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A NOTE ON THE COMPOSITION OF DISTRIBUTIONS

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Let F be a distribution and let f be a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The distribution $(x_+^{2r})_+^{-1/2}$ is evaluated for $r = 1, 2, ...$

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In the following, we let $\mathcal{D}$ be the space of all infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions on $\mathcal{D}$. Further, we let ρ be an infinitely differentiable function having the properties

(*i*) $\rho(x) = 0, |x| \geq 1,$

(*ii*) $\rho(x) \geq 0,$

(*iii*) $\rho(x) = \rho(-x)$

and

(*iv*) $\int_{-1}^{1} \rho(x)\, dx = 1.$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$ in the sense that

$$\lim_{n \to \infty} \langle \delta_n(x), \varphi(x) \rangle = \langle \delta(x), \varphi(x) \rangle$$

for all φ in $\mathcal{D}$, see Gel'fand and Shilov[7].

If now F is an arbitrary distribution in $\mathcal{D}'$, we define

$$F_n(x) = (F * \delta_n)(x) = \langle F(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to F.

The following definition was given in [2].

Definition 1 — Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to the distribution $h(x)$ in $\mathcal{D}'$ on the interval (a, b) if

$$\underset{n \to \infty}{N\text{-}lim} \int_a^b F_n(f(x))\, \varphi(x)dx = \langle h(x), \varphi(x) \rangle$$

for all φ in $\mathcal{D}$ with support contained in the interval (a, b), where

$$F_n(x) = (F * \delta_n)(x)$$

and N is the neutrix having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers with negligible functions finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n, \ln^r n : \lambda > 0,\ r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as n tends to infinity, see van der Corput[1].

This definition was later extended in [6] to cover the case when f is also a distribution.

Definition 2 — Let F and f be distributions in $\mathcal{D}'$. We say that the distribution $D(f(x))$ exists and is equal to the distribution $h(x)$ in $\mathcal{D}'$ on the interval (a, b) if

$$\underset{n \to \infty}{N\text{-lim}} \left[\underset{n \to \infty}{N\text{-lim}} \int_a^b F_n(f_m(x))\, \varphi(x)dx \right] = \langle h(x), \varphi(x) \rangle$$

for all φ in $\mathcal{D}$ with support contained in the interval (a, b), where

$$F_n(x) = (F * \delta_n)(x), f_m(x) = (f * \delta_m)(x)$$

and N is the neutrix defined above.

The following theorems were proved in [2] and [3] respectively:

***Theorem* 1** — *The distributions* $(x_-^{\mu})_-^{\lambda}$ *and* $(x_+^{\mu})_-^{\lambda}$ *exists and*

$$(x_-^{\mu})_-^{\lambda} = (x_+^{\mu})_-^{\lambda} = 0$$

for $\mu > 0,\ \lambda \neq -1, 2, \ldots$ *and* $\lambda\mu \neq -1, -2, \ldots$ *and*

$$(x_-^{\mu})_-^{\lambda} = (-1)^{\lambda\mu}(x_+^{\mu})_-^{\lambda} = \frac{\pi \operatorname{cosec}(\pi\lambda)}{2\mu(-\lambda\mu - 1)!}\,\delta^{(-\lambda\mu - 1)}(x)$$

for $\mu > 0,\ \lambda \neq -1, -2, \ldots$ *and* $\lambda\mu = -1, -2, \ldots$

***Theorem* 2** — *The distribution* $(x^2)_+^{-s-1/2}$ *exists and*

$$(x^2)_+^{-s-1/2} = |x|^{-2s-1} + \frac{2}{(2s)!}[\ln 2 - c(\rho)]\,\delta^{(2s)}(x),$$

for $s = 0, 1, 2, \ldots$

Further results were later proved in [4], [5] and [6].

Before proving our main theorem we note the following lemma which can be proved easily.

