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A SUSCEPTIBLE-INFECTED REMOVAL (SIR) EPIDEMIC MODEL

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Within the framework of SIR epidemic model with time-dependent recovery rate the time-behaviour of infectives for cholera (+ve) and non-choleric diarrhoea has been studied. Here, the population of greater Calcutta has been considered. The infectivity curves for these diseases, as computed from this model, have been fitted with the data available upto 1991 and have been extrapolated upto 2000 years. The steady number of infectives for the forth coming year is being predicted here.

Key Words : SIR; Epidemic; Infectives; Recovery

1. INTRODUCTION

In greater Calcutta, the diseases like cholera (+ve) and non-choleric diarrhoa break out regularly, occasionally in epidemic form. It is interesting to note that in the classical Susceptible-Infected Removal (SIR) models[1&2] for infectious diseases the epidemic can persists only if the susceptibles are being supplied steadily; for example, through births and immigration. Such are the cases which occur in this part of West Bengal, the city areas of Calcutta and its neighbourhood. Therefore, it is an obvious reason to study epidemic disease for this area with such a mathematical model. It is, of course, more appropriate to use a discrete-time epidemic model in stead of a continuous one. That is, in fact, employed here and we study the two types diseases, cholera (+ve) and non-choleric diarrhoea as the data [5] have been available for the period from 1978 to 1991. The census report of the population for Calcutta is also available for the same period. (Census of India, 1981; 1991) [3, 4].

We first convert the continuous SIR model to the discrete SIR model. In this model, it is proposed that rate of transition to the removal or the recovery rate is to be time-dependent instead of usual constant. Of course, it will be show that the rate is a very slowly varying function of time. It is shown that the theoretically calculated epidemic trends are good agreement with the data. As the published census report for this periods after 1991 not yet available we can only extrapolate for that period upto 2000 year with a rate calculated from the previous period.

2. THE MODEL

The continuous SIR model that is relevant to the present situation is

$$\frac{ds}{dt} = -\beta SI + \mu N,$$

$$\frac{dI}{dt} = \beta SI - \gamma I = \beta I\left(S - \frac{\gamma}{\beta}\right)$$

$$= \beta I(S - \rho) \text{ where, } \rho = \frac{\gamma}{\beta}$$

andM $$\frac{dR}{dt} = \gamma I$$

with initial conditions $S(0) = S_0 > 0$,

$$I(0) = I_0 > 0, R(0) = R_0 = 0$$

and where, β = Susceptible rate,

γ = recovery or removal rate,

μ = specific death rate and

$\rho = \frac{\gamma}{\beta}$ is called the effective

removal rate, i.e. the ratio of the rate at which individuals are removed from the infected category to the rate at which they are added to the same category.

We propose the discrete-time version of the model to be expressible through the following equations :

$$\left\{\begin{array}{l} S_{t+1} - S_t = -\beta S_t I_t + \mu N_t \\ S_{t+1} = S_t(1 - \beta I_t) + \mu N_t \\ \text{or,} \\ I_{t+1} - I_t = \beta S_t I_t - \gamma I_t \\ I_{t+1} = I_t(\beta S_t - \gamma + 1) \\ \text{or,} \\ R_{t+1} - R_t = \gamma I_t \\ \text{or,} \\ R_{t+1} = \gamma I_t + R_t \\ S_t \approx N_t \end{array}\right\}, \quad \ldots (1)$$

where $\gamma(t)$ is tal as to be of the form

$$\gamma(t) = \gamma_0\, e^{a_0 + a_1 t + a_2 t^2} \quad \ldots (2)$$

and where γ_0 = 60.287

a_0 = .00017226

a_1 = .0062581

a_2 = .0000026

a_0, a_1, a_2 is obtained by least square fitting, which, in brief, is described in the following section 3.

3. Cholera (+ve) Infectives Among the Calcuttan

TABLE I : *Human population of Calcutta (Data from census report 1981; 1991)*

Year	Population
1978	40, 46, 613
1979	40, 73, 002
1980	40, 99, 563
1981	41, 26, 297
1982	41, 52, 097
1983	41, 78, 059
1984	42, 04, 183
1985	42, 30, 417
1986	42, 56, 922
1987	42, 83, 540
1988	43, 10, 323
1989	43, 37, 275
1990	43, 64, 394
1991	43, 91, 684

TABLE II : *Yearwise infectives of cholera (+ve) in Calcutta (Source: I.D. Hospital[5])*

Year	Infectives
1979	495
1980	426
1981	483
1982	350
1983	442
1984	131
1985	179
1986	323
1987	117
1988	123
1989	56
1990	59

The value of μ can be calculated from the average value of μ_1, μ_2, ..., obtained from table 1 by using the formula

$$\mu_1 = \frac{1}{t} \ln \frac{N_i}{N_{i-1}} \quad (i = 1, 2, ...)$$

(with $t = 1$, time unit being 1 year)

The following Table III gives the value of μ_i (i = 1, 2, ...).

