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OSCILLATORY BEHAVIOR OF QUASILINEAR DIFFERENCE EQUATIONS OF SECOND ORDER

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In this paper, we consider second order quasilinear difference equations of the form

$$\Delta(a_n (\Delta y_n)^{\alpha^*}) + F(n, y_n, y_{n-k}, \Delta y_n) = 0, n \in \mathcal{N}(n_0) \qquad \text{... } (E)$$

and

$$\Delta(a_n(\Delta y_n)^{\alpha^*}) + q_n f(y_n, y_{n-k}) = 0, n \in \mathcal{N}(n_0) \qquad \text{... } (E_1)$$

where $\mathcal{N}(n_0) = \{n_0, n_0 + 1, \ldots\}$ (n_0 is a fixed nonnegative integer) and $\Delta y_n = y_{n+1} - y_n$. We give some oscillation criteria for eq. (E) and establish conditions for the existence of nonoscillatory solutions of eq. (E_1). Examples are inserted to illustrate our results.

Key Words : Quasilinear Difference Equation; Oscillation

1. INTRODUCTION

Consider the second order quasilinear difference equations of the form

$$\Delta(a_n (\Delta y_n)^{\alpha^*}) + F(n, y_{n-k}, \Delta y_n) = 0, n \in \mathcal{N}(n_0) \qquad \text{... (1)}$$

and

$$\Delta(a_n (\Delta y_n)^{\alpha^*}) + q_n f(y_n, y_{n-k}) = 0, \; n \in \mathcal{N}(n_0), \qquad \text{... (2)}$$

where

$$(\Delta y_n)^{\alpha^*} = | \Delta y_n |^{\alpha-1} \; \Delta y_n = | \Delta y_n |^{\alpha} \text{ sgn } \Delta y_n, \; \mathcal{N}(n_0) = \{n_0, n_0 + 1, \ldots\},$$

$n_0 \in \mathcal{N} = \{0, 1, 2, 3, \ldots\}$, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $a, q : \mathcal{N}(n_0) \to (0, \infty)$, $F : \mathcal{N}(n_0)/R^3 \to R$ and $f : R^2 \to R$ are real valued continuous functions, $k \in N$ and $\alpha > 0$ is constant.

By a solution of eq. (1) or (2) we mean a nontrivial sequence $\{y_n\}$ defined on $\mathcal{N}(n_0 - k)$ and satisfying (1) or (2) for all $n \in \mathcal{N}(n_0)$. A solution $\{y_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

Determining oscillation criteria for difference equations has attracted a great deal of attention in the last several years, see, for example [1-9, 12, 15] and the references contained therein. Especially, the oscillatory and asymptotic behaviour of solutions of quasilinear difference equation of the form

$$\Delta(a_n (\Delta y_n)^{\alpha^*}) + F(n, u_n) = 0, \qquad \text{... (3)}$$

have been investigated in [10, 11, 13, 14, 16, 17].

In this paper, our aim is to establish criteria for the oscillation of all solutions of equation (1) which generalizes and improves some of the results obtained for eq. (3). Moreover, a sufficient condition for the existence of nonoscillatory solution of eq. (2) is given.

2. Oscillatory Theorems for Equation (1)

In this section we study the oscillatory behaviour of solutions of eq. (1) by assuming that

$$u f(n, u, v, w) > 0 \text{ for } uv > 0 \text{ and } w \in R \text{ and } n \in \mathcal{N}(n_0).$$

Now we introduce the following conditions:

(c_1) $\mathcal{R}(n) = \sum_{s=n_0}^{n-1} \frac{1}{a_s^{1/\alpha}}$ and $\lim_{n \to \infty} \mathcal{R}(n) = \infty$.

(c_2) There exists a β such that $0 < \beta \le \alpha$ and

$$\sum_{n=n_0+k+1}^{\infty} \mathcal{R}^{\beta}(n-k) \frac{|F(n, v_n, v_{n-k}, \Delta v_n)|}{|v_{n-k}|^{\beta}} = \infty$$

for every positive nondecreasing or negative nonincreasing sequence $\{v_n\}$.

