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OSCILLATION OF IMPULSIVE NEUTRAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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In this paper, oscillation criteria are established for impulsive neutral differential equations with positive and negative coefficients. Some interesting examples are given, which illustrate that impulses play an important role in giving rise to the oscillation of equations.

Key Words : Oscillation; Neutral Differential Equations; Impulse; Coefficients

1. INTRODUCTION

The impulsive delay differential equations are adequate mathematical model of numerous processes and phenomena studied in physics, biology, engineering, etc. In recent years, there is increasing interest on the oscillatory and nonoscillatory properties of this class of equations (see for example, [1-7]), and many results are obtained. However, to the best of our knowledge, there is little in the way of results for the oscillation of impulsive delay differential equations of neutral type[8].

In this paper, we consider the oscillation of all solutions of impulsive neutral delay differential equations with positive and negative coefficients of the form

$$[x(t) - R(t)x(t-r)]' + P(t)\, x(t-\tau) - Q(t)\, x(t-\sigma) = 0,\ t \geq t_0,$$

$$x(t_k^+) = I_k\,(x(t_k)),\ k = 1, 2, \ldots \qquad \ldots (1.1)$$

Our aim is to establish sufficient conditions for the oscillation of all solutions of (1.1). Some example are also given which show that the oscillation of all solutions of (1.1) can be caused by the impulsive perturbations thoûgh the corresponding equation without impulses admits a nonoscillatory solution.

The following assumptions will be used throughout this paper, without further mention.

(A_1) $r > 0$, $\tau \geq \sigma \geq 0$ and $0 \leq t_0 < t_1 < \ldots < t_k < t_{k+1} \to \infty$ as $k \to \infty$;

(A_2) $P, Q, R \in PC([t_0, \infty), R^+)$;

$$H(t) := P(t) - Q(t - \tau + \sigma) \geq 0$$

and

$$H(t) \not\equiv 0 \text{ on } (t_{k-1}, t_k]\ (k \geq 1),$$

where $R^+ = [0, \infty)$, $PC([t_0, \infty), R^+) = \{f : [t_0, \infty) \to R^+ : f(t)$ is continuous for $t_0 \leq t \leq t_1$, $t_k < t \leq t_{k+1}$ and $\lim_{t \to t_k^+} f(t) = f(t_k^+)$ exists $(k = 1, 2, \ldots)\}$;

(A_3) $I_k(x)$ is continuous and there exist positive numbers b_k^*, b_k such that $b_k^* \leq I_k(x)/x \leq b_k$ for $x \neq 0$ and $k = 1, 2, \ldots$.

With (1.1) one associates an initial condition of the form

$$x_{t_0} = \phi(s),\ s \in [-\rho, 0],\ \rho = \max\ \{r, \tau\}, \quad \ldots (1.2)$$

where $x_{t_0} = x(t_0 + s)$ for $-\rho \leq s \leq 0$ and $\phi \in PC([-\rho, 0], R) = \{\phi : [-\rho, 0] \to R : \phi$ is continuous everywhere except at the finite number of points $\bar{s}$ and $\phi(\bar{s}^+)$ and $\phi(\bar{s}^-) = \lim_{s \to \bar{s}^-} \phi(s)$ with $\phi(\bar{s}^-) = \phi(\bar{s})\}$.

A function $x(t)$ is said to be a solution of equation (1.1) satisfying the initial value condition (1.2) if

(i) $x(t) = \phi(t - t_0)$ for $t_0 - \rho \leq t \leq t_0$, $x(t)$ is continuous for $t \geq t_0$ and $t \neq t_k$ $(k = 1, 2, \ldots)$;

(ii) $x(t) - R(t)\, x(t - r)$ is continuously differentiale for

$$t > t_0,\ t \neq t_k,\ t \neq t_k + r,\ t \neq t_k + \tau,\ t \neq t_k + \sigma,\ k = 1, 2, \ldots \text{ and satisfies (1.1);}$$

(iii) $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^-) = x(t_k)$ and satisfy (1.1).

Using the method of steps as in the case without impulses, one can show the global existence and uniqueness of the solution of the initial value problem (1.1) and (1.2).