TABLE III

Human population	$\mu_i = \ln \frac{N_i}{N_{i-1}}$
40, 46, 613	—
40, 73, 002	.006
40, 99, 563	.006
41, 26, 297	.006
41, 52, 097	.006
41, 78, 059	.006
42, 04, 183	.006
42, 30, 417	.006
42, 56, 922	.006
42, 83, 540	.006
43, 36, 275	.006
43, 64, 394	.006
43, 91, 684	.006

Therefore $\langle \mu \rangle$ = 0.006. This value is taken as the value of μ used for calculation of β from tables 1, 2, 3, 4. The constant γ in the model given by the 2nd equation of (1) has been calculated by computing the averaged value of β from the 1st equation of (1) by using the data from tables 2, 4. Then from this averaged value of β, an averaged value of γ has been found. This value of γ has been used as γ_0 in the time dependent $\gamma(t)$ given by (2) for the time-dependent recovery rate discrete time model.

The constants a_0, a_1, a_2 has been calculated by least square fitting. The normal equations for the method are

$$V_1 = a_0 + a_1 + a_2 - .006065796,$$

$$V_2 = a_0 + 2a_1 + 4a_2 - .013224868$$

and $$V_3 = a_0 + 3a_1 + 9a_2 - .018612022$$

where $V_t = a_0 + a_1 t + a_2 t^2 - \ln(\gamma_t/\gamma_0)$ (t = 1, 2, ..., 10).

Residual equations are

$$10a_0 + 55a_1 + 385a_2 - .346942476 = 0,$$

$$55a_0 + 385a_1 + 3025a_2 - 2.426887435 = 0$$

and $$385a_0 + 3025a_1 + 25333a_2 - 19.06442575 = 0.$$

Solving these equations we get the values of a_0, a_1, a_2 cited in section 2.

It is clear from the continuous model that

$$\frac{d}{dt}(S + I + R) = \mu N$$

or,
$$\frac{dN}{dt} = \mu N \text{ as } S + I + R = N(t).$$

The solution of this equation is $N = N_0 e^{\mu t}$.

N_0 being the value of N at $t = 0$.

It is also evident from the discrete time model that

$$S_{t+1} - S_t = -\beta S_t I_t + \mu N_t,$$

$$I_{t+1} - I_t = \beta S_t I_t - \gamma I_t$$

and
$$R_{t+1} - R_t = \gamma I_t$$

and by adding the three equations we have

$$(S_{t+1} + I_{t+1} + R_{t+1}) - (S_t + I_t + R_t) = \mu N_t$$

or
$$N_{t+1} - N_t = \mu N_t$$

or
$$N_{t+1} = (1 + \mu)N_t.$$

In fact, for values of μ the two models give rise to the same growth of population. Also, it is to be noted that $N_t \simeq S_t$ as the I_t is very small compared to N_t.

4. Computation of the Trends and the Form of the Time-Dependent Recovery Rate

The census report depicted in table 1 shows that the growth rate μ has the value μ = .006. With this value the susceptible rate β can be computed from the first of eqs. (1) by using the numbers of susceptibles. With these values one can calculate the number of infectives I_t for each year from 1979 onwards upto 1990. This is depicted in Table IV.

TABLE IV : *Yearwise calculated susceptibles and infectives of cholera (+ve)*

Year	Susceptibles	Infectives
1979	40, 72, 507	495.00
1980	40, 99, 137	426.21
1981	41, 25, 814	488.86
1982	41, 51, 747	366.80
1983	41, 77, 617	445.37
1984	42, 40, 52	142.25
1985	42, 30, 292	185.34
1986	42, 56, 599	330.62
1987	42, 83, 423	125.22
1988	43, 10, 200	119.25
1989	43, 37, 219	54.75
1990	43, 64, 335	60.11
1991	43, 90, 599	60.12

The constants a_0, a_1, a_2 in the expression of $\gamma(t)$ have been computed by comparing with the data and by using the method of least square.

