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STRONGLY NONLINEAR VARIATIONAL-LIKE INEQUALITIES

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The theory of variational inequalities has become a powerful tool in the study of a wide class of problems arising in various areas of mathematical and engineering sciences. Two important generalizations of the variational inequality problem are the nonlinear variational-like inequality problem and the strongly nonlinear variational inequality problem. In this paper we introduce the concept of strongly nonlinear variational-like inequality problem and study the existence and uniqueness of its solution in Banach spaces and topological vector spaces.

Key Words : Topological Vector Spaces; Variational Inequalities; KKM-Maps

1. INTRODUCTION

In the last twenty years nonlinear variational inequalities have assumed great importance, both from the theoretical and practical points of view due to their applicability in the calculus of variations and different branches of engineering and sciences. In the recent years variational inequalities have been extended and generalized in many directions. Two such generalizations are the *nonlinear* variational-like inequalities and the strongly nonlinear variational inequalities. Motivated and inspired by the research work going on in these areas, we introduce the concept of strongly nonlinear variational-like inequality problem as follows : Let (X, X^*) be a dual system of locally convex spaces. Let the value of $f \in X^*$ at $x \in X$ be denoted by (f, x). Let K be a nonempty subset of X, and $T : K \to X^*$, $A : K \to X^*$ and $\theta : K \times K \to X$ be any three maps. We consider the following problem (P_1) of finding $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq (Ax_0, \theta(y, x_0))$$

for all $y \in K$.

We need the following results to prove the main results of this paper.

Definition 1.1[6&10] — Let K be a nonempty subset of a locally convex space X. A point-to-set map $F : K \to 2^X$ is called a KKM-map if for every finite subset $\{x_1, x_2, ..., x_n\}$ of K,

$$\text{Conv}\ (\{x_1, x_2, ..., x_n\}) \subset \bigcup_{i=1}^{n} F(x_i).$$

where Conv(A) denotes the convex hull of any subset A of X.

Theorem 1.2[5&6] — *Let K be an arbitrary nonempty subset of a Hausdorff topological vector space X, and let $F : K \to 2^X$ be a KKM-map. If $F(x)$ is closed for all $x \in K$ and is compact for at least one $x \in K$ then*

$$\bigcap_{y \in K} F(y) \neq \phi.$$

Theorem 1.3[5&6] — *Let K be a nonempty compact subset of a Hausdorff topological vector space X. Let E be a subset of $K \times K$ having the following properties :*

(*i*) *for each $x \in K$, $(x, x) \in E$,*

(*ii*) *for each $y \in K$, the set*

$$E_y = \{x \in K : (x, y) \in E\}$$

is closed,

(*iii*) for each $x \in K$, the set

$$E^x = \{y \in K : (x, y) \notin E\}$$

is convex.

Then there exists $x_0 \in K$ such that

$$\{x_0\} \times K \subset E.$$

2. Existence of Solution

In this section we prove some theorems concerning the existence of solution of the strongly nonlinear variational-like inequality problem introduced in Section 1, under different conditions and settings.

Theorem 2.1 — *Let K be a compact convex subset of a real Banach space X and let X^* be the dual of X. Let $T : K \to X^*, A : K \to X^*$ and $\theta : K \times K \to X$ be continuous maps such that the following conditions are satisfied :*

(*i*) For each $x \in K$,

$$(Tx,\ \theta(x, x)) \geq (Ax,\ \theta(x, x)),$$

and

(*ii*) θ is affine in its first argument.

Then there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq (Ax_0, \theta(y, x_0))$$

for all $y \in K$.

PROOF : Let

$$C = \{(x, y) \in K \times K : (Tx, \theta(y, x)) \geq (Ax, \theta(y, x))\}.$$

C is nonempty since by (i), $(x, x) \in C$ for each $x \in K$. Since T, A and θ are continuous it is clear that the map

$$x \mapsto (Tx - Ax, \theta(y, x))$$

of K into $\boldsymbol{R}$ is continuous from the subspace topology of K to the usual topology of $\boldsymbol{R}$ for each $y \in K$. Hence the set

$$\begin{aligned} C_y &= \{x \in K : (x, y) \in C\} \\ &= \{x \in K : (Tx, \theta(y, x)) \geq (Ax, \theta(y, x))\} \end{aligned}$$

is closed for each $y \in K$.

