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A PARADOX IN A FIXED CHARGE TRANSPORTATION PROBLEM

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This paper discusses a paradox in a Fixed Charge Transportation Problem. A paradox arises when the fixed charge transportation problem admits of a total cost which is lower than the optimum cost, by transporting larger quantities of goods over the same routes. A sufficient condition for the existence of a paradox is established. Paradoxical Range of Flow is obtained for any given flow in which the corresponding objective function value is less than the optimum value of the fixed charge transportation problem. The results are illustrated with the help of an example.

Key Words : Fixed Charge Transportation Problem; Paradox

INTRODUCTION

A paradox arises when a transportation problem admits of a total cost which is lower than the optimum and is attainable by shipping larger quantities of goods over the same routes that were previously designated as optimal. This unusual phenomenon was noted by Szwarc[11]. A paradox in linear fractional transportation problem with mixed constraints was discussed by Gupta *et al.*[5]. In this paper, such a paradox is discussed for a Fixed Charge Transportation Problem.

The fixed charge transportation problem was originally formulated by Dantzig and Hirsch[6] in 1954. The fixed charge transportation problem is an extension of the classical transportation problem in which a fixed cost is incurred for every origin. Many distribution problems in practice can only be modelled as fixed charge transportation problems. For example rail, roads and trucks have invariably used frieght rates which consist of a fixed cost and a variable cost. The fixed cost may represent the cost of renting a vehicle, landing fees in an airport, set up costs for machines in manufacturing environment etc.

This paper is divided into three sections. In Section-I, the condition for the existence of the paradox is developed. Section-II gives a method to determine the best paradoxical pair; while Section-III shows how to get a paradoxical solution for a specified flow.

THEORETICAL DEVELOPMENT

Consider the following Fixed Charge Transportation Problem (P^0)

$$(P^0) : \min \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} f_i \right\}$$

subject to $\sum_{j=1}^{n} x_{ij} \leq a_i;\ i = 1, 2, ..., m$

and $\sum_{i=1}^{m} x_{ij} = b_j;\ j = 1, 2, ..., n,\ x_{ij} \geq 0;\ \forall\ i$ and j,

where

$I = \{1, 2, ..., m\}$ is the index set of m origins,

$J = \{1, 2, ..., n\}$ is the index set of n destinations,

x_{ij} = the amount transported from the ith origin to the jth destination,

c_{ij} = the variable cost per unit amount transported from the i^{th} origin to the jth destination,

f_i = the fixed cost associated with origin i,

a_i = the maximum quantity available at the ith origin

and

b_j = the demand at the jth destination.

For formulation of $f_i (i = 1, 2, ..., m)$ we assume that f_i $(i = 1, 2, ..., m)$ has p number of steps so that

$$f_i = \sum_{l=l}^{p} \delta_{il} f_{il},\ i = 1, 2, ..., m,$$

where

$$\delta_{il} = 1, \text{ if } \sum_{j=1}^{n} x_{ij} > a_{il},\ i = 1, 2, ..., m;\ l = 1, 2, ..., p$$

$$= 0, \text{ otherwise}$$

Here $0 = a_{i1} < a_{i2} < \ldots < a_{ip}$

$a_{i1}, a_{i2}, \ldots, a_{ip}$ $(i = 1, 2, ..., m)$ are constants and f_{i1l} $(l = 1, 2, ..., p; i = 1, 2, ..., m)$ are fixed costs.

Note — The manner in which the fixed costs are defined indicates that corresponding to the source i if $\sum_{j=1}^{n} x_{ij} > a_{i1}$ and less than a_{i2} then fixed cost is f_{i1}; if $\sum_{j=1}^{n} x_{ij} > a_{i2}$ and less than a_{i3} then the fixed cost is $f_{i1} + f_{i2}$. Proceeding like this when $\sum_{j=1}^{n} x_{ij} > a_{ip}$, then the fixed cost is $f_{i1} + f_{i2} + \dots + f_{ip}$.