In fact these values of the constants a_0, a_1, a_2 give rise to an excellent agreement of the computed values of infectives with the data. The figures 1 and 2 show how the computed values of the infectives agree with the data.

For non-choleric diarrhoea the recovery rate $\gamma(t)$ is taken to be of the form

$$\gamma(t) = \gamma_0 e^{a_0 + a_1 t + a_2 t^2}$$

where

$$\gamma_0 = .339091,$$

$$a_0 = 0.0580855,$$

$$a_1 = .0668599$$

and $$a_2 = -\ .002356813.$$

The constants, a_0, a_1, a_2 is obtained by least square fitting. The infectives of non-choleric diarrhoea is given the following table (Source : I.D. Hospital, Cal-10)

TABLE V : *Yearwise infectives of non-choleric diarrea*

Year	Infectives
1979	5714
1980	6054
1981	6532
1982	7095
1983	7587
1984	14861
1985	10316
1986	11454
1987	10973
1988	16037
1989	14372
1990	17968

5. COMPUTATION OF THE TRENDS AND THE FORM OF THE TIME-DEPENDENT RECOVERY RATE FOR THE NON-CHOLERIC DIARRHOEA

The census report depicted in Table I shows that the growth rate μ has the value μ = .006 with this value the susceptible rate β can be computed from the first of eqs. (1) using the numbers of