As is costumary, a solution of (1.1) is said to be nonoscillatory of it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

2. LEMMAS

In the following, we let

$$0 < p \leq R(t) + \int_{t-\tau+\sigma}^{t} Q(s)\, ds \leq 1. \quad \ldots (2.1)$$

$$z(t) = x(t) - R(t)\, x(t - r) - \int_{t-\tau+\sigma}^{t} Q(s)\, x(s - \sigma) ds. \quad \ldots (2.2)$$

Lemma 2.1 — Assume that $b_0 = 1, b_k \le 1$ and

$$R(t_k^+) \ge R(t_k), \text{ for } k \in E_{1k} = \{k \ge 1 : t_k - r \ne t_i, i < k\} \qquad \dots (2.3)$$

$$\bar{b}_k R(t_k^+) \ge R(t_k), \text{ for } k \in E_{2k} = \{k \ge 1 : t_k - r = t_i, i < k\} \qquad \dots (2.4)$$

where $\bar{b}_k = b_i^*$ when $t_k - r = t_i\ (i < k)$. Let $x(t)$ be a solution of (1.1) such that $x(t-\rho) > 0$ for $t \ge t_0$. Then $z(t) > 0$ for $t \ge t_0$ and $z(t_k^+) \le b_k z(t_k)$ for $k = 1, 2, \dots$.

PROOF : By (1.1) and (2.2), we get

$$z'(t) = -H(t)x(t-\tau) \le 0. \ t_k < t \le t_{k+1}, k \ge 0. \qquad \dots (2.5)$$

From (2.2), we have

$$z(t_k^+) = x(t_k^+) - R(t_k^+)\, x(t_k^+ - r) - \int_{t_k - \tau + \sigma}^{t_k} Q(s)x(s-\sigma)\, ds. \qquad \dots (2.6)$$

If $k \in E_{1k}$, then

$$z(t_k^+) = I_k(x(t_k)) - R(t_k^+)\, x(t_k - r) - \int_{t_k - \tau + \sigma}^{t_k} Q(s)x(s-\sigma)ds$$

$$\le x(t_k) - R(t_k)\, x(t_k - r) - \int_{t_k - \tau + \sigma}^{t_k} Q(s)x(s-\sigma)ds$$

$$= z(t_k).$$

If $k \in E_{2k}$, then

$$z(t_k^+) = I_k(x(t_k)) - R(t_k^+)\, I_i(x(t_k - r)) - \int_{t_k - \tau + \sigma}^{t_k} Q(s)\, x(s-\sigma)\, ds$$

$$\le b_k x(t_k) - R(t_k^+)\, b_i^*\, x(t_k - r) - \int_{t_k - \tau + \sigma}^{t_k} Q(s)x(s-\sigma)ds$$

$$\le x(t_k) - R(t_k^+)\, \bar{b}_k x(t_k - r) - \int_{t_k - \tau + \sigma}^{t_k} Q(s)x(s-\sigma)ds$$

$$\le z(t_k).$$

Since $E_{1k} \cup E_{2k} = \{1, 2, ...\}$. So we have

$$z(t_k^+) \le z(t_k),\ k = 1, 2, \ldots .$$

Thus, $z(t)$ is nonincreasing on $[t_0, \infty)$.

We first claim that $z(t_k) \ge 0$ for $k \ge 1$. Otherwise, suppose that there exists some $m \ge 1$ such that $z(t_m) = -\mu < 0$. Then $z(t) \le -\mu < 0$ for $t \ge t_m$. Therefore, from (2.2), we have

$$x(t) \le -\mu + R(t)\, x(t-r) + \int_{t-\tau+\sigma}^{t} Q(s)x(s-\sigma)ds,\ t \ge t_m. \qquad \ldots (2.7)$$

We consider two possible cases:

Case I — $\limsup_{t\to\infty} x(t) = \infty$. Then there exists a sequence of points $\{s_n\}_{n=1}^{\infty}$ such that $s_n \ge t_m + \rho$, $x(s_n) \to \infty$ as $n \to \infty$ and $x(s_n) = \max\{x(t) : t_m \le t \le s_n\}$. From (2.7) and (2.1), we obtain

$$x(s_n) \le -\mu + R(s_n)\, x(s_n - r) + \int_{s_n-\tau+\sigma}^{s_n} Q(s)x(s-\sigma)ds$$

$$\le -\mu + \left(R(s_n) + \int_{s_n-\tau+\sigma}^{s_n} Q(s)ds \right) x(s_n)$$

$$\le -\mu + x(s_n),$$

which is a contradiction.