Let an optimal basic feasible solution of (P^0) yield value Z^0 of the objective function and $F^0 = \sum_{i=1}^{m} a_i' = \sum_{j=1}^{n} b_j'$ be the corresponding flow where $a_i' \leq a_i$, $i = 1, 2, \dots, m$, $b_j' = b_j$, $j = 1, 2, \dots, n$. A paradox exists if more than F^0 is flown at an objective function value less than Z^0. It may be observed that flow can be increased by an increase of a certain a_i' and b_j'. This gives rise to the following problem (P^1) :

$$(P^1) : \text{Min} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} f_i \right\}$$

$$\text{subject to } \sum_{j=1}^{n} x_{ij} \geq a_i'; \; i = 1, 2, \dots, m$$

$$\text{and} \quad \sum_{i=1}^{m} x_{ij} \geq b_j'; \; j = 1, 2, \dots, n, \; x_{ij} \geq 0; \; \forall \; i \text{ and } j.$$

The feasible region of (P^1) being larger than that of (P^0), it follows that the minimum objective function value Z^1 of (P^1) is not greater than Z^0 i.e., $Z^1 \leq Z^0$. So more may be flown than that in (P^0) at an objective function value less than that of (P^0). Hence, a paradox may arise in this case.

Definitions

1. Paradoxical Pair — An objective function-flow pair (Z, F) of problem (P^1) is called a paradoxical pair if $Z < Z^0$ and $F > F^0$.

2. Best Paradoxical Pair — The paradoxical pair (Z^*, F^*) is called the best paradoxical pair if

$Z^* = \text{Min } \{Z : (Z, F) \text{ is a paradoxical pair}\}$

$F^* = \text{Max } \{F : (Z, F) \text{ is a paradoxical pair}\}$.

3. Paradoxical Range of Flows — Paradoxical range of flows is $[F^0, F^*]$ where F^* is the flow corresponding to the best paradoxical pair.

All objective function-flow pairs in this range are paradoxical pairs.

SECTION — I : SUFFICIENT CONDITION FOR THE EXISTENCE OF A PARADOXICAL SOLUTION

Let X^0 be the BFS of problem (P^0) w.r.t. the variable cost only and let u_i and v_j be the corresponding dual variables. Let this X^0 be also local optimal solution of (P^0) (refer algorithm on page 7). Let Z^0 be the corresponding value of the objective function and $F^0 = \sum_{i=1}^{m} a_i' = \sum_{j=1}^{n} b_j'$ be the corresponding flow, where $a_i' \leq a_i$, $i = 1, 2, ..., m$; $b_j' = b_j$, $j = 1, 2, ..., n$. Then

$$Z^0 = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} f_i$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - u_i - v_j) x_{ij} + \sum_{i=1}^{m} f_i + \sum_{i=1}^{m} \sum_{j=1}^{n} (u_i + v_j) x_{ij}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - u_i - v_j) x_{ij} + \sum_{i=1}^{m} f_i + \sum_{i=1}^{m} u_i \left(\sum_{j=1}^{n} x_{ij} \right) + \sum_{j=1}^{n} v_j \left(\sum_{i=1}^{m} x_{ij} \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - u_i - v_j) x_{ij} + \sum_{i=1}^{m} f_i + \sum_{i=1}^{m} a_i' u_i + \sum_{j=1}^{n} b_j' v_j.$$

Since at a basic feasible solution $c_{ij} = u_i + v_j$, we get

$$Z^0 = \sum_{i=1}^{m} a_i' u_i + \sum_{j=1}^{n} b_j' v_j + \sum_{i=1}^{m} f_i \quad .$$

Suppose now that a_p' is replaced by $a_p' + \lambda$ and b_q' is replaced by $b_q' + \lambda$, where $\lambda > 0$ is such that the same basis remains optimal, then the new value Z^1 of the objective function is given by

$$Z^1 = \sum_{\substack{i=1 \\ i \neq p}}^{m} a_i' u_i + (a_p' + \lambda) u_p + \sum_{\substack{j=1 \\ j \neq q}}^{n} b_j' v_j + (b_q' + \lambda) v_q + \sum_{i=1}^{m} f_i \quad ,$$

where $\sum_{i=1}^{m} f_i'$ is the new fixed cost.

$$Z^1 = \sum_{i=1}^{m} a_i' u_i + \sum_{j=1}^{n} b_j' v_j + \lambda(u_p + v_q) + \sum_{i=1}^{m} f_i'$$

and

$$Z^1 - Z^0 = \lambda(u_p + v_q) + \sum_{i=1}^{m} f_i' - \sum_{i=1}^{m} f_i$$

$$= \lambda(u_p + v_q) + d,$$

where d is change in the fixed cost, Now

$Z^1 < Z^0$ if $\lambda(u_p + v_q) + d < 0$.