(c_3) There exists a θ such that $0 < \theta < \alpha$ and

$$\sum_{n=n_0+k+1}^{\infty} \mathcal{R}^{\alpha-\theta}(n-k) \frac{|F(n, v_n, v_{n-k}, \Delta v_n)|}{|v_{n-k}|^{\alpha}} = \infty.$$

for every positive nondecreasing or negative nonincreasing sequence $\{v_n\}$.

(c_4) For every positive nondecreasing or negative nonincreasing sequence $\{v_n\}$ there exists a constant $\beta > \alpha$, such that

$$\sum_{n=n_0+k+1}^{\infty} \mathcal{R}^{\alpha}(n-k) \frac{|F(n, v_n, v_{n-k}, \Delta v_n)|}{|v_{n-k}|^{\beta}} = \infty.$$

(c$_5$) $\lim\limits_{n \to \infty} \dfrac{a_n}{a_{n-k}} = c \neq 0.$

(c$_6$) For every positive nondecreasing or negative nonincreasing sequence $\{v_n\}$ there exists a constant $\beta > \alpha$ such that

$$\sum_{n=n_0+k+1}^{\infty} \mathcal{R}^{\alpha}(n) \frac{|F(n, v_n, v_{n-k}, \Delta v_n)|}{|v_{n-k}|^{\beta}} = \infty.$$

Theorem 1 — *If contains* (c_1) *and* (c_2) *hold then every solution of equation (1) oscillates.*

PROOF : Let $\{y_n\}$ be a nonoscillatory solution of eq. (1). Without loss of generality, we may assume that $y_n < 0$ and $y_{n-k} < 0$ for all $n \in N(n_1)$, $n_1 \in N(n_0 + k + 1)$. From the hypotheses we have $\Delta(a_n(\Delta y_n)^{\alpha^*}) > 0$. There are two possible cases :

Case 1 — $a_n(\Delta y_n)^{\alpha^*} > 0$ for $n \in N(n_2), n_2 \in N(n_1)$. This implies that $\Delta y_n \geq \frac{1}{a_n^{1/\alpha}} a_n^{1/\alpha} \Delta y_{n_2}$, $n \in \mathcal{N}(n_2)$. Summing the inequality from n_2 to $n - 1$, we obtain

$$y_n - y_{n_2} \geq a_{n_2}^{1/\alpha} \Delta y_{n_2} R(n). \qquad \text{... (4)}$$

Letting $n \to \infty$ in (4), we get $y_n > 0$, a contradiction.

Case 2 — $a_n(\Delta y_n)^{\alpha^*} < 0$ for $n \in N(n_3), n_3 \in N(n_1)$. Then $\Delta y_n < 0$ for $n \in N(n_3)$. From equation (1), we have

$$a_n |\Delta y_n|^{\alpha-1} \Delta y_n \leq \sum_{i=n}^{\infty} F(i, y_i, y_{i-k}, \Delta y_i), \; n \in \mathcal{N}(n_3). \qquad \text{... (5)}$$

From the monotone property of $\{a_n |\Delta y_n|^{\alpha-1} \Delta y_n\}$ and $k \geq 0$ we have

$$a_{n-k} |\Delta y_{n-k}| \Delta y_{n-k} \leq a_n |\Delta y_n|^{\alpha-1} \Delta y_n.$$

From (5), we obtain

$$|\Delta y_{n-k}|^{\alpha-1} \Delta y_{n-k} \leq \frac{1}{a_{n-k}} \sum_{i=n}^{\infty} F(i, y_i, y_{i-k}, \Delta y_i)$$

and hence

$$\Delta y_{n-k} \leq -\frac{1}{a_{n-k}^{1/\alpha}} \left(-\sum_{i=n}^{\infty} F(i, y_i, y_{i-k}, \Delta y_i) \right)^{\frac{1}{\alpha}}.$$

