Indian J. pure appl. Math., 31(7) : 849-855, July 2000

ENTIRE FUNCTIONS SHARING ONE VALUE IM*

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(*Received* 9 *February* 1999; *after revision* 27 *July* 1999; *accepted* 14 *October* 1999)

In this paper, we establish a unicity theorem corresponding to the results of Cluine and Hayman. Let f and g be two nonconstant entire functions, $n \geq 12$ an integer, and let $a \neq 0$ be a finite constant. If $f^n f'$ and $g^n g'$ share the value a IM, then either $f = tg$ for some $(n+1)th$ root of unity t or $f(z) = c_1 e^{-cz}$ and $g(z) = c_2 e^{cz}$, where c, c_1 and c_2 are constants that satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

Key Words : Entire Function; Sharing Value; Unicity

1. INTRODUCTION

In this paper, the term "meromorphic" will always mean meromorphic in the complex plane C. We say that two meromorphic functions f and g share a finite value a IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). It is assumed that the reader is familiar with the standard natations of value distributions theory that can be found, for instance, in [4] or [6]. We denote by $S(r, f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\}$$

as $r \to \infty$, possibly outside a set of finite measure.

In addition, we shall also use the following notations.

Let f and g be two nonconstant meromorphic functions such that f and g share 1 IM. We denote by $\overline{N}_L\left(r, \frac{1}{f-1}\right)$ the counting function for 1 - points of both f and g about which f has larger multiplicity than g, with multiplicity being not counted, and denote by $N_{11}\left(r, \frac{1}{f-1}\right)$ the counting function for common simple 1 - points of both f and g. In the same way, we can define

*Supported in part by NSF of Jiangsu Province.

$\overline{N}_L\left(r,\frac{1}{g-1}\right)$ and $N_{11}\left(r,\frac{1}{g-1}\right)$. Especially, if f and g share 1 CM, then

$$\overline{N}_L\left(r,\frac{1}{f-1}\right)=\overline{N}_L\left(r,\frac{1}{g-1}\right)=0.$$

For any constant c, we define

$$\Theta(c,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}\left(r,\frac{1}{f-c}\right)}{T(r,f)}.$$

Combining the results of Clunie[1] and Hayman[3], we have the following result.

***Theorem* A** — *Let f be an entire function, $n\geq 1$. If $f^n f'\neq 1$, then f is a constant.*

It is interesting to establish the unicity theorem corresponding to the above result. In 1996 Fang and Hua[2] proved

***Theorem* B** — *Let f and g be two nonconstant entire functions, $n\geq 6$. If $f^n f'$ and $g^n g'$ share the value of 1 CM, then either $f^n f' g^n g'=1$ or $f=tg$ for a constant t with $t^{n+1}=1$.*

In this paper, we shall prove the following result :

***Theorem* 1** — *Let f and g be two nonconstant entire functions, $n\geq 12$ an integer, and let $a\neq 0$ be a finite constant. If $f^n f'$ and $g^n g'$ share the value a IM, then either $f=tg$ for some $(n+1)$th root of unity t or $f(z)=c_1 e^{-cz}$ and $g(z)=c_2 e^{cz}$, where c, c_1 and c_2 are constants that satisfy $(c_1 c_2)^{n+1} c^2=-a^2$.*

Remark 1 : That $a\neq 0$ in Theorem 1 is necessary. For example, let $f=e^{e^z}$ and $g=e^z$, we know that $f^n f'$ and $g^n g'$ share 0 IM for any integer n, but f and g do not satisfy the conclusion of Theorem 1.

2. Lemmas

To prove our results, we need some lemmas.

By Milloux inequality (see [6]), we have

Lemma 1 — Let f be a nonlinear entire function, and let $a\neq 0$ be finite complex number. Then

$$T(r,f)\leq 2\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f'-a}\right)-N_0\left(r,\frac{1}{f''}\right)+S(r,f),$$

where $N_0\left(r,\frac{1}{f''}\right)$ denotes the counting function corresponding to the zeros of f'' that are not zeros of f and $f'-a$.