Case II — $\limsup_{t\to\infty} x(t) = l < \infty$. Choose a sequence of points $\{s_n\}_{n=1}^{\infty}$ such that $x(s_n) \to l$ as $n \to \infty$ and $x(\xi_n) = \max\ \{x(s) : s_n - \rho \le s \le s_n - \delta\}$, where $\delta = \min\ \{r, \sigma\}$. Then $\xi_n \to \infty$ as $n \to \infty$ and $\limsup_{t\to\infty} x(\xi_n) \le l$. Thus, we have

$$x(s_n) \le -\mu + \left(R(s_n) + \int_{s_n-\tau+\sigma}^{s_n} Q(s)ds \right) x(\xi_n)$$

$$\le -\mu + x(\xi_n)$$

Taking the superior limit as $n \to \infty$ we get $l \le -\mu + l$, which is also a contradiction. Combining the cases I and II we see that $z(t_k) \ge 0\ (k \ge 1)$. From (2.5), $z(t_0) \ge 0$.

To prove $z(t) > 0$ for $t \geq t_0$, we first prove that $z(t_k) > 0\ (k \geq 0)$. If it is not true, then there exists some $m \geq 0$ such that $z(t_m) = 0$. Thus, from (2.5), we have

$$z(t_{m+1}) = z(t_m^+) - \int_{t_m}^{t_{m+1}} H(s)x(s-\tau)ds$$

$$\leq z(t_m) - \int_{t_m}^{t_{m+1}} H(s)x(s-\tau)ds < 0.$$

This contradiction shows that $z(t_k) > 0\ (k \geq 0)$. Therefore, from (2.5), $z(t) \geq z(t_{k+1}) > 0$ for $t \in (t_k, t_{k+1}]\ (k \geq 0)$.

Finally, since $b_k \leq 1$, if $k \in E_{1k}$, then $R(t_k^+) \geq R(t_k) \geq b_k R(t_k)$. Thus, from (2.6), we have

$$z(t_k^+) = I_k(x(t_k)) - R(t_k^+)\, x(t_k - r) - \int_{t_k-\tau+\sigma}^{t_k} Q(s)x(s-\sigma)ds$$

$$\leq b_k\, x(t_k) - b_k\, R(t_k)x(t_k - r) - x(t_k - r) - b_k \int_{t_k-\tau+\sigma}^{t_k} Q(s)x(s-\sigma)ds$$

$$= b_k\, z(t_k)$$

If $k \in E_{2k}$, then $R(t_k^+)\, \bar{b}_k \geq R(t_k) \geq b_k\, R(t_k)$. Thus

$$z(t_k^+) = I_k(x(t_k)) - R(t_k^+)\, I_i\, (x(t_k - r)) - \int_{t_k-\tau+\sigma}^{t_k} Q(s)x(s-\sigma)ds$$

$$\leq b_k\, x(t_k) - R(t_k^+)\, \bar{b}_k\, x(t_k - r) - b_k \int_{t_k-\tau+\sigma}^{t_k} Q(s)x(s-\sigma)ds$$

$$\leq b_k\, z(t_k)$$

Therefore, $z(t_k^+) \leq b_k z(t_k)$ for $k = 1, 2, \ldots$. The proof is complete.

Lemma 2.2 — Let all the assumptions of Lemma 2.1 hold with $\sigma > 0$. Let $\delta = \min\{r, \sigma\}$, $m = \min\{k \geq 1 : t_k > t_0 + \tau\}$ and assume that (1.1) has a solution $x(t)$ such that $x(t-\rho) > 0$ for $t \geq t_0$. Then the second order impulsive differential inequality

$$\left.\begin{array}{ll} y''(t) + \rho^{-1} H(t) p^{(t-\tau)/\delta} y(t) \le 0, & t \ge t_0 + \tau \\ y(t_k^+) = y(t_k), & k = m, m+1, ... \\ y'(t_k^+) \le b_k y'(t_k), & k = m, m+1, ... \end{array}\right\} \qquad ... (2.8)$$

has a solution $y(t)$ such that $y(t) > 0$ and $y'(t^+) > 0$ for $t > t_0 + \tau$, where $y'(t^+) = y'(t)$ when $t \ne t_k$.