Summing the last inequality from n_3 to $n-1$, we obtain

$$y_{n-k} \leq y_{n-k} - y_{n_3-k} \leq -\sum_{i=n_3}^{n-1} \frac{1}{a_{i-k}^{1/\alpha}} \left(-\sum_{s=n}^{\infty} F(s, y_s, y_{s-k}, \Delta y_s) \right)^{1/\alpha}$$

$$\leq -\sum_{i=n_3}^{n-1} \frac{1}{a_{i-k}^{1/\alpha}} \left(-\sum_{s=n}^{\infty} F(s, y_s, y_{s-k}, \Delta y_s) \right)^{1/\alpha}.$$

$$= -\left[\mathcal{R}(n-k) - \mathcal{R}(n_3-k)\right] \left(-\sum_{s=n}^{\infty} F(s, y_s, y_{s-k}, \Delta y_s) \right)^{1/\alpha}.$$

Since both sides of the above inequality are negative, we have

$$(\mathcal{R}(n-k) - \mathcal{R}(n_3-k))^{\beta} |y_{n-k}|^{-\beta} \leq \left(\left| \sum_{s=n}^{\infty} F(s, y_s, y_{s-k}, \Delta y_s) \right| \right)^{-\beta/\alpha}.$$

Let

$$H(n) = -\sum_{s=n}^{\infty} F(s, y_s, y_{s-k}, \Delta y_s) > 0.$$

Multiplying the last inequality by $-F$, we have

$$(\mathcal{R}(n-k) - \mathcal{R}(n_{3-k}))^{\beta} \frac{F(n, y_n, y_{n-k}, \Delta y_n)}{|y_{n-k}|^{\beta}} \leq - \left| \sum_{s=n}^{\infty} F(s, y_s, y_{s-k}, \Delta y_s) \right|^{-\beta/\alpha}$$

$$F(n, y_n, y_{n-k}, \Delta y_n)$$

$$= -H(n)^{-\beta/\alpha} \Delta H(n).$$

Summing the above inequality from n_3 to $n-1$, we have

$$-\sum_{s=n_3}^{n-1} (\mathcal{R}(s-k) - \mathcal{R}(n_3-k))^{\beta} \frac{F(s, y_s, y_{s-k}, \Delta y_s)}{|y_{s-k}|^{\beta}} \leq -\sum_{s=n_3}^{n-1} \frac{\Delta H(s)}{H(s)^{\beta/\alpha}}. \qquad \text{... (6)}$$

Set $G(t) = H(n) + (t-n)\,\Delta H(n)$, then $G(n) = H(n) > 0$ $G(n+1) = H(n+1) > 0$, $G'(t) = \Delta H(n) < 0$, $0 < G(n+1) \leq G(t) \leq G(n)$ for $n \leq t \leq n+1$

$$\sum_{s=n_3}^{n-1} \frac{\Delta H(s)}{H^{\beta/\alpha}(s)} = \sum_{s=n_3}^{n-1} \int_s^{s+1} \frac{G'(t)}{G^{\beta/\alpha}(s)} dt \geq \sum_{s=n_3}^{n-1} \int_s^{s+1} \frac{G'(t)}{G^{\beta/\alpha}(t)} dt \qquad \ldots (7)$$

$$= \frac{1}{1-\frac{\beta}{\alpha}} \sum_{s=n_3}^{n-1} \left(G^{1-\frac{\beta}{\alpha}}(s+1) - G^{1-\frac{\beta}{\alpha}}(s) \right)$$

$$= \frac{1}{1-\frac{\beta}{\alpha}} \left(G^{1-\frac{\beta}{\alpha}}(n) - G^{1-\frac{\beta}{\alpha}}(n_3) \right). \qquad \ldots (8)$$

From (6) and (8), we have

$$-\sum_{s=n_3}^{n-1} (\mathcal{R}(s-k) - \mathcal{R}(n_3-k))^{\beta} \frac{F(s, y_s, y_{s-k}, \Delta y_s)}{|y_{s-k}|^{\beta}} \leq \frac{G^{1-\frac{\beta}{\alpha}}(n_3)}{1-\frac{\beta}{\alpha}}. \qquad \ldots (9)$$