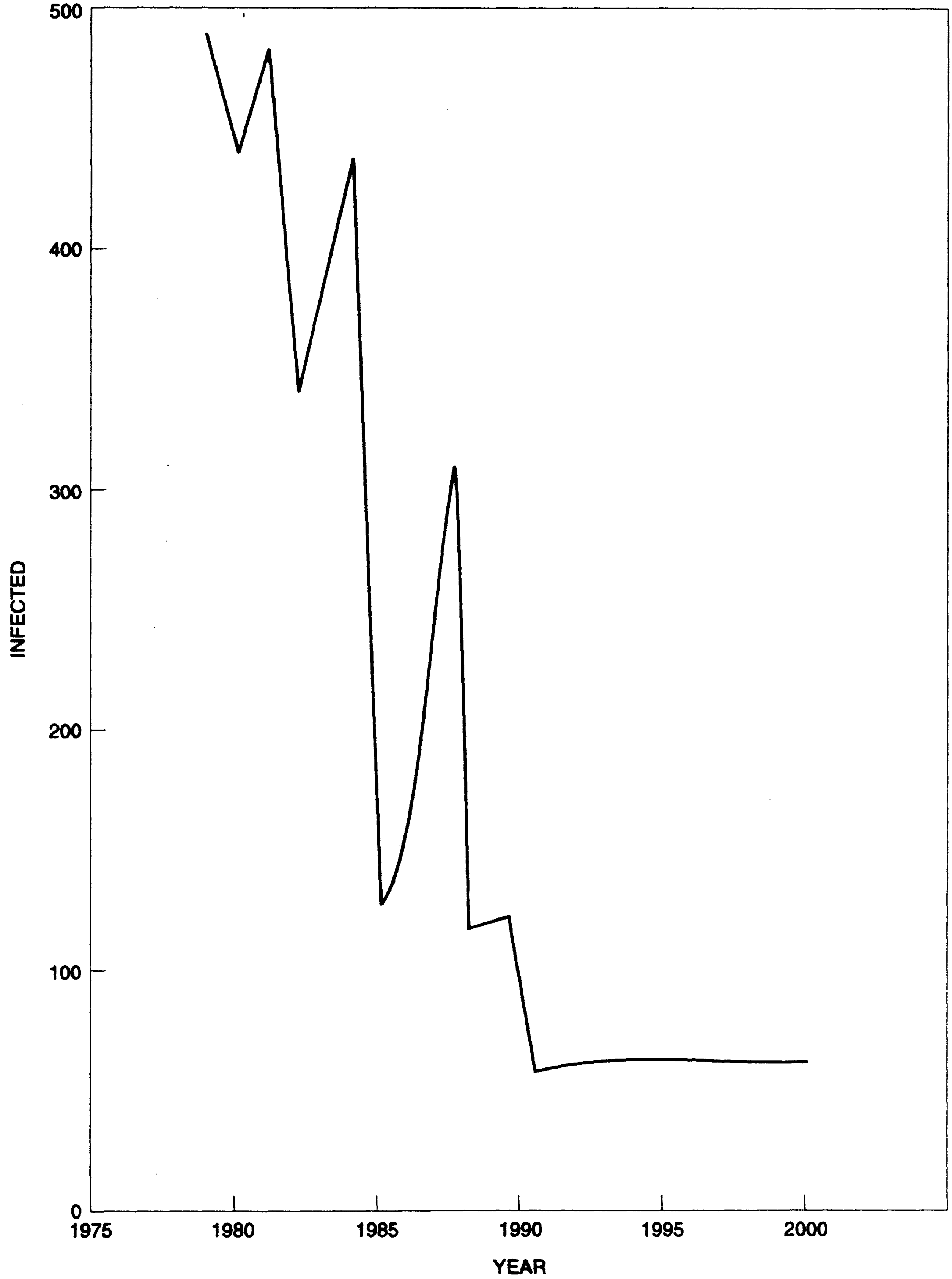

FIG. 1. The infectivity curve shows the variation of infectives of cholera (+ve) with the time. The graph is plotted against data.

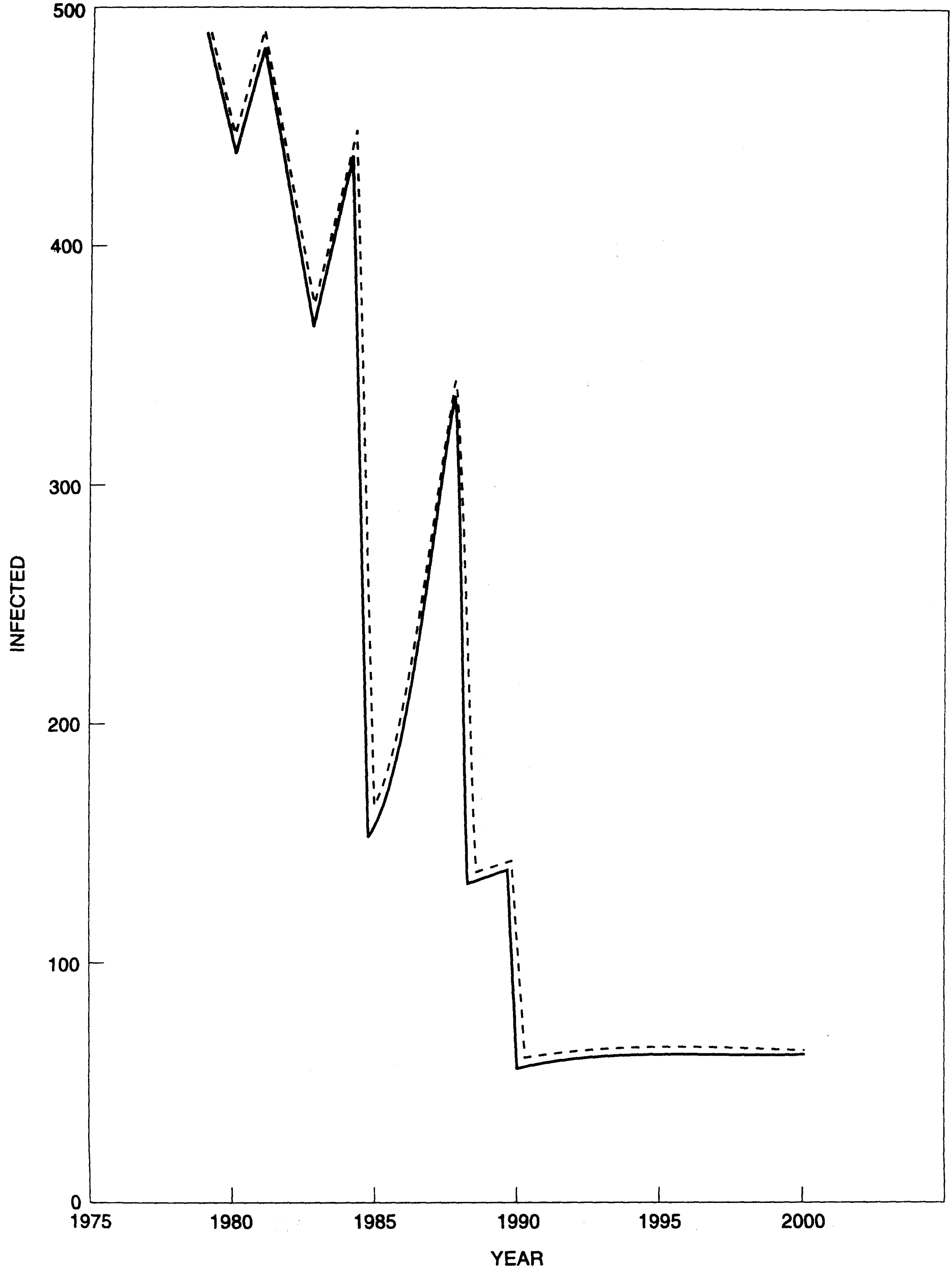

FIG. 2. The infectivity curve shows the variation of infectives of cholera (+ve) upto 2000 years. The graph is obtained from SIR model.