PROOF : By Lemma 2.1, we have $z(t) > 0$ for $t \ge t_0$ and (2.5) holds. Set $M = \min \{x(t) : t_0 - \rho \le t \le t_0\}$, then $M > 0$. From the proof of Lemma 2.1, we know that $z(t)$ is nonincreasing. Thus

$$z(t) \ge \frac{1}{\rho} \int_t^{t+\rho} z(s)ds, \ t \ge t_0. \qquad ... (2.9)$$

Since $\delta > 0$, for $t \in [t_0, t_0 + \delta]$, we have

$$x(t) = z(t) + R(t)x(t-r) + \int_{t-\tau+\sigma}^{t} Q(s)x(s-\sigma)ds$$

$$\ge z(t) + M\left(R(t) + \int_{t-\tau+\sigma}^{t} Q(s)ds \right)$$

$$\ge \frac{1}{\rho} \int_t^{t+\rho} z(s)ds + pM.$$

Similarly, for $t \in [t_0 + \delta, t_0 + 2\delta]$, we have

$$x(t) \ge \frac{1}{\rho} \int_t^{t+\rho} z(s)ds + R(t)\left(\frac{1}{\rho} \int_{t-r}^{t-r+\rho} z(s)ds + pM \right)$$

$$+ \int_{t-\tau+\sigma}^{t} Q(s)\left(\frac{1}{\rho} \int_{s-\sigma}^{s-\sigma+\rho} z(u)du + pM \right) ds$$

$$\ge \frac{1}{\rho} \int_t^{t+\rho} z(s)ds + \left(R(t) + \int_{t-\tau+\sigma}^{t} Q(s)ds \right) \frac{1}{\rho} \int_{t-\delta}^{t} z(s)\, ds + p^2 M$$

$$\ge \frac{p}{\rho} \int_{t-\delta}^{t+\rho} z(s)ds + p^2 M.$$

By induction, for $t \in [t_0 + n\delta, t_0 + (n+1)\delta]$, we obtain

$$x(t) \geq \frac{p^n}{\rho} \int_{t-n\delta}^{t+\rho} z(s)ds + p^{n+1} M, \; n = 1, 2, \ldots .$$

Thus, for $t \in [t_0, \infty)$, we have

$$x(t) \geq \frac{1}{\rho} p^{(t-t_0)/\delta} \int_{t_0+\delta}^{t+\rho} z(s)ds + p^{(t-t_0)/\delta+1} M.$$

Therefore, for $t > t_0 + \tau$

$$x(t-\tau) \geq \frac{1}{\rho} p^{(t-\tau-t_0)/\delta} \int_{t_0+\delta}^{t-\tau+\rho} z(s)ds + p^{(t-\tau-t_0)/\delta+1} M.$$

Since $t_0 \geq 0$ and $\rho \geq \tau$, we have

$$x(t-\tau) \geq \frac{1}{\rho} p^{(t-\tau)/\delta} \int_{t_0+\delta}^{t} z(s)ds + p^{(t-\tau)/\delta+1} M. \qquad \ldots (2.10)$$

Substituting (2.10) into (2.5) leads to

$$z'(t) + H(t)p^{(t-\tau)/\delta} \left(\frac{1}{\rho} \int_{t_0+\delta}^{t} z(s)ds + pM \right) \leq 0, \; t \geq t_0 + \tau.$$

Let

$$y(t) = \frac{1}{\rho} \int_{t_0+\delta}^{t} z(s)ds + pM.$$

Then $y(t_k^+) = y(t_k), y'(t_k^+) = \rho^{-1} z(t_k^+) \leq \rho^{-1} b_k z(t_k) = b_k y'(t_k)$ for $k = m, m+1, \ldots$. Thus $y(t) > 0, y'(t^+) > 0$ for $t > t_0 + \tau$ and $y(t)$ satisfies (2.8). The proof is complete.

