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OPTIMAL CONTROL FOR QUASI-STATIC PROBLEM WITH VISCOUS BOUNDARY CONDITIONS

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In this work we assume less simple boundary conditions of dynamic type and are to some extent related to a viscous force acting the boundary of a viscoelastic body occupying bounded region $\Omega \in R^3$. So the existence and uniqueness of solution depending on the class of admissible boundary conditions and constraint imposed on the boundary, by apply the Fourier transform method the quasi-static problem with dynamic boundary conditions still admits one and only one solution[1]. In this work we using the variational methods for solving the problem, hence we find a necessary and sufficient conditions for the existence of a unique optimal control in which minimizes the cost function $J(v)$ on the set of admissible control U_{ad} for the quasi-static problem with dynamic boundary conditions.

Key Words : Viscoelastic Body; Quasi-static Problem with Viscous Boundary Condition; Optimal Control

1. INTRODUCTION

A linearly viscoelastic body is described by the constitutive equation :

$$T(x, t) = G_0(x)E(x, t) + \int_0^\infty \dot{G}(x, s)E(x, t-s)ds, \qquad \text{... (1)}$$

where T is the Cauchy stress tensor, $E = \frac{1}{2}(\nabla u + \nabla u^T)$, u is the displacement vector and

$$G(x, t) = G_0(x) + \int_0^t \dot{G}(x, s)\, ds, \qquad t > 0 \qquad \text{... (2)}$$

is a symmetric fourth-order tensor representing the relaxation function of the viscoelastic material. The quasi-static behaviour of a continum medium is described by the equation :

$$\nabla . T(x, t) + f(x, t) = 0 \qquad (x, t) \in \Omega \times R \qquad \text{... (3)}$$

together with suitable boundary conditions.

For materials of type (1), (3) turns out to be an integro-differential equation of elliptic type, depending on time, whose integral Kernel is $\dot{G}$.

We assume that G satisfies the fading memory principle at least in weak form and we impose on G the restriction dictated by the second Law of Thermodynamics in the Clausius form. These restrictions on G are summarized in the following

(i) $\dot{G}(x,.) \in L^1(R^+)\ \forall_x \in \overline{\Omega};\ G(.,t), \dot{G}(.,t) \in C^1(\Omega) \quad \forall t \geq 0$

(ii) $G(x, t)$ is a symmetric tensor in $\overline{\Omega} \times R^+$; moreover, $G_0(x)$ and $G_\infty = \lim_{t \to \infty} G(x,t)$ are positive definite, that is there exists two positive constants γ_0 and γ_∞ such that :

$$G_0(x)E \,.\, E \geq \gamma_0 \,|\, E \,|^2, \qquad G_\infty(x)\, E \,.\, E \geq \gamma_\infty \,|\, E \,|^2 \qquad \forall x \in \Omega.$$

(iii) The sine Fourier transform of $\dot{G}$:

$$\dot{G}_s\,(x, w) = \int_0^\infty \dot{G}(x, s)\, sinwsds$$

is such that :

$$\dot{G}_s(x, w)E \,.\, E < 0 \qquad \forall x \in \overline{\Omega}, \qquad \forall w > 0, \qquad \forall E \neq 0. \qquad \dots (5)$$

A quasi-static problem for a viscoelastic body with homogeneous Dirichlet boundary conditions has recently been studied in [2] and [3].

Here, we consider a boundary condition related to a viscous force acting on the boundary.

We find the set of equations and inequalities which describe the optimal control of this system.

2. System Description and Some Existence Theorems

Consider the quasi-static problem of a viscoelastic body of the type (1) occupying a bounded open domain of Ω of R^1 when a subset of Γ_0 of its boundary $\partial\Omega$ is fixed and remaining part $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ is subject to a viscous force. In this case quasi-static behaviour of the material is governed by the following system :

$$\left.\begin{array}{ll} -\nabla \,.\, T(x, t) = f(x, t) & (x, t) \in \Omega \times R, \\ y(x, t) = 0 & (x, t) \in \Gamma_0 \times R \\ \lambda\,(x)\dot{y}(x, t) + T(x, t)\,.\, n = 0 & (x, t) \in \Gamma_1 \times R, \end{array}\right\} \qquad \dots (6)$$