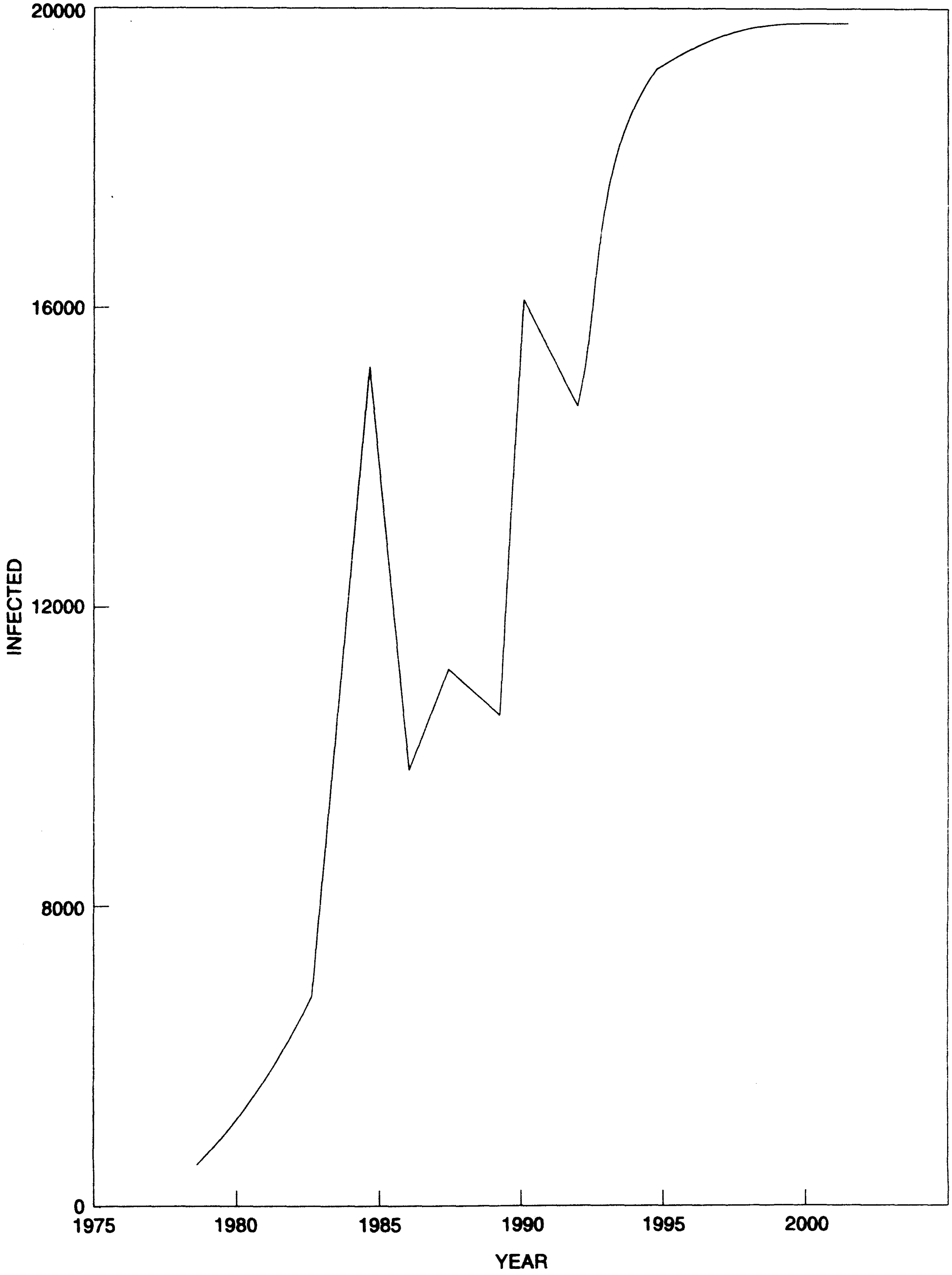

FIG. 3. Plot of non-choleric diarrhoea is obtained from data.