Lemma 2.3 — Let all the assumptions of Lemma 2.1 hold with $p = 1$. Assume that (1.1) has a solution $x(t)$ such that $x(t-\rho) > 0$ for $t \geq t_0$. Then there exists some $T > t_0$ such that the second order impulsive differential inequality

$$\left.\begin{array}{ll} y''(t) + \rho^{-1} H(t)y(t) \leq 0, & t \geq T+\tau, t \neq t_k, \\ y(t_k^+) = y(t_k), & k = m, m+1, \ldots \\ y'(t_k^+) \leq b_k y'(t_k), & k = m, m+1, \ldots \end{array}\right\} \qquad \ldots (2.11)$$

has a solution $y(t)$ such that $y(t) > 0$ and $y'(t^+) > 0$ for $t > T$, where $m = \min \{k \geq 1 : t_k > T + \tau\}$ and $y'(t^+) = y'(t)$ when $t \neq t_k$.

PROOF : By Lemma 2.1, we have $z(t) > 0$ for $t \geq t_0$ and (2.5) holds. Set $M = 2^{-1} \min \{x(t) : t_0 - \rho \leq t \leq t_0\}$, then $M > 0$ and $x(t) > M$ for $t_0 - \rho \leq t \leq t_0$. We claim that

$$x(t) > M, \ t \in (t_0, t_1]. \qquad \text{... (2.12)}$$

If (2.12) does not hold, then there exists a $t^* \in (t_0, t_1]$ such that $x(t^*) = M$ and $x(t) > M$ for $t_0 - \rho \leq t < t^*$. From (2.2), we have

$$M = z(t^*) + R(t^*)\, x(t^* - r) + \int_{t^* - \tau + \sigma}^{t^*} Q(s) x(s - \sigma) ds$$

$$> \left(R(t^*) + \int_{t^* - \tau + \sigma}^{t^*} Q(s) ds \right) M = M,$$

which is a contradiction and so (2.12) holds. Noting that $z(t_1^+) \geq z(t_2) > 0$, we have

$$x(t_1^+) = x(t_1^+) + R(t_1^+)\, x(t_1^+ - r) + \int_{t_1 - \tau + \sigma}^{t_1} Q(s)\, x(s - \sigma) ds$$

$$> R(t_1^+)\, x(t_1^+ - r) + \int_{t_1 - \tau + \sigma}^{t_1} Q(s) x(s - \sigma) ds.$$

If $1 \in E_{1k}$, then

$$x(t_1^+) > \left(R(t_1) + \int_{t_1 - \tau + \sigma}^{t_1} Q(s) ds \right) M = M.$$

If $1 \in E_{2k}$, then

$$x(t_1^+) > R(t_1^+)\, \overline{b}_1 x(t_1 - r) + \int_{t_1 - \tau + \sigma}^{t_1} Q(s) x(s - \sigma) ds$$

$$\geq \left(R(t_1) + \int_{t_1 - \tau + \sigma}^{t_1} Q(s) ds \right) M = M.$$

Therefore, $x(t_1^+) > M$. Repeating the above asrgument, by induction, we obtain

$$x(t) > M, \ t \geq t_0 - \rho,$$

$$x(t_k^+) > M, \ k = 1, 2, \dots .$$

Because $z(t) > 0$ and $z(t)$ is nonincreasing, we can let $\lim_{t \to \infty} z(t) = a$. There is two possible cases:

Case I — $a = 0$. There exists a $T_1 > t_0$ such that $z(t) \le M/2$ for $t \ge T_1$. Then for any $\bar{t} > T_1$, we have

$$\frac{1}{\rho} \int_{\bar{t}}^{t+\rho} z(s)ds \le M < x(t), t \in [\bar{t}, \bar{t} + \rho].$$

Case II — $a > 0$. Then $z(t) \ge a$ for $t \ge t_0$. From (2.2), we get

$$x(t) \ge a + R(t)\, x(t-r) + \int_{t-\tau+\sigma}^{t} Q(s)x(s-\sigma)ds \ge a + M, t \ge t_0.$$