Lemma 1 — Let φ be a function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. Then

$$\langle x_+^{-r}, \varphi(x) \rangle = \int_0^1 x^{-r}\left[\varphi(x) - \sum_{i=0}^{r-1}\frac{\varphi^{(i)}(0)}{i!}x^i\right]dx - \sum_{i=0}^{r-2}\frac{\varphi^{(i)}(0)}{i!\,(r-i-1)} - \frac{\phi(r-1)}{(r-1)!}\varphi^{(r-1)}(0),$$

for $r = 1, 2, \ldots$, where

$$\phi(r) = \begin{cases} \sum_{i=1}^{r} 1/i, & r \geq 1, \\ 0, & r = 0. \end{cases}$$

In the above lemma, the distribution x_+^{-r} is defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!}(\ln x_+)^{(r)},$$

for $r = 1, 2, \ldots$ and not as in Gel'fand and Shilov[7].

We now prove the following theorem.

***Theorem* 3** — *Let F(x) denote the function* $x_+^{-1/2}$. *Then the distribution* $F(x_+^{2r})$ *exists and*

$$F(x_+^{2r}) = x_+^{-r} + \frac{(-1)^{r-1}[\ln 2 - c(\rho) + r\phi(r-1)]}{r!}\,\delta^{(r-1)}(x), \qquad \ldots (1)$$

for $r = 1, 2, \ldots$

PROOF : We have

$$F_n(x) = \begin{cases} \int_{-1/n}^{1/n} (x-t)^{-1/2}\,\delta_n(t)\,dt, & x > 1/n, \\ \int_{-1/n}^{x} (x-t)^{-1/2}\,\delta_n(t)dt, & -1/n < x < 1/n, \\ 0, & x < -1/n. \end{cases}$$

It follows that

$$\int_{-1}^{1} F_n(x_+^{2r})\,x^i\,dx = \int_{0}^{1} F_n(x^{2r})\,x^i\,dx + \int_{-1}^{0} F_n(0)\,x^i\,dx$$

$$= \int_{0}^{n^{-1/2r}} \int_{-1/n}^{x^{2r}} (x^{2r}-t)^{-1/2}\,x^i\,\delta_n(t)\,dt\,dx$$

$$+ \int_{n^{-1/2r}}^{1} \int_{-1/n}^{1/n} (x^{2r}-t)^{-1/2}\,x^i\,\delta_n(t)\,dt\,dx$$

$$+ \int_{-1}^{0} \int_{-1/n}^{0} (-t)^{-1/2}\,x^i\,\delta_n(t)\,dt\,dx$$

$$= \int_{0}^{1/n} \delta_n(t) \int_{t^{1/2r}}^{1} (x^{2r}-t)^{-1/2}\,x^i\,dx\,dt$$

$$+ \int_{-1/n}^{0} \delta_n(t) \int_{0}^{1} (x^{2r}-t)^{-1/2}\,x^i\,dx\,dt$$

$$+ \frac{(-1)^i}{i+1} \int_{-1/n}^{0} (-t)^{-1/2}\,\delta_n(t)\,dt \qquad \ldots (2)$$

$$= \int_{0}^{1/n} \delta_n(t) \int_{t^{1/2r}}^{1} [(x^{2r}-t)^{-1/2} + (x^{2r}+t)^{-1/2}]\,x^i\,dx\,dt$$

$$+ \int_{0}^{1/n} \delta_n(t) \int_{0}^{t^{1/2r}} (x^{2r}+t)^{-1/2}\,x^i\,dx\,dt$$

$$+ \frac{(-1)^i}{i+1} \int_0^{1/n} t^{-1/2}\, \delta_n(t)\, dt$$

$$= I_1 + I_2 + I_3.$$

Putting $nt = u$ and $n^{1/2r} x = \upsilon$, we have

$$I_1 = n^{(r-i-1)/2r} \int_0^1 \rho(u) \int_{u^{1/2r}}^{n^{1/2r}} [(\upsilon^{2r} - u)^{-1/2} + (\upsilon^{2r} + u)^{-1/2}]\, \upsilon^i \, d\upsilon\, du$$

$$= 2n^{(r-i-1)/2r} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{2k} \int_0^1 u^{2k} \rho(u) \int_{u^{1/2r}}^{n(1/2r} \upsilon^{-4rk-r+i}\, d\upsilon\, du$$

$$= 2n^{(r-i-1/2r} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{2k} \int_0^1 \frac{n^{(-4rk-r+i+1)/2r} - u^{(-4rk-r+i+1)/2r}}{-4rk-r+i+1} u^{2k} \rho(u)\, du$$

and it follows that

$$\underset{n\to\infty}{N\text{–}\lim}\; I_1 = -\frac{2}{r-i-1} \int_0^1 \rho(u)\, du = -\frac{1}{r-i-1} \qquad \dots (3)$$

for $i = 0, 1, \dots, r-2$.