Thus if $\exists$ a cell (p, q) with $\lambda(u_p + v_q) + d < 0$, then the new value Z^1 of the objective function is less than Z^0. Hence, the flow is increased by λ but cost is reduced i.e., a paradox exists. This result can be stated as

Theorem 1 — *Let X^0 be an optimal basic feasible solution of problem (P^0) with objective function value Z^0 and flow $F^0 = \Sigma\ a_i' = \Sigma\ b_j'$, where $a_i' \leq a_i$, i = 1, 2, ..., m, $b_j' = b_j$, j = 1, 2, ..., n. If $\exists$ a cell (p, q) such that $\lambda(u_p + v_q) + d < 0$, $\lambda > 0$, then there may exist a paradox on changing a_p' by $a_p' + \lambda$ and b_q' by $b_q' + \lambda$; basis remaining the same.*

Remark 1 : The condition $\lambda(u_p + v_q) + d < 0, \lambda > 0$ implies $u_p + v_q < 0$ because $\lambda > 0$ and change in fixed cost is non-negative. Thus to obtain paradoxical solution we consider only those cells (p, q) for which $u_p + v_q < 0$.

Algorithm to find 'Paradoxical Solution'

Step 1 — Find a basic feasible solution of (P^0) with respect to variable cost only.

Step 2 — Find the corresponding fixed cost. Let it be denoted by f (current), where

$$f \text{ (current)} = \sum_{i=1}^{m} f_i$$

Also, find $A_{ij} = (c_{ij})^1 \times (E_{ij})$,

where $(c_{ij})^1 = c_{ij} - u_i - v_j\ \forall\ (i, j) \notin B$, B being the current basis; u_i, v_j are the dual variables i = 1, 2, ..., m + 1; j = 1, 2, ..., n + 1 and A_{ij} is the change in cost that occurs on introducing a non-basic cell (i, j), with value (E_{ij}) into the basis by making re-allocations.

Step 3 — (*a*) Find f_{ij} (Diff) = f_{ij} (NB) - f (current), where f_{ij} (NB) is the total fixed cost obtained on introducing the cell (i, j) into the basis.

(*b*) Find $\Delta_{ij} = f_{ij}$ (Diff) + $A_{ij}\ \forall\ (i, j) \notin B$

If all $\Delta_{ij} \geq 0$, then the current solution is the local optimal solution to (P^0). To test for the existence of paradox go to step-4.

Otherwise find min $\{\Delta_{ij}, \Delta_{ij} < 0, (i, j) \notin B\}$. The cell associated with the minimum value Δ_{ij} enters the basis. Go to step-2.

Step 4 — Let $F^0 = \sum_{i=1}^{m} a_i' = \sum_{j=1}^{n} b_j'$ be the optimal flow where $a_i' \le a_i$, $i = 1, 2, ..., m$, $b_j' = b_j$, $j = 1, 2, ..., n$. Choose a cell (p, q) with $u_p + v_q < 0$. Replace a_p' by $a_p' + \lambda$ and b_q' by $b_q' + \lambda$, $\lambda > 0$. Keeping the same basis, find the new basic feasible solution. For this $\lambda > 0$, if the condition $\lambda(u_p + v_q) + d < 0$ is satisfied then corresponding to this basic feasible solution the value of the objective function reduces and the flow increases i.e., a paradox exists.

SECTION — II : BEST PARADOXICAL PAIR

If a paradox exists, one would obviously be interested in the 'Best Paradoxical Pair'. Theorem 2 below proves that the optimal basic feasible solution of (P^1) yields the best paradoxical pair.

Theorem 2 — *Global optimal basic feasible solution of* (P^1) *yields the Best Paradoxical Pair.*

PROOF : Let $X^\alpha = \left\{ x_{ij}^\alpha \right\}$ be an optimal BFS of problem (P^1). Let corresponding to this solution, we have

$$\sum_{j=1}^{n} x_{ij}^\alpha = a_i^\alpha \ge a_i', \; i = 1, 2, ..., m$$

and

$$\sum_{i=1}^{m} x_{ij}^\alpha = b_j^\alpha \ge b_j', \; j = 1, 2, ..., n.$$

Let F^α and Z^α be respectively the corresponding optimal flow and the optimal value of the objective function.