We assert that for each $x \in K$, the set

$$\begin{aligned} C^x &= \{y \in K : (x, y) \notin C\} \\ &= \{y \in K : (Tx, \theta(y, x)) < (Ax, \theta(y, x))\} \end{aligned}$$

is convex : let $y_1, y_2 \in C^x, 0 < t < 1$ and

$$z = (1 - t)\, y_1 + t y_2.$$

By *(ii)* we have

$$\begin{aligned} (Tx, \theta(z, x)) &= (1 - t)\,(Tx, \theta(y_1, x)) + t(Tx, \theta(y_2, x)) \\ &< (1 - t)\,(Ax, \theta(y_1, x)) + t(Ax, \theta(y_2, x)) \\ &= (Ax, \theta((1 - t)\, y_1 + t y_2, x) \\ &= (Ax, \theta(z, x)). \end{aligned}$$

Thus $z \in C^x$ which proves that C^x is convex.

Now all conditions of Theorem 1.3 are fulfilled, and hence there exists $x_0 \in K$ such that

$$\{x_0\} \times K \subset C,$$

which is equivalent to

$$(Tx_0, \theta(y, x_0)) \geq (Ax_0, \theta(y, x_0))$$

for all $y \in K$. This completes the proof of Theorem 2.1.

The following theorem generalizes Theorem 2.1 by considering the set K to be locally compact and convex.

Theorem 2.2. — *Let K be a locally compact and convex subset of a real Banach space X and let X^* be the dual of X. Let $T: K \to X^*$, $A: K \to X^*$ and $\theta: K \times K \to X$ be continuous maps, and $r > 0$ be such that*

(*i*) *for each $x \in K$,*

$$(Tx, \theta(x, x)) = (Ax, \theta(x, x)),$$

(*ii*) *there exists $u \in K$ with $\| u \| < r$ such that*

$$(Tx, \theta(u, x)) \leq (Ax, \theta(u, x))$$

for all $x \in K$ with $\| x \| = r$,

(*iii*) *θ is affine in its first argument.*

Then there exists $x_0 \in K$ such that $\| x_0 \| \leq r$ and

$$(Tx_0, \theta(y, x_0)) \geq (Ax_0, \theta(y, x_0)) \qquad \dots (1)$$

for $y \in K$.

PROOF : Let

$$K_r = \{x \in K : \| x \| \leq r\}.$$

Since K is locally compact, it is clear that K_r is a compact and convex subset of K. Hence by Theorem 2.1, there exists $x_r \in K_r$ such that

$$(Tx_r, \theta(y, x_r)) \geq (Ax_r, \theta(y, x_r)) \qquad \dots (2)$$

for all $y \in K_r$.

We distinguish two cases :

Case I — $\| x_r \| < r$. In this case for each $y \in K$, there exists $\lambda \in (0, 1)$ such that

$$y_r = \lambda y + (1 - \lambda)\, x_r \in K_r.$$

Now putting $y = y_r$ in (2) we get

$$(Tx_r, \theta(y_r, x_r)) \geq (Ax_r, \theta(y_r, x_r)). \qquad \dots (3)$$

Since θ is affine in its first argument, (3) is equivalent to

$$\lambda(Tx_r, \theta(y, x_r)) + (1 - \lambda)\,(Tx_r, \theta(x_r, x_r))$$

$$\geq \lambda(Ax_r, \theta(y, x_r)) + (1 - \lambda)\,(Ax_r, \theta(x_r, x_r)). \qquad \dots (4)$$

An application of hypothesis (i) of the theorem to (4) gives

$$(Tx_r, \theta(y, x_r)) \geq (Ax_r, \theta(y, x_r)).$$

Since $y \in K$ is arbitrary, it follows that x_r is a solution of (1).