Next we have

$$I_2 = n^{(r-i-1)/2r} \int_0^1 \rho(u) \int_0^{u^{1/2r}} (\upsilon^{2r} + u)^{-1/2}\, \upsilon^i \, d\upsilon\, du$$

and it follows that

$$\underset{n\to\infty}{N\text{–}\lim}\; I_2 = 0 \qquad \dots (4)$$

for $i = 0, 1, \dots, r-2$.

Finally we have

$$I_3 = \frac{(-1)^i n^{1/2}}{i+1} \int_0^1 \rho(u) du$$

and it follows that

$$\underset{n\to\infty}{N\text{–}\lim}\; I_3 = 0 \qquad \dots (5)$$

for $i = 0, 1, \dots, r-2$.

It now follows from eqs. (3), (4) and (5) that

$$\int_{-1}^{1} F_n(x_+^{2r})\, x^i\, dx = -\frac{1}{r-i-1} \qquad \dots (6)$$

for $i = 0, 1, \dots, r-2$.

We now consider the case $i = r - 1$. We have

$$\int_{t^{1/2r}}^{1} (x^{2r} - t)^{-1/2}\, x^{r-1}\, dx = \frac{\ln[1 + (1-t)^{1/2}] - \ln t^{1/2}}{r},$$

$$\int_{0}^{1} (x^{2r} - t)^{-1/2}\, x^{r-1}\, dx = \frac{\ln[1 + (1-t)^{1/2}] - \ln|t|^{1/2}}{r}$$

and
$$\int_{-1/n}^{0} (-t)^{-1/2}\, \delta_n(t)\, dt = n^{1/2} \int_{-1}^{0} (-u)^{-1/2}\, \rho(u)\, du.$$

It follows from eq. (2) that

$$\int_{-1}^{1} F_n(x_+^{2r})\, x^{r-1}\, dx = \frac{1}{r} \int_{-1/n}^{1/n} \ln[1 + (1-t)^{1/2}]\, \delta_n(t)\, dt$$

$$- \frac{1}{2} \int_{-1/n}^{0} (-t)^{-1/2}\, \delta_n(t)\, dt = n^{1/2} \int_{-1}^{0} (-u)^{-1/2}\, \rho(u)\, du$$

and so

$$\underset{n\to\infty}{N\text{-lim}} \int_{-1}^{1} F_n(x_+^{2r})\, x^{r-1}\, dx = \frac{\ln 2 - c(\rho)}{r}, \qquad \dots (7)$$

since

$$\int_{-1/n}^{1/n} \ln|t|\, \delta_n(t)\, dt = \int_{-1}^{1} \ln|u|\, \rho(u)\, du - \ln n \int_{-1}^{1} \rho(u)\, du.$$

When $i = r$, we have

$$\int_{0}^{n^{-1/2r}} |x^r F_n(x_+^{2r})|\, dx = \int_{0}^{n^{-1/2r}} \int_{-1/n}^{x^{2r}} x^r (x^{2r} - t)^{-1/2}\, \delta_n(t)\, dt\, dx$$

$$= n^{-1/2r} \int_0^1 \int_{-1}^{\upsilon^{2r}} \upsilon^r(\upsilon^{2r} - u)^{-1/2} \rho(u)\, du d\upsilon$$

$$= O(n^{-1/2r})$$

and it follows that if ψ is an arbitrary continuous function then

$$\lim_{n \to \infty} \int_0^{n^{-1/2r}} x^r F_n(x_+^{2r})\, \psi(x) dx = 0. \qquad \text{... (8)}$$