By induction, it is easy to see that $x(t) \ge na + M$ for $t \ge t_0 + (n-1)\rho$, and so $\lim_{t \to \infty} x(t) = \infty$, which implies that there exists a $T > T_1$ such that

$$\frac{1}{\rho} \int_{T}^{t+\rho} z(s)ds \le 2z(T) < x(t), t \in [T, T + \rho].$$

Combining the cases I and II we see that

$$x(t) > \frac{1}{\rho} \int_{T}^{t+\rho} z(s)ds, \quad t \in [T, T+\rho].$$

Let $l = \min \{k \ge 0 : t_k > T + \rho\}$, we claim that

$$x(t) > \frac{1}{\rho} \int_{T}^{t+\rho} z(s)ds, \; t \in (T+\rho, t_l]. \qquad \text{... (2.13)}$$

Otherwise, there exists a $t^* \in (T + \rho, t_l]$ such that

$$x(t^*) = \frac{1}{\rho} \int_{T}^{t^*+\rho} z(s)ds; \;\; x(t) > \frac{1}{\rho} \int_{T}^{t+\rho} z(s)ds \text{ for } t \in (T+\rho, t^*).$$

Then, from (2.2) and (2.9), we have

$$\frac{1}{\rho} \int_{T}^{t^*+\rho} z(s)ds = z(t^*) + R(t^*)\, x(t^* - r) + \int_{t^*-\tau+\sigma}^{t^*} Q(s)x(s-\sigma)ds$$

$$> \frac{1}{\rho}\int_{t^*}^{t^*+\rho} z(s)ds + \left(R(t^*) + \int_{t^*-\tau+\sigma}^{t^*} Q(s)ds \right)\frac{1}{\rho}\int_{T}^{t^*} z(s)ds$$

$$= \frac{1}{\rho}\int_{T}^{t^*+\rho} z(s)ds.$$

This is a contradiction and so (2.13) holds. Thus if $l \in E_{1k}$, we have

$$x(t_l^+) = z(t_l^+) + R(t_l^+)\, x(t_l^+ - r) + \int_{t_l-\tau+\sigma}^{t_l} Q(s)x(s-\sigma)ds$$

$$\geq x(t_l^+) + R(t_l)\, x(t_l - r) + \int_{t_l-\tau+\sigma}^{t_l} Q(s)x(s-\sigma)ds$$

$$> \frac{1}{\rho}\int_{t_l}^{t_l+\rho} z(s)ds + \left(R(t_l) + \int_{t_l-\tau+\sigma}^{t_l} Q(s)ds \right)\frac{1}{\rho}\int_{T}^{t_l} z(s)ds$$

$$= \frac{1}{\rho}\int_{T}^{t_l+\rho} z(s)ds.$$

Similarly, when $l \in E_{2k}$, we have also

$$x(t_l^+) > \frac{1}{\rho}\int_{T}^{t_l+\rho} z(s)ds.$$

Repeating the above procedure, by induction, we can see that

$$x(t) > \frac{1}{\rho}\int_{T}^{t+\rho} z(s)ds, \ t \geq T.$$

Thus, for $t > T + \tau$, we obtain

$$x(t-\tau) > \frac{1}{\rho}\int_{T}^{t} z(s)ds. \qquad \text{... (2.14)}$$

Substituting (2.14) into (2.5) leads to

$$z'(t) + H(t)\left(\frac{1}{\rho}\int_{T}^{t} z(s)ds \right) \leq 0, \ t > T + \tau.$$

Set

$$y(t) = \frac{1}{\rho} \int_T^t z(s)ds, \quad t > T + \tau.$$

Then $y(t_k^+) = y(t_k), y'(t_k^+) = \rho^{-1} z(t_k^+) \le \rho^{-1} b_k z(t_k) = b_k y'(t_k)$ for $k = m, m+1, \ldots$. Thus $y(t) > 0, y'(t^+) > 0$ for $t > T + \tau$ and $y(t)$ satisfies (2.11). The proof is complete.

Lemma 2.4 — Consider the impulsive differential inequality

$$\left.\begin{array}{ll} y''(t) + G(t)\, y(t) \le 0, & t \ge t_0, t \ne t_k, \\ y(t_k^+) \ge y(t_k), & k = 1, 2, \ldots \\ y'(t_k^+) \le c_k\, y'(t_k), & k = 1, 2, \ldots \end{array}\right\} \qquad \ldots (2.15)$$

where $0 \le t_0 < t_1 < \ldots < t_k < t_{k+1} \to \infty$ as $k \to \infty$, $G(t) \in PC([t_0, \infty), R^+)$ and $c_k > 0$.

