{2\pi\alpha i}\zeta, |\zeta| < 2. \qquad \ldots (3)$$

Extend ψ_j such that it maps coformally $C \setminus V_j$ onto $2 < |\zeta|$ and $\psi_j(\infty) = \infty$. We draw in V_j a closed Jordan curve $\gamma_j := \psi_j^{-1}(|\zeta| = 1)$ and in U_j a closed Jordan curve $\tilde{\gamma}_j := \varphi_j^{-1}(|\zeta| = 1)$. Modify $\varphi_j^{-1} \circ \psi_j$ to obtain a quasiconformal map $\tilde{\psi}_j : \hat{C} \to \hat{C}$ such that $\tilde{\psi}_j|_{\gamma_j} = \varphi_j^{-1} \circ \psi_j|_{\gamma_j}$ and $\tilde{\psi}_j = \varphi_j^{-1} \circ \psi_j$ in some neighbourhood of $\hat{C} \setminus (V_j \cap \psi_j^{-1} \circ \varphi_j(U_j))$.

Define

$$g := \begin{cases} f, & \text{in } C \setminus \bigcup_{j=1}^{\infty} \text{int}\,\gamma_j, \\ \tilde{\psi}_j^{-1} \circ P_j \circ \tilde{\psi}_j, & \text{in int}\,\gamma_j, j = 1, 2, \ldots. \end{cases}$$

Then it follows from (2) and (3) that g is well defined in C. Let E be the ellipse field which is circles in $C \setminus \bigcup_{j=1}^{\infty} \text{int}\,\gamma_j$ and which is mapped to circles by $\tilde{\psi}_j$ in int γ_j. It is clear that E is invariant under g and has the dilatation $\|\mu\|_\infty < 1$. Then there exists a quasiconformal map φ which fixes 0, 1 and ∞ such that $\tilde{f} = \varphi^{-1} \circ g \circ \varphi$ is a transcendental meromorphic function. Put $W_j := \varphi^{-1}(V_j \setminus \overline{\text{int}\gamma_j})$. Then $\tilde{f}(W_j) = W_j$, and W_j is contained in a Siegel disk or Herman ring of $\tilde{f}$.

Since $\tilde{f}$ has a superattracting fixed point $b_j := \varphi^{-1}(a_j)$ in $\varphi^{-1}(\text{int}\,\gamma_j)$, W_j must be contained in a Herman ring. Thus $\tilde{f}$ has an infinite number of Herman rings.

Theorem 2 follows.

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