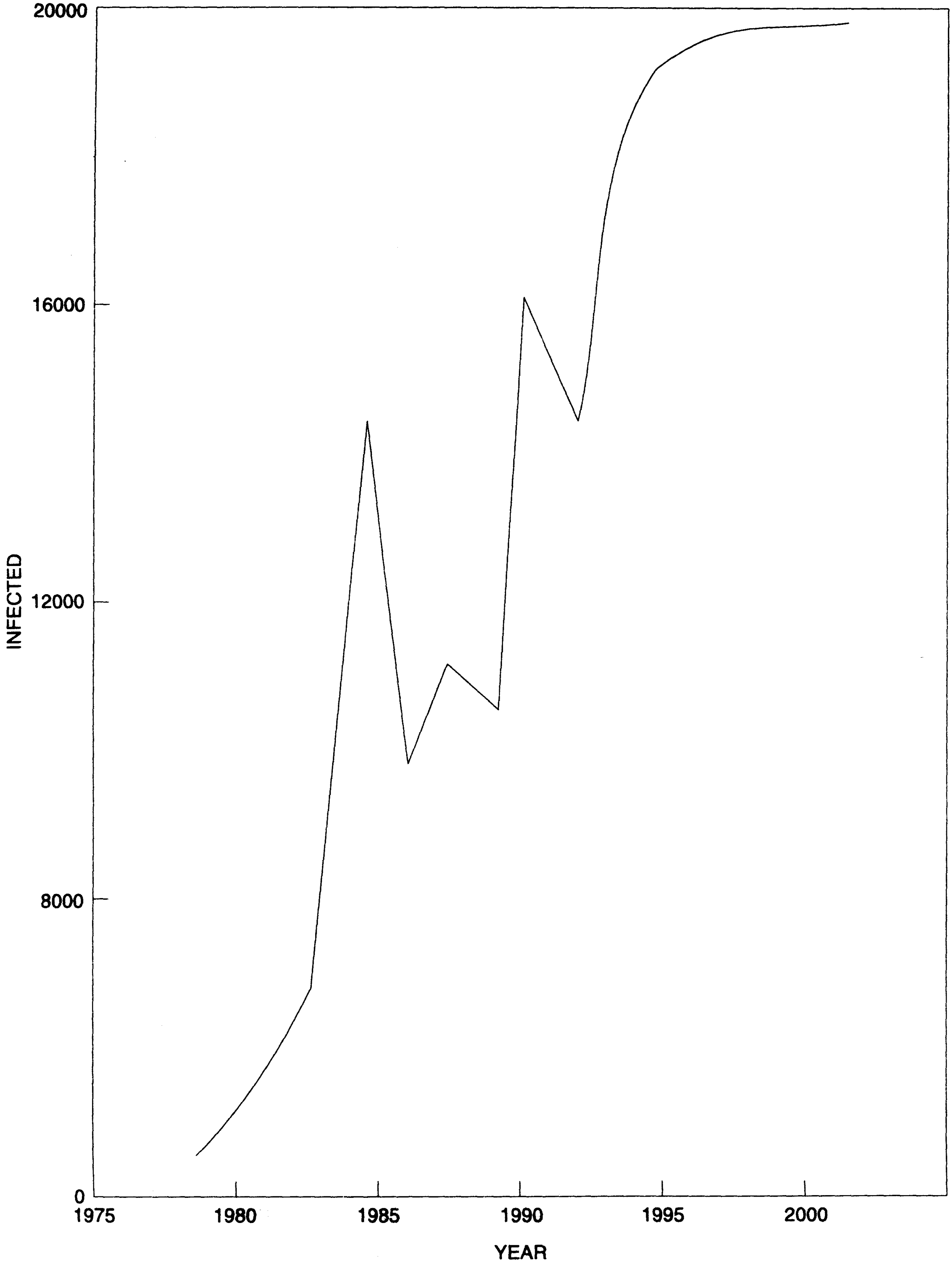

FIG. 4. Plot of non-choleric diarrhoea is obtained from the SIR model.