Further,

$$\int_{-1}^0 x^r F_n(x_+^{2r})\, \psi(x) dx = \int_{-1}^0 x^r \psi(x) \int_{-1/n}^0 (-t)^{-1/2} \delta_n(t)\, dt\, dx$$

$$= n^{1/2} \int_{-1}^0 x^r \psi(x) \int_{-1}^0 (-u)^{-1/2} \rho(u)\, du\, dx$$

and it follows that

$$N\text{–}\lim_{n \to \infty} \int_{-1}^0 x^r F_n(x_+^{2r})\, \psi(x)\, dx = 0. \qquad \text{... (9)}$$

If now η is chosen so that $n^{-1/2r} < \eta < 1$, we have on making the substitution $nt = u$ again

$$\int_{n^{-1/2r}}^{\eta} | x^r F_n(x_+^{2r}) |\, dx = \int_{n^{-1/2r}}^{\eta} \int_{-1/n}^{1/n} x^r (x^{2r} - t)^{-1/2} \delta_n(t)\, dx\, dt$$

$$= \sum_{i=0}^{\infty} \binom{-\frac{1}{2}}{i} \int_{-1}^{1} \rho(u) \int_{n^{-1/2r}}^{\eta} \frac{(-u)^i}{n^i x^{2ri}}\, dx\, du$$

$$= \sum_{i=0}^{\infty} \binom{-\frac{1}{2}}{i} \int_{-1}^{1} \rho(u) \frac{(-u)^i n^{-i} (\eta^{1-2ri} - n^{-1/2r})}{1 - 2ri}\, du.$$

It follows that

$$\lim_{n \to \infty} \int_{n^{-1/2r}}^{\eta} | x^r F_n(x_+^{2r}) |\, dx = \eta$$

and so if ψ is a continuous function then

$$\lim_{n \to \infty} \int_{n^{-1/2r}}^{\eta} | x^r F_n(x_+^{2r}) \psi(x) | \, dx = O(\eta). \qquad \dots (10)$$

Now let φ be an arbitrary function $\mathcal{D}$ with support contained in the interval $[-1, 1]$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i + \frac{\varphi^{(r)}(\xi x)}{r!} x^r,$$

where $0 < \xi < 1$. Then

$$\langle F_n(x_+^{2r}), \varphi(x) \rangle = \int_{-1}^{1} F_n(x_+^{2r}) \varphi(x) dx$$

$$= \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} \int_{-1}^{1} F_n(x_+^{2r}) x^i \, dx + \frac{1}{r!} \int_{-1}^{0} x^r F_n(x_+^{2r}) \varphi^{(r)}(\xi x) \, dx$$

$$+ \frac{1}{r!} \int_{0}^{n^{-1/2r}} x^r F_n(x_+^{2r}) \varphi^{(r)}(\xi x) \, dx + \frac{1}{r!} \int_{n^{-1/2r}}^{\eta} x^r F_n(x_+^{2r}) \varphi^{(r)}(\xi x) \, dx$$

$$+ \frac{1}{r!} \int_{\eta}^{1} x^r F_n(x_+^{2r}) \varphi^{(r)}(\xi x) \, dx.$$

Taking the neutrix limit as n tends to infinity, using eqs. (6)-(10). and noting that the sequence $\{x^r F_n(x_+^{2r})\}$ converges uniformly to 1 on the interval $[\eta, 1]$, it follows that

$$\underset{n \to \infty}{N\text{-lim}} \; \langle F_n(x_+^{2r}), \varphi(x) \rangle = - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!\,(r-i-1)} + \frac{\ln 2 - c(\rho)}{r!} \varphi^{(r-1)}(0)$$

$$+ \int_{\eta}^{1} \varphi^{(r)}(\xi x) \, dx + O(\eta)$$

$$= - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!\,(r-i-1)} + \frac{\ln 2 - c(\rho)}{r!} \varphi^{(r-1)}(0)$$