Lemma 2 [See (7)] — Let f be a nonconstant meromorphic function, and let a_j be distinct finite complex numbers such that $a_j\neq 0$ ($j=1, 2, ..., q$). Then

$$\sum_{j=1}^{q}\left(N\left(r,\frac{1}{f-a_j}\right)-\overline{N}\left(r,\frac{1}{f-a_j}\right)\right)\leq\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+m\left(r,\frac{f'}{f}\right)+O(1)$$

Lemma 3 — [See (5)] Let f and g be two nonconstant entire functions, and let n be a positive integer. If $f^n f' g^n g' = 1$, then $f = c_1 e^{-cz}$ and $g = c_2 e^{cz}$, where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -1$.

Lemma 4 — Let F and G be two nonlinear entire functions such that $\Theta(0, F) > \frac{11}{12}$ and $\Theta(0, G) > \frac{11}{12}$. If F' and G' share the value 1 *IM*, then either $F'G' = 1$ or $F = G$.

PROOF : Set

$$H=\frac{F'''}{F''}-2\frac{F''}{F'-1}-\frac{G'''}{G''}+2\frac{G''}{G'-1}. \quad \ldots (1)$$

Suppose that $H \not\equiv 0$. If z_0 is a common simple zero of $F' - 1$ and $G' - 1$, by a simple computation on local expansions, we know that z_0 is a zero of H. Thus we have

$$N_{11}\left(r,\frac{1}{F'-1}\right)\leq N\left(r,\frac{1}{H}\right)\leq N(r,H)+S(r,F)+S(r,G). \quad \ldots (2)$$

By our assumptions, the poles of H only occur at the zeros of F'' and G'' and the common 1 points of F and G with not the same multiplicity. So we obtain

$$N(r,H)\leq\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_L\left(r,\frac{1}{F'-1}\right)+\overline{N}_L\left(r,\frac{1}{G'-1}\right)$$
$$+N_0\left(r,\frac{1}{F''}\right)+N_0\left(r,\frac{1}{G''}\right), \quad \ldots (3)$$

where $N_0\left(r,\frac{1}{F''}\right)$ denotes the counting function corresponding to the zeros of F'' that are not zeros of F and $F' - 1$, $N_0\left(r,\frac{1}{G''}\right)$ denotes the counting function corresponding to the zeros of G'' that are not zeros of G and $G' - 1$. By Lemma 1, we have

$$T(r,F)+T(r,G)\leq 2\overline{N}\left(r,\frac{1}{F}\right)+2\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F'-1}\right)+\overline{N}\left(r,\frac{1}{G'-1}\right)$$
$$-N_0\left(r,\frac{1}{F''}\right)-N_0\left(r,\frac{1}{G''}\right)+S(r,F)+S(r,G). \quad \ldots (4)$$

Noting that F' and G' share 1 IM, we have

$$\overline{N}\left(r,\frac{1}{F'-1}\right)+\overline{N}\left(r,\frac{1}{G'-1}\right)\leq N_{11}\left(r,\frac{1}{F'-1}\right)$$
$$+\overline{N}_L\left(r,\frac{1}{F'-1}\right)+N\left(r,\frac{1}{G'-1}\right).$$

Since

$$N\left(r, \frac{1}{G'-1}\right) \le T(r, G') + O(1) \le m(r, G) + m\left(r, \frac{G'}{G}\right) + O(1)$$

$$= T(r, G) + S(r, G),$$

then we have

$$+ \overline{N}\left(r, \frac{1}{F'-1}\right)\overline{N}\left(r, \frac{1}{G'-1}\right) \le N_{11}\left(r, \frac{1}{F'-1}\right)$$

$$+ \overline{N}_L\left(r, \frac{1}{F'-1}\right) + T(r, G) + S(r, G) \qquad \text{... (5)}$$

Combining (2), (3), (4) and (5), we get

$$T(r, F) \le 3\overline{N}\left(r, \frac{1}{F}\right) + 3\overline{N}\left(r, \frac{1}{G}\right) + 2\overline{N}_L\left(r, \frac{1}{F'-1}\right)$$

$$+ \overline{N}_L\left(r, \frac{1}{G'-1}\right) + S(r, G) + S(r, G). \qquad \text{... (6)}$$