Letting $n \to \infty$ in the above inequality and using the condition $0 < \beta < \alpha$, one can conclude that the left side of (9) is convergent. This contradicts (c_2). For $\alpha = \beta$, we have from (7)

$$\sum_{s=n_3}^{n-1} \frac{\Delta H(S)}{H^{\beta/\alpha}(s)} = \sum_{s=n_3}^{n-1} \frac{G'(t)}{G(t)} dt = \log G(n) - \log G(n_3). \qquad \ldots (10)$$

From (6) and (10), we have

$$-\sum_{s=n_3}^{n-1} (\mathcal{R}(s-k) - R(n_3-k))^{\beta} \frac{F(s, y_s, y_{s-k}, \Delta y_s)}{|y_{s-k}|^{\beta}} \leq \log G(n_3). \qquad \ldots (11)$$

Letting $n \to \infty$ in the above inequality, we find that the left side of (11) is convergent. This again contradicts condition (c_2). Thus the proof is complete.

Example 1 — Consider the difference equation

$$\Delta(a_n(\Delta y_n)^{\alpha^*}) + q_n |y_{n-k}|^{\beta} \operatorname{sgn} y_{n-k} = 0, \qquad \ldots (12)$$

where β is a constant such that $0 < \beta < \alpha$, $\{a_n\}$ satisfies condition (c1) and k is a positive integer. To apply Theorem 1 to eq. (12), we need to assume that $q_n > 0$ for $n \in N(n_0)$, and

$$\sum_{n=n_0+k+1}^{\infty} \mathcal{R}^{\beta}(n-k)\, q_n = \infty.$$

Then all solutions of eq. (12) oscillate.

Theorem 2 — *If condition* (c_1) *and* (c_3) *hold, then every solution of eq.* (1) *oscillates.*

PROOF : In fact

$$\sum_{n=n_0+k+1}^{\infty} \mathcal{R}^{\alpha-\theta}(n-k)\frac{|F(n, v_n, v_{n-k}, \Delta v_n)|}{|v_{n-k}|^{\alpha-\theta}} = \sum_{n=n_0+k+1}^{\infty} \mathcal{R}^{\alpha-\theta} \quad \text{... (13)}$$

Since $|v_{n-k}|^{\theta} \geq |v_{n_0+k+1-k}|^{\theta} = |v_{n_0+1}|^{\theta} > 0$ for $n \in \mathcal{N}(n_0+k+1)$. From (13) and ($c_3$), we get condition ($c_2$). Then from Theorem 1, we obtain Theorem 2.

Example 2 — Consider the difference equation

$$\Delta(a_n(\Delta y_n)^{\alpha^*}) + \sum_{i=0}^{m} q_{in} y_{n-k}^{2i+1} = 0, n \in \mathcal{N}(n_0) \quad \text{... (14)}$$

where $\{a_n\}$ satisfies condition (c_1), k is a nonnegative integer, $\{q_{in}\}$ are positive real sequences and $\alpha \geq 1$. Further assume that

$$\sum_{n=n_0+k+1}^{\infty} \mathcal{R}^{\alpha-\theta}(n-k)\left(\sum_{i=0}^{m} q_{in}\right) = \infty, \ 0 < \theta < \alpha. \quad \text{... (15)}$$

Then by Theorem 2, every solution of eq. (14) oscillates.