Case II — $\|x_r\| = r$. In this case putting $y = u$, in (2) we get

$$(Tx_r, \theta(u, x_r)) \geq (Ax_r, \theta(u, x_r)). \quad \text{... (5)}$$

But by hypothesis (ii) we have

$$(Tx_r, \theta(u, x_r)) \leq (Ax_r, \theta(u, x_r)). \quad \text{... (6)}$$

Combining (5) and (6) we get

$$(Tx_r, \theta(u, x_r)) = (Ax_r, \theta(u, x_r)). \quad \text{... (7)}$$

Now let $z \in K$. Choose $t > 0$ sufficiently small so that

$$u_t = tz + (1 - t)\, u \in K_r.$$

Putting $y = u_t$ in (2) we get

$$(Tx_r, \theta(u_t, x_r)) \geq (Ax_r, \theta(u_t, x_r)). \quad \text{... (8)}$$

Since θ is affine in its first argument, (8) is equivalent to

$$t(Tx_r, \theta(z, x_r)) + (1 - t)\, (Tx_r, \theta(u, x_r))$$

$$\geq t(Ax_r, \theta(z, x_r)) + (1 - t)\, (Ax_r, \theta(u, x_r)). \quad \text{... (9)}$$

An application of (7) to (9) gives

$$(Tx_r, \theta(z, x_r)) \geq (Ax_r, \theta(z, x_r)).$$

Since $z \in K$ is arbitrary, it follows that in this case also x_r is a solution of (1). This completes the proof of Theorem 2.2.

The following result also generalizes Theorem 2.1 by considering the set K to be closed convex and bounded.

Theorem 2.3 — *Let K be a nonempty closed convex and bounded subset of a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*, A : K \to X^*$ and $\theta : K \times K \to X$ be continuous maps such that*

(*i*) *for each* $x \in K$,

$$(Tx, \theta(x, x)) = (Ax, \theta(x, x)),$$

(*ii*) *for all* $x, y \in K$,

$$(Tx, \theta(y, x)) + (Ty, \theta(x, y))$$

$$\leq (Ax, \theta(y, x)) + (Ay, \theta(x, y)),$$

(*iii*) *θ is affine in its first argument.*

Then there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq (Ax_0, \theta(y, x_0))$$

for all $y \in K$.

We need the following lemma, which is a generalization of a lemma due to Minty[7] (see also Browder[3] and Chipot[4]) to prove Theorem 2.3.

Lemma 2.4 — *Let K be a nonempty closed and convex subset of a real Banach space X* and let X^* be the dual of X. Let $T : K \to X^*, A : K \to X^*$ and $\theta : K \times K \to X$ be continuous maps such that

(a) for each $x \in K$,

$$(Tx, \theta(x, x)) = (Ax, \theta(x, x)),$$

(b) for all $x, y \in K$,

$$(Tx, \theta(y, x)) + (Ty, \theta(x, y))$$

$$\leq (Ax, \theta(y, x)) + (Ay, \theta(x, y)),$$

(c) θ is affine in its first argument.

Then the following are equivalent:

(A) $x_0 \in K, (Tx_0, \theta(y, x_0)) \geq (Ax_0, \theta(y, x_0))$ for all $y \in K$.

(B) $x_0 \in K, (Ty, \theta(x_0, y)) \leq (Ay, \theta(x_0, y))$ for all $y \in K$.

PROOF : First assume that (A) holds. By (b), we have

$$(Tx_0, \theta(y, x_0)) + (Ty, \theta(x_0, y))$$

$$\leq (Ax_0, \theta(y, x_0)) + (Ay, \theta(x_0, y)).$$

Thus,

$$(Tx_0, \theta(y, x_0)) - (Ax_0, \theta(y, x_0))$$

$$\leq - [(Ty, \theta(x_0, y)) - (Ay, \theta(x_0, y))].$$

Since the left-hand side of the above inequality is non-negative by (A), (B) follows.