Consider the following problem $(P^1)'$:

$$(P^1)' : \text{Min} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} f_i \right\}$$

subject to
$$\sum_{j=1}^{n} x_{ij} = a_i^\alpha + p_i \ge a_i' ; \; i = 1, 2, ..., m$$

$$\sum_{i=1}^{m} x_{ij} = b_j^\alpha + q_j \ge b_j'; \; j = 1, 2, ..., n$$

$$x_{ij} \ge 0; \; i = 1, 2, ..., m; \; j = 1, 2, ..., n,$$

where

$$\sum_{i=1}^{m} p_i = 0 = \sum_{j=1}^{n} q_j \; .$$

Let $X^{\alpha'} = \left\{ x_{ij}^{\alpha'} \right\}$ be the optimal solution of problem $(P^1)'$. Then $X^{\alpha'}$ will be a feasible solution of (P^1). But X^{α} is the optimal solution of P^1. Therefore, $Z^{\alpha'} \geq Z^{\alpha}$, where $Z^{\alpha'}$ is the value of the objective function of problem (P^1) at the feasible solution $X^{\alpha'}$.

$\Rightarrow$ *No optimal feasible solution of* $(P^1)'$ *can yield the objective function value less than* Z^{α}.

Thus there does not exist any solution of problem (P^1) which gives value less than Z^{α} and flow greater than F^{α}. Hence, optimal solution of (P^1) yields the best paradoxical pair.

To solve (P^1) we construct and solve the Related Fixed Charge Transportation Problem (RP^1) with an additional supply point and an additional destination.

$$(RP^1) : \text{Min} \left\{ \sum_{i \in I'} \sum_{j \in J'} c'_{ij} y_{ij} + \sum_{i \in I'} f'_i \right\}$$

subject to
$$\sum_{j \in J'} y_{ij} = a''_i,\ i \in I' = I \cup \{m+1\},\ I = \{1, 2, \ldots, m\}$$

$$\sum_{i \in I'} y_{ij} = b''_j,\ j \in J' = J \cup \{n+1\},\ J = \{1, 2, \ldots, n\}$$

and

$$y_{ij} \geq 0;\ i \in I', j \in J',$$

where

$$a''_i = a'_i,\ i \in I, \qquad a''_{m+1} = \sum_{j \in J} b'_j + 1$$

$$b''_j = b'_j,\ j \in J, \qquad b''_{n+1} = \sum_{i \in I} a'_i + 1$$

$$c'_{ij} = c_{ij},\ i \in I, j \in J, \qquad c'_{m+1,j} = \min_{i \in I} c_{ij}$$

$$c'_{-i,n+1} = \min_{j \in J} c_{ij}, \qquad c'_{m+1,n=1} = 0$$

$$f'_i = f_i,\ i \in I, \qquad f'_{m+1} = 0.$$

On solving (RP^1) let $Y^0 = \{y_{ij}\}$, $i \in I', j \in J'$ be the optimal solution. The optimal solution $X^0 = \{x_{ij}\}$ to the problem (P^1) is then derived by the following transformation. For $i \in I$, let

$$w^0_{is} = y^0_{is} + y^0_{i,n+1}$$

and

$$w^0_{ij} = y^0_{ij},\ j \in J, j \neq s,$$

where s is the destination satisfying

$$c_{is} = \min_{j \in J} c_{ij}.$$

Similarly, for $j \in J$

$$x^0_{rj} = w^0_{rj} + y^0_{m+1,j}$$

and

$$x^0_{ij} = w^0_{ij},\ i \in I, i \neq r,$$

where r is an origin satisfying

$$c_{rj} = \min_{i \in I} c_{ij}.$$

Then, $\{x^0_{ij}\}$ is the optimal solution to (P^1).

Remark 2 :[3] The a''_{m+1} and b''_{n+1} have been so defined to balance the related transportation problem (RP^1). The '+1' has been taken to ensure allocation in $(m + 1, n + 1)^{\text{th}}$ cell which is a must in order to move from an optimal solution to (RP^1) to an optimal solution of (P^1).