and

where $\lambda\,(x) > 0$, $\forall x \in \Gamma_1$ represents the viscosity coefficient corresponding the dissipative stress on Γ_1. Substituting the constitutive eq. (1) into (6), we obtain the following integro-differential problem :

$$\left.\begin{aligned}
&-\nabla.\left(G_0(x)\nabla y(x,t)+\int_0^\infty \dot{G}(x,s)\nabla y(x,t-s)ds\right) \\
&\quad = f(x,t) && (x,t)\in \Omega\times R \\
&y(x,t)=0 && (x,t)\in \Gamma_0\times R \\
&\lambda(x)\dot{y}(x,t)+G_0(x)\nabla y(x,t).n+ \\
&\int_0^\infty \dot{G}(x,s)\nabla y(x,t-s).nds=0, && (x,t)\in \Gamma_1\times R
\end{aligned}\right\} \quad \ldots (7)$$

we make use the functional space $V=H_0^1(\Omega\cap\Gamma_1)$ which denotes the closure, in H^1 norm, of $C^\infty(\Omega)$ functions whose support is a compact subset of $\Omega\times\Gamma_1$.

Following [6] a function y is said to be a weak solution of the quasi-static problem (7), with $f\in L^2(R, L^2(\Omega))$, if $y\in L^2(R, V)$ and if $\forall\psi\in V$ and $\forall t\in R$, it satisfies the following variational form :

$$\left.\begin{aligned}
&\int_\Omega\left(\Gamma_0(\xi)\nabla\psi(\xi,\tau)+\int_0^\infty \Gamma(\xi,\sigma)\nabla\psi(\xi,\tau-\sigma)\delta\sigma\right).\nabla\psi(\xi)\delta\xi \\
&+\frac{\delta}{\delta\tau}\int_{\Omega_1} Iv\tau\ \lambda(\xi)\psi(\xi,\tau)\ \psi(\delta)\delta\Gamma=\int_\Omega\int(\xi,\tau)\ \psi(\xi)\delta\xi
\end{aligned}\right\} \quad \ldots (8)$$

To prove the existence and uniqueness of solutions for (7), we consider the Fourier transform of (7). We identify $\dot{G}(x,.)$ with its casual extension to R, that is we consider $\dot{G}(x,s)=0$ when $s<0$, so that the Fourier transform $\dot{G}$ of $\dot{G}$ can be written as

$$\dot{G}(x,w)=\dot{G}(x,w)-i\dot{G}_s(x,w),$$

where $\dot{G}_s$ and $\dot{G}_c$ are sine and cosine Fourier transforms. Therefore, if G denotes the following fourth-order symmetric tensor

$$G(x,w)=G_0(x)+\dot{G}(x,w),$$

the transformed problem of (6) is defined for each $w\in R$ as :

$$\left.\begin{aligned}
&-\nabla.(G(x,w)\nabla\hat{y}(x,w))=\hat{f}(x,w) && x\in\Omega \\
&\hat{y}(x,w)=0 && x\in\Gamma_0 \\
&i.w\nabla(x)\hat{y}(x,w)+(G(x,w)\nabla\hat{y}(x,w)).n=0 && x\in\Gamma_1
\end{aligned}\right\} \quad \ldots (9)$$

In this way, we reduce the study of an intrgro-differential problem to a family of elliptic problems depending on a parameter $w\in R$.

We say that $\hat{y}$ is a weak solution of problem (9) for each fixed w, if $\hat{y} \in V$ and for each $\psi \in V$, we have :

$$\left.\begin{aligned} \int_{\Omega} G(x, w) \nabla \hat{y}(x) \nabla \psi^{*}(x)dx + i.w \int_{\Gamma_1} \lambda(x) \hat{y}(x) \psi^{*}(x) d\Gamma \\ = \int_{\Omega} \hat{f}(x, w) \psi^{*}(x) dx. \end{aligned}\right\}$$

holds for every $v \in V$.

Remark 1: Observe that the Fourier transform and its inverse are continuous mappings of $L^2(\Omega)$ into itself. Therefore if y is a weak solution of (7) then $\hat{y}(., w)$ is a weak solution of (9) for almost all $w \in R$, whereas if $\hat{y}(., w)$ is a weak solution of (9) almost all $w \in R$ and $\hat{y} \in L^2(R, V)$), then y is a weak solution of (7).