Conversely assume that (B) holds. Let $y \in K, t \in (0, 1)$ and

$$y_t = (1 - t)\, x_0 + ty.$$

Putting $y = y_t$ in (B) we get

$$(Ty_t, \theta(x_0, y_t)) \leq (Ay_t, \theta(x_0, y_t)). \quad \ldots (10)$$

By (a) we have

$$(Ty_t, \theta(y_t, y_t)) = (Ay_t, \theta(y_t, y_t)). \quad \text{... (11)}$$

Since θ is affine in its first argument, (11) is equivalent to

$$\begin{aligned}(1-t)\,(Ty_t, \theta(x_0, y_t)) + t(Ty_t, \theta(y, y_t))\\ = (1-t)\,(Ay_t, \theta(x_0, y_t)) + t(Ay_t, \theta(y, y_t)).\end{aligned} \quad \text{... (12)}$$

Substitution of (10) in (12) gives

$$(Ty_t, \theta(y, y_t)) \geq (Ay_t, \theta(y, y_t)). \quad \text{... (13)}$$

Since T, A and θ are continuous and $y_t \to x_0$ as $t \to 0$, taking limit as $t \to 0$ in (13) we get

$$(Tx_0, \theta(y, x_0)) \geq (Ax_0, \theta(y, x_0)).$$

Since $y \in K$ is arbitrary, (A) follows. This completes the proof of the lemma.

2.5. PROOF OF THEOREM 2.3 — Define two point-to-set maps $E, F : K \to 2^X$ as

$$E(y) = \{x \in K : (Tx, \theta(y, x)) \geq (Ax, \theta(y, x))\}$$

and

$$F(y) = \{x \in K : (Ty, \theta(x, y)) \leq (Ay, \theta(x, y))\}.$$

We claim that E s a KKM-map. Indeed if we assume that E is not a KKM-map then there exist

$$\{x_1, x_2, \ldots, x_n\} \subset K,$$

$$t_i \geq 0, \quad \text{with} \sum_{i=1}^{n} t_i = 1$$

such that

$$u = \sum_{i=1}^{n} t_i x_i \notin \bigcup_{i=1}^{n} E(x_j).$$

Then of course, $u \notin E(x_j)$ for any $j \in \{1, 2, \ldots, n\}$, which, in turn, implies that

$$(Tu, \theta(x_j, u)) < (Au, \theta(x_j, u))$$

for every $j = 1, 2, \ldots, n$. Now multiplying both sides by t_i, summing over $i = 1$ to $i = n$ and using the fact that θ is affine in its first argument, we obtain

$$(Tu, \theta(u, u)) < (Au, \theta(u, u)),$$

which is a contradiction to hypothesis (i) of the theorem.

We next prove that for each $y \in K, E(y) \subset F(y)$: If $x \in E(y)$, then

$$(Tx, \theta(y, x)) - (Ax, \theta(y, x)) \geq 0. \qquad \dots (14)$$

But by hypothesis (ii) of the theorem

$$(Tx, \theta(y, x)) - (Ax, \theta(y, x))$$

$$\leq -[(Ty, \theta(x, y)) - (Ay, \theta(x, y))]. \qquad \dots (15)$$

An application of (14) to (15) gives

$$(Ty, \theta(x, y)) - (Ay, \theta(x, y)) \leq 0,$$

showing that $x \in F(y)$. Since x is arbitrary, the assertion follows. Thus F is also a KKM-map.

We further observe that for each $y \in K, F(y)$ is nonempty, (since $y \in F(y)$). Again since T, A and θ are continuous and θ is affine in its first argument, it follows that for each y, $F(y)$ is a closed and convex subset of K. Since K is weakly compact (being closed convex and bounded subset of a reflexive Banach space), $F(y)$ is also weakly compact for each $y \in K$. Now by Theorem 1.2 we have,

$$\bigcap_{y \in K} F(y) \neq \phi.$$

But by Lemma 2.4,

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} E(y).$$

Thus there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq (Ax_0, \theta(y, x_0))$$

for all $y \in K$. This completes the proof.

The following theorem generalizes Theorem 2.3 to an unbounded closed and convex subset K of X.