Remark 3 : If cell $(i, n + 1)$ is basic in an optimal solution and the corresponding cell in the ith row with the same cost is non basic then their roles can always be interchanged so that in the new solution there will not be any change in the objective function value. Thus our shifting is justified.

In the next section, we discuss how to find a paradoxical solution for a specified flow in a given paradoxical range of flows.

SECTION — III : PARADOXICAL SOLUTION FOR A SPECIFIED FLOW IN $[F^0, F^1]$

Let $F^0 = \sum_{i=1}^{m} a_i' = \sum_{j=1}^{n} b_j'$ be the flow corresponding to the optimal basic feasible solution X^0 of (P^0) where $a_i' \leq a_i$, $i = 1, 2, \ldots, m$, $b_j' = b_j, j = 1, 2, \ldots, n$. Also let F^1 be the flow corresponding to the optimal basic feasible solution X^1 of (P^1). Then $[F^0, F^1]$ is the 'Paradoxical Range of Flows'.

Quite often finding the best objective function value for a given flow in $[F^0, F^1]$ is of great importance to the decision maker.

Let the specified flow be $F \in [F^0, F^1]$. The 'Paradoxical Solution' for F is given by the optimal solution of (P^2).

$$(P^2) : \text{Min} \left\{ \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{i \in I} f_i \right\}$$

subject to $\sum_{j \in J} x_{ij} \geq a_i', i \in I = \{1, 2, ..., m\}$

$$\sum_{i \in I} x_{ij} \geq b_j', j \in J = \{1, 2, ..., n\}$$

$$\sum_{i \in I} \sum_{j \in J} x_{ij} = F\left(F > \sum_{i \in I} a_i' = \sum_{j \in J} b_j'\right)$$

$$x_{ij} \geq 0, i \in I, j \in J.$$

Note that problem (P^2) is different from (P^1) because of the flow constraint.

To solve (P^2) we consider the following related problem (RP^2) with an additional origin and an additional destination.

$$(RP^2): \text{Min}\left\{\sum_{i \in I'} \sum_{j \in J'} c_{ij}' y_{ij} + \sum_{i \in I'} f_i'\right\}$$

subject to $\sum_{j \in J'} y_{ij} = a_i'', i \in I' = I \cup \{m+1\}$

$$\sum_{i \in I'} y_{ij} = b_j'', j \in J' = J \cup \{n+1\}$$

$$y_{ij} \geq 0; i \in I', j \in J'$$

where

$$a_i'' = a_i', i \in I, \ a_{m+1}'' = F - \sum_{i \in I} a_i',$$

$$b_j'' = b_j', j \in J, \ b_{n+1}'' = F - \sum_{j \in J} b_j',$$

$$c_{ij}' = c_{ij}, i \in I, j \in J, \ c_{i,n+1}' = \min_{j \in J} c_{ij}, i \in I,$$

$$c_{m+1,j}' = \min_{i \in I} c_{ij}, j \in J$$

and $c_{m+1,n+1}' = M$,

where M is a large positive number

$$f_i' = f_i, i \in I, \ f_{m+1}' = 0.$$

Definition — A feasible solution $\{y_{ij}\}$, $i \in I', j \in J'$ to (RP^2) is called a corner feasible solution if $y_{m+1,n+1} = 0$.

***Theorem* 3** — *No non-corner feasible solution to* (RP^2) *can lead to a feasible solution to* (P^2).

This theorem has been proved in [7] for a transportation problem with a flow constraint. It can easily be extended to the fixed charge transportation problem with a flow constraint. This theorem explains the choice of $c'_{m+1,n+1}$ in (RP^2).

Remark 4 : Optimal corner feasible solution to (RP^2) provides an optimal solution to (P^2) by using the same transformation as given in Section II.

CONCLUDING REMARK

The sufficient condition given in the theorem tells us when it is possible to improve a solution of problem (P^0) by an increase of a certain a_i and b_j.

This procedure, however, does not always lead to an optimal solution of problem (P^1).

Numerical Example

Consider the problem (P^0) for $m = n = 3$.