Theorem 1 — *For each fixed $w \in R$, if G satisfies (i)-(iii) and if means $(\Gamma_0) \neq 0$, then* $\forall \hat{f}(., w) \in L^2(\Omega)$ there exists a unique weak solution $\hat{y}(., w) \in V$ of (9).

$$\alpha(y, \psi; w) = \int_{\Omega} G(x, w) \nabla y(x) \nabla \psi^{*}(x) dx + iw \int_{\Gamma_1} \lambda(x) y(x) \psi^{*}(x) d\Gamma$$

is coercive in V i.e. there exists constant $C(w)$ such that

$$\alpha(y, y; w) \geq C(w) \| y \|^2_{H_0^1(\Omega)} \qquad \ldots (11)$$

the following theorem is also proved in [1].

Theorem 2: *Under the hypotheses of theorem (1) on G* and Γ_0, for every $f \in L^2$ *$(R, L^2(\Omega))$, the integro-differential problem (7) has a unique weak solution $y \in L^2(R, V)$.*

Remark 2 — In problem (7), if $\Gamma_0 = \phi$, then $V = H^1(\Omega)$. In order to ensure existence of solution, it is no longer sufficient that $f \in L^2(R, L^2(\Omega))$, but we must require also that the impulse

$$g(x, t) = \int_{-\infty}^{t} f(x, s) ds \qquad \ldots (12)$$

belongs to the space $L^2(R, L^2(\Omega))$ and then we have the following theorem[1].

Theorem 3 — *Under the hypotheses of Theorem (1) on G, for each f such that g defined by (12), belongs to $H^1(R, L^2(\Omega))$, the integro-differential problem (7) with $\Gamma_0 = \phi$ has unique solution $y \in L^2(R, H^1(\Omega))$.*

Remark 3: If G, λ and $\partial \Omega$ are sufficiently regular, we can state that the weak solution y of (7) with $\Gamma_0 = \phi$ is induced a classical solution.

Theorem 4 — *If G satisfied (i)-(iii), $\lambda \in C^1(\partial \Omega)$ and $\partial \Omega$ is of C^2-class, then for every f such that g defined by (12), belongs to $H^1(R, L^2(\Omega))$, the integro-differentail problem (7) with* $\Gamma_0 = \phi$ has *unique solution $y \in L^2(R, H^2(\Omega))$.*

Now we can formulate our control problem.

3. CONTROL PROBLEM

The space $L^2(R, L^2(R))$ is the space of controls.

For a control $u \in L^2(R, L^2(R))$, the state $y(u)$ of the system is given by the solution of :

$$\left.\begin{array}{ll} -\nabla . T(x, t) = f(x, t, u) + u & (x, t) \in \Omega \times R \\ y(x, t, u) = 0 & (u, t) \in \Gamma_0 \times R \\ \lambda(x)\dot{y}(x, t, u) + T(x, t, u) . n = 0 & (x, t) \in \Gamma_1 \times R \end{array}\right\} \quad \ldots (13)$$

The observation equation is given by

$$z(u) = y(x, t, u)$$

and the cost function is given by :

$$J(v) = \int_\Omega | y(x, t, v) - z_d |^2 dx + (N, v, v) \quad \ldots (14)$$

where N is hermitian positive definite operator i.e.

$$(Nu, u) \geq c \| u \|^2 \quad \ldots (15)$$

The control problem is to find

$$\left.\begin{array}{ll} u \in U_{ad} & \text{such that} \\ J(u) \leq J(v) & v \in U_{ad} \end{array}\right\},$$

where U_{ad} is a closed convex subset of $L^2(R, L^2(\Omega))$. Under the above consideration, we may apply theorems in [4] - [5] and [7] to obtain our main theorem :

Theorem 5 — *Assume that (11) and (15) hold. If the cost function is given by (14), then the optimal control exists and characterized by following equations and inequalities :*

$$\left.\begin{array}{ll} -\nabla . T(x, t) = f(x, t, u) + u & (x, t) \in \Omega \times R \\ y(x, t, u) = 0 & (u, t) \in \Gamma_0 \times R \\ \lambda(x)\dot{y}(x, t, u) + T(x, t, u) . n = 0 & (x, t) \in \Gamma_1 \times R \end{array}\right\}$$

$$\int_\Omega p(u)(v - u)dx + (Nu, v - u) \geq 0 \qquad v \in U_{ad}$$

together with (13), where $p(u)$ is the adjoint state.

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