Theorem 2.6 — *Let K be a nonempty closed convex and unbounded subset of a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*, A : K \to X^*$ and $\theta : K \times K \to X$ be continuous maps and $r > 0$ be such that*

(*i*) for each $x \in K$,

$$(Tx, \theta(x, x)) = (Ax, \theta(x, x)),$$

(*ii*) for all $x, y \in K$,

$$(Tx, \theta(y, x)) + (Ty, \theta(x, y))$$

$$\leq (Ax, \theta(y, x)) + (Ay, \theta(x, y)),$$

(*iii*) there exists $u \in K$ with $\| u \| < r$ such that

$$(Tx, \theta(u, x)) \leq (Ax, \theta(u, x)),$$

for all $x \in K$ with $\| x \| = r$,

(*iv*) θ is affine in its first argument.

Then there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq (Ax_0, \theta(y, x_0)),$$

for all $y \in K$.

PROOF : Let

$$K_r = \{x \in K : \| x \| \leq r\}.$$

Clearly K_r is a closed convex and bounded subset of K. By Theorem 2.3, there exists $x_r \in K_r$ such that

$$(Tx_r, \theta(y, x_r)) \geq (Ax_r, \theta(y, x_r))$$

for all $y \in K_r$.

The remaining part of the proof is exactly same as the proof of the corresponding part of Theorem 2.2; hence it is omitted.

3. UNIQUENESS OF SOLUTIONS

The following theorem deals with the uniqueness of solution to the strongly nonlinear variational-like inequality problem P_1.

Theorem 3.1 — *Let K be a nonempty subset of a real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$, $A : K \to X^*$ and $\theta : K \times K \to X$ be three maps such that the following hold :*

(*i*) For all $x, y \in K$,

$$(Tx, \theta(y, x)) + (Ty, \theta(x, y))$$

$$\leq (Ax, \theta(y, x)) + (Ay, \theta(x, y)),$$

(*ii*) equality does not hold in (i) unless $x = y$.

Then there can be at most one solution of P_1.

PROOF : Let x_1 and x_2 be two solutions of P_1. Then

$$(Tx_1, \theta(y, x_1)) \geq (Ax_1, \theta(y, x_1)),$$

and

$$(Tx_2, \theta(y, x_2)) \geq (Ax_2, \theta(y, x_2))$$

for all $y \in K$. Putting $y = x_2$ in the former inequality, $y = x_1$ in the later and then adding the resulting inequalities we get

$$(Tx_1, \theta(x_2, x_1)) + (Tx_2, \theta(x_1, x_2))$$

$$\geq (Ax_1, \theta(x_2, x_1)) + (Ax_2, \theta(x_1, x_2)).$$

Now putting $x = x_1$ and $y = x_2$ in the inequality of (i) we get

$$(Tx_1, \theta(x_2, x_1)) + (Tx_2, \theta(x_1, x_2))$$

$$\leq (Ax_1, \theta(x_2, x_1)) + (Ax_2, \theta(x_1, x_2)).$$

Combining the last two inequalities, we get

$$(Tx_1, \theta(x_2, x_1)) + (Tx_2, \theta(x_1, x_2))$$

$$= (Ax_1, \theta(x_2, x_1)) + (Ax_2, \theta(x_1, x_2)).$$

Applying the hypothesis (ii) of the theorem to the above equation, we get $x_1 = x_2$. This completes the proof.

4. Special Cases

Case 1 — If $\theta(x, y) = x - y$ for all $x, y \in K$ then P_1 reduces to the problem of finding $x_0 \in K$ such that

$$(Tx_0, y - x_0) \geq (Ax_0, y - x_0)$$

for all $y \in K$, which is called the strongly nonlinear variational inequality problem, studied by Isac[6], and Nanda[8].

Case 2 — If $g : K \to X$ is any function and $\theta(x, y) = g(x) - g(y)$ for all x, $y \in K$ then P_1 reduces to the problem of finding $x_0 \in K$ such that

$$(Tx_0, g(y) - g(x_0)) \geq (Ax_0, g(y) - g(x_0))$$

for all $y \in K$, which is called the strongly nonlinear implicit variational inequality problem, studied by Noor[9] in connection with the solution of the differential equations of odd order.

Case 3 — If $Ax = 0$ for all $x \in K$, then P_1 reduces to the problem of finding $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$, which is called the variational-like inequality problem, studied by Behera and Panda[1&2], and Siddiqi, Khaliq and Ansari[10] and Siddiqi, Ansari and Kazmi[11].

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