Table I gives the value of variable cost c_{ij} ($i = 1, 2, 3$; $j = 1, 2, 3$) as also the values of a_i ($i = 1, 2, 3$) and b_j ($j = 1, 2, 3$)

TABLE I : *Variable cost, c_{ij}, a_i*

				$a_i \downarrow$
$c_{ij} \rightarrow$	5	9	9	22
	4	6	2	10
	2	1	1	11
$b_j \rightarrow$	5	8	18	

The fixed costs are

$$f_{11} = 10, f_{12} = 5, f_{13} = 5$$

$$f_{21} = 12, f_{22} = 5, f_{23} = 5$$

$$f_{31} = 20,\ f_{32} = 10,\ f_{33} = 5,$$

f_i is given by

$$f_i = \sum_{i=1}^{3} \delta_{il} f_{il};\ i = 1, 2, 3,$$

where

$$\delta_{i1} = 1, \text{ if } \sum_{j=1}^{3} x_{ij} > 0, \text{ for } i = 1, 2, 3$$

$$= 0, \text{ otherwise}$$

$$\delta_{i2} = 1, \text{ if } \sum_{j=1}^{3} x_{ij} > 7, \text{ for } i = 1, 2, 3$$

$$= 0, \text{ otherwise}$$

$$\delta_{i3} = 1, \text{ if } \sum_{j=1}^{3} x_{ij} > 11, \text{ for } i = 1, 2, 3$$

$$= 0, \text{ otherwise.}$$

Here f_i $(i = 1, 2, 3)$ has three steps.

As $\sum_{i=1}^{3} a_i > \sum_{j=1}^{3} b_j$, we add a dummy destination in Table I with $c_{i4} = 0$, $i = 1, 2, 3$. A basic feasible solution of problem (P^0) is given in Table II.

TABLE II : *Basic feasible solution of* (P^0)

					u_i
	5 **5**	9	9 **5**	0 **12**	0
	4	6	2 **10**	0	–7
	2	1 **8**	1 **3**	0	–8
v_j	5	9	9	0	

In this basic feasible solution the fixed cost is f (current) = 62.

On applying step 2 and step 3, we get the values A_{ij}, f_{ij} (NB), Δ_{ij} which are displayed in Table III.

TABLE III : *Values of A_{ij}, f_{ij} (NB) and Δ_{ij}*

(i, j)	(1, 2)	(2, 1)	(2, 2)	(2, 4)	(3, 1)	(3, 4)
A_{ij}	0	30	32	70	15	24
f_{ij} (NB)	62	62	62	50	62	67
Δ_{ij}	0	30	32	58	15	29

As $\Delta_{ij} \geq 0 \ \forall \ (i, j) \notin B$, the solution given in Table II is an optimal solution of (P^0). Here $a'_1 = 10$, $a'_2 = 10$, $a'_3 = 11$.

Now $u_3 = -8$, $v_1 = 5$, $u_3 + v_1 = -3 < 0$.

We can assign any value to λ varying between 1 and 5. Let $\lambda = 3$

Increase a'_3 and b'_1 each by 3.

Then the new fixed cost = 67.

Therefore, $d = 67 - 62 = 5$.

$$\lambda(u_3 + v_1) + d = 3(-3) + 5 = -9 + 5 = -4 < 0.$$

Thus a paradox exists in this case.

Verification

Objective function value of the original problem, $Z^0 = 163$.

Corresponding flow $F^0 = 31$.

Objective function value after increment = 159.

Corresponding Flow = 34.

Best Paradoxical pair is found by solving the problem (P^1) for $m = n = 3$.

Form the corresponding problem (RP^1).

The optimal solution got on solving (RP^1) is given in Table IV.

TABLE IV : *Optimal solution to problem* RP^1

					u_i
	5 **5**	9	9	5 **5**	5
	4	6	2 **10**	2	1
	2	1 **8**	1 **3**	1	0
	2	1	1 **5**	0 **27**	0
v_j	0	1	1	0	

On making the transformation, the optimal solution to problem (P^1) is given in Table V.

TABLE V : *Optimal solution to problem* (P^1)

5 **10**	9 **0**	9
4	6	2 **10**
2	1 **8**	1 **8**

Here the objective function value $Z^1 = 153$ and flow $F^1 = 36$.

Thus the paradoxical range of flows is $[F^0, F^1] = [31, 36]$.

'Paradoxical Solution' for flow $F = 35$, is given in Table VI.

TABLE VI : *Paradoxical solution for flow* F = 35

5 **9**	9 **0**	9 **1**
4	6	2 **10**
2	1 **8**	1 **7**

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