If

$$\sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \frac{1}{c_0 c_1 \ldots c_i} G(y)dt = \infty, \qquad \ldots (2.16)$$

where $c_0 = 1$. Then the inequality (2.15) has no solution $y(t)$ such that $y(t) > 0$ and $y'(t) > 0$ for $t \ge t_0$.

The conclusion of this Lemma follows from the similar arguments to that in the Theorem 1 in [9] and by letting $\varphi(x) = x$. We omit the details.

3. Theorems and Proofs

In this section, we suppose that all assumptions of Lemma 2.1 and condition (2.1) are satisfied.

Theorem 3.1 — *Let* $m = \min \{k \ge 1 : t_k > t_0 + \tau\}$. Assume $\sigma > 0$ and

$$\int_{t_0+\tau}^{t_m} G(t)dt + \sum_{i=0}^{\infty} \frac{1}{b_m \ldots b_{m+i}} \int_{t_{m+i}}^{t_{m+i+1}} G(t)\, dt = \infty, \qquad \ldots (3.1)$$

where

$$G(t) = \rho^{-1} H(t) p^{(t-\tau)/\delta}. \qquad \ldots (3.2)$$

Then all solutions of (1.1) oscillate.

PROOF : Suppose that (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that $x(t - \rho) > 0, t \ge t_0$. Then, by Lemma 2.2, the second order impulsive differential inequality (2.8) has a solution $y(t)$ such that $y(t) > 0$ and $y'(t^+) > 0$ for $t > t_0 + \tau$.

On the other hand, by Lemma 2.4, the second order impulsive differential inequality (2.8) has no solution $y(t)$ such that $y(t) > 0$ and $y'(t^+) > 0$ for $t > t_0 + \tau$, which is a contradiction. The proof is complete.

The proof of the following Theorem 3.2 is same as Theorem 3.1 except substituting Lemma 2.2 by Lemma 2.3, and so is omitted.

Theorem 3.2 — *Let* $m = \min\{k \geq 1 : t_k > t_0 + \tau\}$. *Assume* $p = 1$ *and*

$$\sum_{i=0}^{\infty} \frac{1}{b_m \ldots b_{m+i}} \int_{t_{m+i}}^{t_{m+i+1}} H(t)\, dt = \infty, \qquad \ldots (3.3)$$

Then all solutions of (1.1) oscillate.

Remark 3.1. Note that $\sigma > 0$ (and so $\delta > 0$) is a essential condition in Theorem 3.1. However, we do not require $\sigma > 0$ in Theorem 3.2. Therefore, Theorem 3.2 cannot be seen as a corollary of Theorem 3.1.

Corollary 3.1 — Assume $\sigma > 0$ and that there exists a constant $\beta > 0$ such that

$$\frac{1}{b_k} > \left(\frac{t_{k+1}}{t_k}\right)^{\beta}, \quad k = 1, 2, \ldots, \qquad \ldots (3.4)$$

$$\int_{t_0}^{\infty} t^{\beta} H(t)\, p^{(t-\tau)/\delta}\, dt = \infty. \qquad \ldots (3.5)$$

Then all solutions of (1.1) oscillate.

PROOF : From (3.4), we have

$$\int_{t_0+\tau}^{m} G(t)\, dt + \sum_{i=0}^{\infty} \frac{1}{b_m \ldots b_{m+i}} \int_{t_{m+i}}^{t_{m+i+1}} G(t)\, dt$$

$$\geq \frac{1}{b_m} \int_{t_m}^{t_{m+1}} G(t) dt + \ldots + \frac{1}{b_m \ldots b_{m+n}} \int_{t_{m+n}}^{t_{m+n+1}} G(t)\, dt$$

$$\geq \frac{1}{t_m^{\beta}} \left(\int_{t_m}^{t_{m+1}} t_{m+1}^{\beta}\, G(t) dt + \ldots + \int_{t_{m+n}}^{t_{m+n+1}} t_{m+n+1}^{\beta}\, G(t)\, dt \right)$$