$$+ \int_{0}^{1} \varphi^{(r)}(\xi x) \, dx,$$

since η can be made arbitrarily small. Thus, on using Lemma 1, we have

$$N\text{-}\lim_{n \to \infty} \langle F_n(x_+^{2r}), \varphi(x) \rangle = \int_0^1 x^{-r} \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i \right] dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)}$$

$$+ \frac{\ln 2 - c(\rho)}{r!} \varphi^{(r-1)}(0)$$

$$= \langle x_+^{-r}, \varphi(x) \rangle + \frac{(-1)^{r-1}[\ln 2 - c(\rho) + r\phi(r-1)]}{r!} \langle \delta^{(r-1)}(x), \varphi(x) \rangle,$$

proving eq. (1) on the interval [– 1, 1]. However, since $F(x_+^{2r}) = x_+^{-r}$ on any closed interval not containing the origin, we have proved eq. (1) on the real line.

Corollary 3.1 — The distribution $F(x_-^{2r})$ exists and

$$F(x_-^{2r}) = x_-^{-r} + \frac{\ln 2 - c(\rho) + r\phi(r-1)}{r!} \delta^{(r-1)}(x), \quad \text{... (11)}$$

for r = 1, 2, ...

PROOF : Replacing x by $-x$ eq. (1), we have

$$F((-x)_+^{2r}) = (-x)_+^{-r} + \frac{(-1)^{r-1}[\ln 2 - c(\rho) + r\phi(r-1)]}{r!} \delta^{(r-1)}(-x)$$

and so

$$F(x_-^{2r}) = x_-^{-r} + \frac{\ln 2 - c(\rho) + r\phi(r-1)}{r!} \delta^{(r-1)}(x),$$

proving eq. (11).

Corollary 3.2 — Let G denote the function $x_-^{-1/2}$. Then the distribution $G(-x_+^{2r})$ exists and

$$G(-x_+^{2r}) = x_+^{-r} + \frac{(-1)^{r-1}[\ln 2 - c(\rho) + r\phi(r-1)]}{r!} \delta^{(r-1)}(x), \quad \text{... (12)}$$

for r = 1, 2, ...

PROOF : We have $G(-x) = F(x)$ and so

$$G(-x_+^{2r}) = F(x_+^{2r}) = x_+^{-r} + \frac{(-1)^{r-1}[\ln 2 - c(\rho) + r\phi(r-1)]}{r!} \delta(x),$$

from equation (1), proving eq. (12).

Corollary 3.3 — The distribution $G(-x_-^{2r})$ exists and

$$G(-x_-^{2r}) = x_-^{-r} + \frac{\ln 2 - c(\rho) + r\phi(r-1)}{r!} \delta^{(r-1)}(x), \quad \text{... (13)}$$

for r = 1, 2, ...

PROOF : Replacing x by $-x$ eq. (12) we have

$$G(-(-x)_+^{2r}) = (-x)_+^{-r} + \frac{(-1)^{r-1}[\ln 2 - c(\rho) + r\phi(r-1)]}{r!}\delta^{(r-1)}(-x)$$

and so

$$G(-x_-^{2r}) = x_-^{-r} + \frac{\ln 2 - c(\rho) + r\phi(r-1)}{r!}\delta^{(r-1)}(x),$$

proving eq. (13).

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REFERENCES

1. J. G. van der Corput, *J. Analyse Math.*, **7** (1959-60), 291-398.
2. B. Fisher, *Rostock. Math. Kolloq.*, **28** (1985), 75-86.
3. B. Fisher and M. Itano, *Mem. Fac. Integrated Arts Sci., Hiroshima Univ., Ser. IV*, **11** (1986), 1-15.
4. B. Fisher, H. Kou and E. Özçağ, *Proc. Fifth Int. Conf. Complex Analysis and Applications with Symposium on Gneralized Functions, Varna, Bulgaria*, 1991, Publishing House of Bulgarian Academy of Sciences, 1993, 63-72.
5. B. Fisher, and E. Özçağ, *Publ. Math. Debrecen*. **43** (1993), 265-72.
6. H. Kou and B. Fisher, *Publ. Math. Debrecen*, **40** (1990), 279-90.
7. I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol. I, Academic Press, 1964.