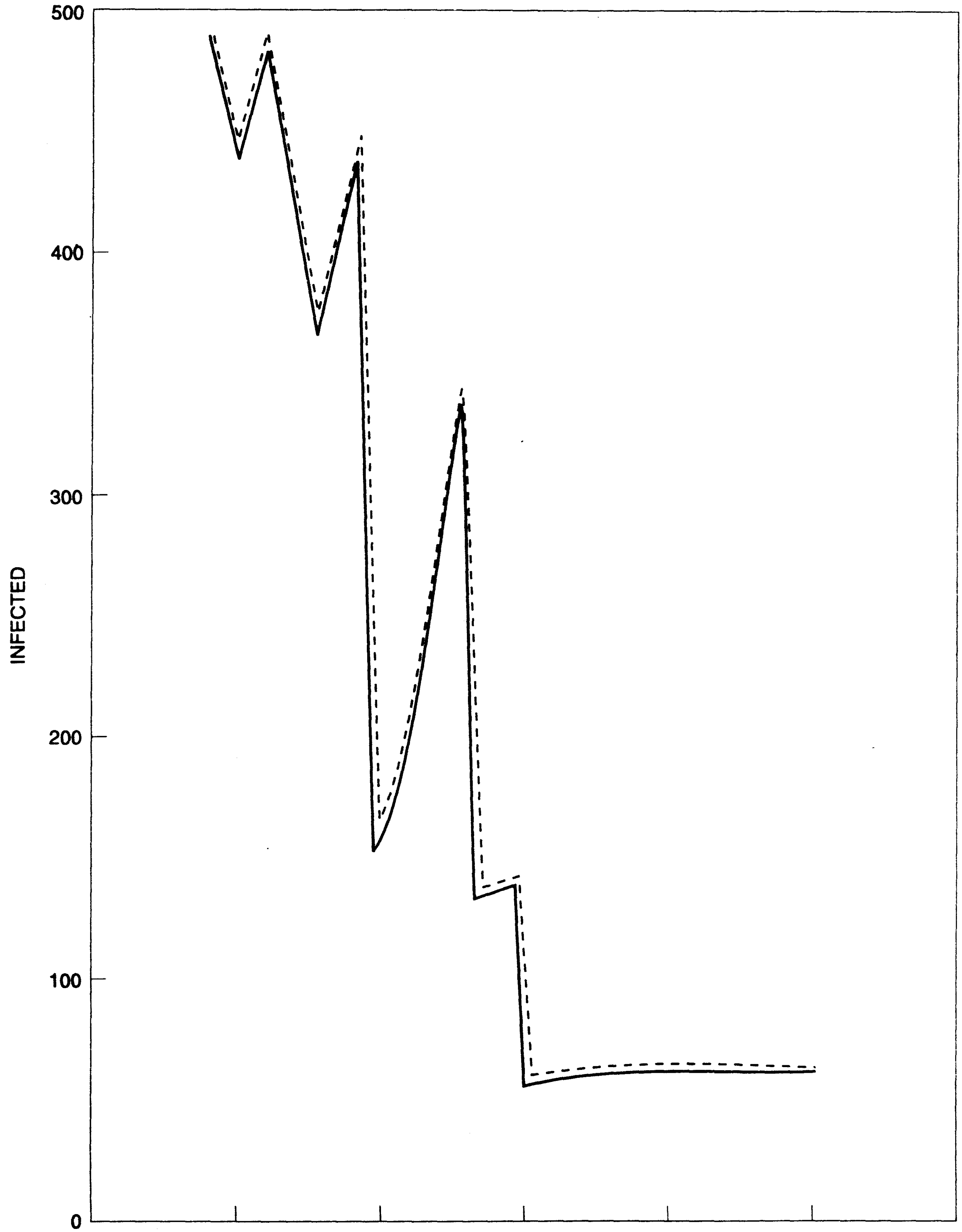

FIG. 5. Comparison of infectively curves showing the variations of infectivity of cholera (+ve) with the time as obtained from data (solid line) and from SIR model (dashed line).