$$\geq \frac{1}{t_m^{\beta}} \left(\int_{t_m}^{t_{m+1}} t^{\beta} G(t) dt + \ldots + \int_{t_{m+n}}^{t_{m+n+1}} t^{\beta} G(t)\, dt \right)$$

$$= \frac{1}{t_m^{\beta}} \int_{t_m}^{t_{m+n+1}} t^{\beta} G(t) dt.$$

Let $n \geq \infty$ from (3.2) and (3.5), we see that (3.1) holds. According to Theorem 3.1, all solutions of (1.1) oscillate. The proof is complete.

Similarly, we have

Corollary 3.2 — Let $m = \min\{k \geq 1 : t_k > t_0 + \tau\}$. Assume p = 1 and that there exists a constant $\beta > 0$ such that (3.4) holds and

$$\int_{t_m}^{\infty} t^{\beta} H(t)dt = \infty. \qquad \dots (3.6)$$

Then all solutions of (1.1) oscillate.

4. Examples

Example 4.1 — Consider the equation

$$[\, x(t) - 0.5x(t-1)\,]' + \frac{t(t-1)\ln(t-1) - t + 2}{2t(t-1)\ln(t-2)} x(t-2) - 0.5x(t-1) = 0, \ t \geq 4, \qquad \dots (4.1)$$

$$x(t_k^+) = \frac{k}{k+1} x(t_k), \qquad k = 1, 2, \dots, \qquad \dots (4.2)$$

where $t_k = 2k$. It is easy to see that

$$R(t) + \int_{t-\tau+\sigma}^{t} Q(s)ds = 0.5 + 0.5 \equiv 1,$$

$$H(t) = P(t) - Q(t-\tau+\sigma) = \frac{\ln\frac{t-1}{t-2} - \frac{t-2}{t(t-1)}}{2\ln(t-2)},$$

$$\frac{1}{b_k} = \frac{k+1}{k} = \frac{t_{k+1}}{t_k}.$$

Since $x - 1/2x^2 \leq \ln(1+x) \leq x$ for $|x| < 1$, we have

$$H(t) \geq \frac{\frac{1}{t-2} - \frac{1}{2}\frac{1}{(t-2)^2} - \frac{t-2}{t(t-1)}}{2\ln(t-2)} = \frac{5t^2 - 19t + 16}{4t(t-1)(t-2)^2\ln(t-2)}.$$

Thus

$$\int_4^{\infty} t^{\beta} H(t)dt \geq \frac{1}{4}\int_4^{\infty} t \cdot \frac{5t^2 - 19t + 16}{t(t-1)(t-2)^2\ln(t-2)} dt = \infty.$$

By Corollary 3.2, all solutions of (4.1) and (4.2) oscillate.

Remark 4.1: We note that eq. (4.1) has an eventually positive solution $x(t) = \ln t$. Therefore, the oscillatory properties of all solutions of eqs. (4.1) and (4.2) are caused by the impulsive perturbations.

Example 4.2 — Consider the equation

$$\left[x(t)-\frac{t-1}{2t}x(t-1.5)\right]'+\left(\frac{1}{2}+2^t t^{-3/2}\right)x(t-2)-\frac{1}{2}x(t-1)=0,\ t\geq 1, \quad \dots (4.3)$$

$$x(t_k^+)=\frac{k}{k+1}x(t_k),\quad k = 1, 2, \dots, \quad \dots (4.4)$$

where $t_k = k$. Since

$$R(t)+\int_{t-\tau+\sigma}^{t} Q(s)ds=\frac{t-1}{2t}+\frac{1}{2}=1-\frac{1}{2t},$$

$$H(t)=P(t)-Q(t-\tau+\sigma)=2^t t^{-3/2},$$

we have $p = 1/2$, thus

$$G(t)=2^{-1}(1/2)^{t-2}2^t t^{-3/2}=2t^{-3/2},$$

and so

$$\int_1^{\infty} t^{\beta}G(t)dt=2\int_1^{\infty} t\cdot t^{-3/2}dt=\infty.$$

According to Corollary 3.1, all solutions of (4.3) and (4.4) oscillate.

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