Theorem 3 — *If condition* (c_1) *and* (c_4) *hold, then every solution of eq.* (1) *oscillates.*

PROOF : In fact

$$\sum_{n=n_0+k+1} \mathcal{R}^{\alpha}(n-k)\frac{|F(n, v_n, v_{n-k}, \Delta v_n)|}{|v_{n-k}|^{\beta}}$$

$$= \sum_{n=n_0+k+1}^{\infty} \frac{\mathcal{R}^{\beta}(n-k)}{\mathcal{R}^{\beta-\alpha}(n-k)} \frac{|F(n, v_n, v_{n-k}, \Delta v_n)|}{|v_{n-k}|^{\beta}}. \quad \text{... (16)}$$

Since $R^{\beta-\alpha}(n-k) \geq R^{\beta-\alpha}(n_0+k+1-k) = R^{\beta-\alpha}(n_0+1)$ for $n \ N(n_0+k+1)$. From (16) and condition (c_4), we get condition (c_2). Then from Theorem 1, we obtain Theorem 3.

Example 3 — Consider the difference equation

$$\Delta(a_n(\Delta y_n)^{\alpha^*} + \sum_{i=1}^{m} q_{in} y_{n-k}^{2i+1} = 0 \ n \in \mathcal{N}(n_0) \quad \text{... (17)}$$

where $\{a_n\}$ satisfies condition (c_1) and k is a nonnegative integer.

If $q_{qn} \geq 0, i \in \{1, 2, ..., m\}$ and $\beta > \alpha > 1$ such that

$$\sum_{n=n_0+k+1}^{\infty} \mathcal{R}^{\alpha}(n-k)\left(\sum_{i=1}^{m} q_{in}\right) = \infty$$

then by Theorem 3, every solution of equation (17) oscillates.

Theorem 4 — *If conditions (c1), (c5) and (c6) hold, then every solution of eq. (1) oscillates.*

PROOF : From the conditions $R(n+1) > R(n)$ and $\lim_{n \to \infty} \mathcal{R}(n) = \infty$, we have by Cauchy-Stolz theorem

$$\lim_{n \to \infty} \frac{\mathcal{R}(n-k)}{\mathcal{R}(n)} = \lim_{n \to \infty} \frac{\Delta \mathcal{R}(n-k)}{\Delta \mathcal{R}(n)} = \lim_{n \to \infty} \frac{\dfrac{1}{a_{n-k}^{1/\alpha}}}{\dfrac{1}{a_n^{1/\alpha}}} = \left(\lim_{n \to \infty} \frac{a_n}{a_{n-k}}\right)^{1/\alpha} = c^{1/\alpha} \neq 0.$$

Thus $R(n)$ and $R(n-k)$ are of the same order if $n \to \infty$, so we obtain condition (c_4) from condition (c_6). The proof of the theorem then follows by the application of Theorem 3.

Example 4 — Consider the difference equation

$$\Delta(n^2 \mid \Delta y \mid \Delta y_n) + \frac{n^3 (n-1)^3 [(2n+3)^2 (n+1)^2 + (2n+1)^2 (n+2)^2]}{(n+1)^2 (n+2)^2}$$

$$(y_n + y_{n-1})^3 = 0, \; n \in \mathcal{N}(2) \qquad \text{... (18)}$$

where

$$a_n = \frac{1}{n^2}, \; \alpha = 2, \; \beta = 3, \; q_n = \frac{n^3(n-1)^3 [(2n+3)^2 (n+1)^2 + (2n+1)^2 (n+2)^2]}{(n+1)^2 (n+2)^2}$$

and $k = 1$. It is easy to see that eq. (18) satisfy the conditions of Theorem 4. Therefore, every solution of equation (18) oscillates. In fact, eq. (18) has a oscillatory solution $\{y_n\} = \left\{\frac{(-1)^n}{n}\right\}$.

Remark : If $a_n = 1$, $k = 0$ and $\alpha = 1$, then conditions (c_2), (c_3) and (c_4) become the conditions (c_1), (c_3) and (c_2) respectively of Theorem 1 in [9].

3. EXISTENCE OF NONOSCILLATORY SOLUTION FOR EQUATION (2)

In this section we obtain conditions for the existence of nonoscillatory solution of eq. (2).

Theorem 5 — *Equation (2) has a nonoscillatory solution if*

$$uf(u, v) > 0 \text{ for } uv > 0$$

and

$$\sum_{n=n_0+1}^{\infty} \mathcal{R}(n+1)\, q_n^{1/\alpha} < \infty, \qquad \text{... (19)}$$

where $\alpha \geq 1$.