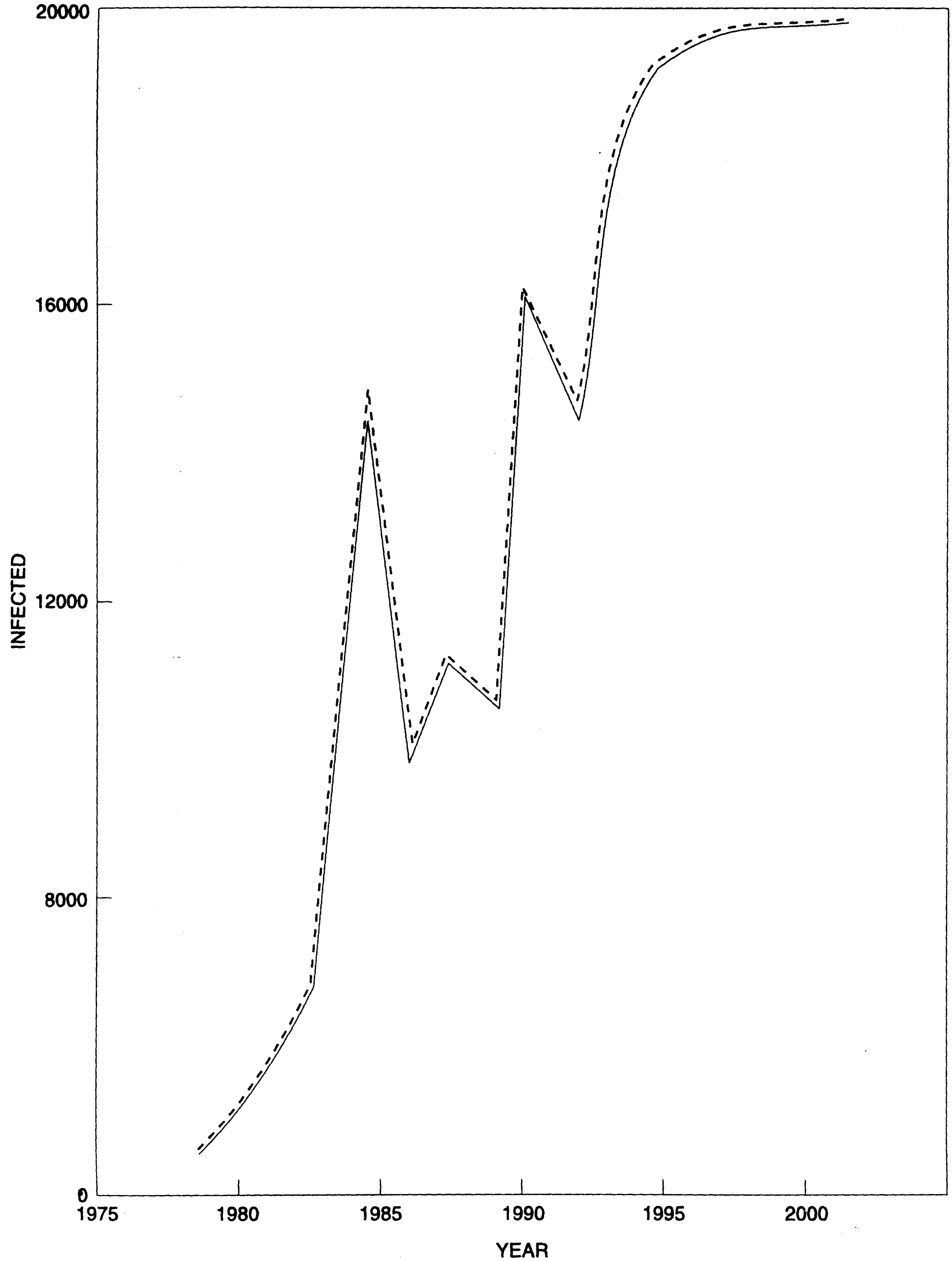

FIG. 6. Comparison of infectivity curves showing the variations of infectives of non-choleric diarrhoea with the time as obtained from data (solid line) and from SIR model (dashed line).

susceptibles. With these values one can calculate the number of infectives I_t for each year from 1979 onwards upto 1990. The susceptible is depicted in the following table :

TABLE VI : *Yearwise calculated susceptibles and infectives of non-choleric diarrea*

Year	Susceptible	Infectives
1979	40, 67, 288	5714.00
1980	40, 93, 509	6053.99
1981	41, 19, 765	6532.00
1982	41, 45, 002	7095.00
1983	41, 70, 472	7585.00
1984	41, 89, 322	14857.08
1985	42, 20, 155	10313.28
1986	42, 45, 468	11450.98
1987	42, 72, 567	10970.10
1988	42, 94, 286	16032.77
1989	43, 22, 903	14368.21
1990	43, 46, 426	17963.26

The values of the constants a_0, a_1, a_2 give rise to an excellent agreement of the computed values of the infectives with the data. The figures 3 and 4 shows how the computed values of the infectives agree with the data.

6. CONCLUSION AND DISCUSSION

We extrapolate the data upto 2000 years and see that in case of both the diseases the number of infectives are almost in a steady state i.e. the relative number of infectives decreases since the population increases in this period. It is to be understood as the general health awareness among the people and also because of the measure taken by the authorities in-regards to the epidemic spread of the diseases. This feature of the trend for the disease as obtained in this model is interesting one to be observed for the next period of time.

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