PROOF : Let $M = \text{Max}\ \{f(u, v) : 1/2 \leq u, v \leq 1\}$ and choose $n_1 \in N(n_0 + 1)$ large enough so that

$$2M^{1/\alpha} \sum_{n=n_0+1}^{\infty} \mathcal{R}(n)\, q_n^{1/\alpha} \leq 1.$$

Consider the Banach space B_{n_1} of all bounded real sequences where $n \in N(n_1)$ with supremum norm $\|y\| = \sup_{n \in N(n_0)} |y_n|$. Let

$$S = \{y \in B_{n_1} : 1/2 \leq y_n \leq 1, n \in N(n_0)\}$$

and define an operator $T : S \rightarrow B_{n_1}$ by the formula

$$Ty_n = 1/2 + R(n)\left[\sum_{i=n}^{\infty} q_i f(y_i, y_{i-k})\right]^{1/\alpha}$$

$$+ \sum_{i=n_1+1}^{\infty} \mathcal{R}(i+1)\left\{\left(\sum_{s=i}^{\infty} q_i f(y_s, y_{s-k})\right)^{1/\alpha} - \left(\sum_{s=i+1}^{\infty} q_s f(y_s, y_{s-k})\right)^{1/\alpha}\right\},$$

$n \in \mathcal{N}(n_1 + 1)$

$= 1,\ n_0 \leq n \leq n_1.$

Clearly, S is bounded, closed and convex subset of B_{n_1}. In addition, the operator T is continuous and relatively compact. Finally for any $y \in S$, we have

$$Ty_n \geq 1/2$$

and

$$Ty_n \leq 1/2 + \left[\sum_{n=n_1+1}^{\infty} \mathcal{R}^{\alpha}(n+1)\, q_n f(y_n, y_{n-k})\right]^{1/\alpha}$$

$$+\sum_{i=n_1+1}^{\infty} \mathcal{R}(i+1)\left\{\left(\sum_{s=i}^{\infty} q_s f(y_s, y_{s-k})\right)^{1/\alpha} - \left(\sum_{s=i+1}^{\infty} q_s f(y_s, y_{s-k})\right)^{1/\alpha}\right\}. \quad \text{... (20)}$$

Since $x^r - y^r \le (x-y)^r$ for $x \ge y \ge 0$ and $0 < r \le 1$, we have from (20)

$$Ty_n \le 1/2 + \left[\sum_{n=n_i+1}^{\infty} \mathcal{R}^{\alpha}(n+1)\, q_n f(y_n, y_{n-k})\right]^{1/\alpha}$$

$$+ \sum_{n=n_1+1}^{\infty} \mathcal{R}(n+1)\,[q_n f(y_n, y_{n-k})]^{1/\alpha}$$

$$\le 1/2 + \sum_{n=n+1}^{\infty} \mathcal{R}+(n+1)\,(q_n f(y_n, y_{n-k}))^{1/\alpha} + \sum_{n=n_1+1}^{\infty} \mathcal{R}(n+1)\,(q_n\,(f(y_n, y_{n-k}))^{1/\alpha}$$

$$\le 1/2 + 2M^{1/\alpha} \sum_{n=n_1+1}^{\infty} \mathcal{R}\,(n+1)\, q_n^{1/\alpha} \le 1.$$

Thus, $TS \subseteq S$. Therefore, by Schauder's fixed point theorem, T has a fized point $y \in S$. It is clear that $y = \{y_n\}$ is a nonoscillatory solution of equation (2) for $n \in N(n_1+1)$. This completes the proof of the theorem.

Remark : If $f(u, v) = (u+v)^{2m+1}$, m is a positive integer and $\alpha = 1$, then Theorem 5 and Theorem 4 give a necessary and sufficient condition for the oscillation of all solutions of eq. (2), see [Theorem 5, [2]).

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