Noting that

$$\overline{N}\left(r, \frac{1}{F'}\right) \le \overline{N}\left(r, \frac{1}{F'}\right) + \overline{N}\left(r, \frac{1}{F}\right) \le T\left(r, \frac{F'}{F}\right) + \overline{N}\left(r, \frac{1}{F}\right) + O(1)$$

$$= 2\overline{N}\left(r, \frac{1}{F}\right) + S(r, F), \qquad \text{... (7)}$$

By Lemma 2, we can obtain that

$$\overline{N}_L\left(r, \frac{1}{F'-1}\right) \le \overline{N}\left(r, \frac{1}{F'}\right) + S(r, F)$$

$$\le 2\overline{N}\left(r, \frac{1}{F}\right) + S(r, F), \qquad \text{... (8)}$$

Similarly, we have

$$\overline{N}\left(r, \frac{1}{G'}\right) \le 2\overline{N}\left(r, \frac{1}{G}\right) + S(r, G), \qquad \text{... (9)}$$

$$\overline{N}_L\left(r, \frac{1}{G'-1}\right) \le 2\overline{N}\left(r, \frac{1}{G}\right) + S(r, G) \qquad \text{... (10)}$$

Substituting (8) and (10) in (6), we get

$$T(r, F) \le 7\overline{N}\left(r, \frac{1}{F}\right) + 5\overline{N}\left(r, \frac{1}{G}\right) + S(r, F) + S(r, G). \qquad \text{... (11)}$$

In the same way, we can get

$$T(r, G) \le 7\overline{N}\left(r, \frac{1}{G}\right) + 5\overline{N}\left(r, \frac{1}{F}\right) + S(r, F) + S(r, G). \quad \text{... (12)}$$

Thus, we have

$$T(r, F) + T(r, G) \le 12\overline{N}\left(r, \frac{1}{G}\right) + 12\overline{N}\left(r, \frac{1}{F}\right) + S(r, F) + S(r, G).$$

Hence

$$(12\Theta(0, F) - 11)T(r, F) + (12\Theta(0, G) - 11)T(r, G) \le S(r, F) + S(r, G), \quad \text{... (13)}$$

which is impossible since $\Theta(0, F) > \frac{11}{12}$ and $\Theta(0, G) > \frac{11}{12}$, therefore $H \equiv 0$. Then we deduce from (1) that

$$\frac{1}{F' - 1} = \frac{bG' + a - b}{G' - 1}, \quad \text{... (14)}$$

where a and b are finite complex numbers satisfying $b^2 + (a - b)^2 \neq 0$.

Next we prove either $F'G' = 1$ or $F = G$. We consider three cases.

Case 1 : $b \neq 0$ and $a = b$. From (14), we know that $G' \neq 0$, then there exists an entire function $P(z)$ such that $G' = e^{P(z)}$ and

$$F' = 1 + \frac{1}{b} - \frac{1}{b}e^{-P(z)}.$$

If $b = -1$, then $F'G' = 1$. This is what we wanted. If $b \neq -1$, then $F' - \left(1 + \frac{1}{b}\right) = -\frac{1}{b}e^{-P} \neq 0$. By Lemma 1, we have

$$T(r, F) \le 2N\left(r, \frac{1}{F}\right) + S(r, F)$$

i.e.,

$$(2\Theta(0, F) - 1)T(r, F) \le S(r, F), \quad \text{... (15)}$$

which contradicts

$$\Theta(0, F) > \frac{11}{12}.$$

Case 2 : $b \neq 0$ and $a \neq b$. From (14) we see that $G' + \frac{a - b}{b} \neq 0$, Using Lemma 1, we have

$$T(r, G) \le 2N\left(r, \frac{1}{G}\right) + S(r, G),$$

then similarly as in Case 1, we can get a contradiction.

Case 3 : $b = 0$ and $a \neq b$. Then from (14) we have

$$F = \frac{1}{a-b} G + L(z), \qquad \dots (16)$$

where $L(z) = \left(1 - \frac{1}{a-b}\right) z + c$, c is a finite constant. If $L(z) \equiv 0$, then $a - b = 1$, and so, from (16) we have $F = G$. If $L(z) \not\equiv 0$, by the second fundamental theorem on three small functions, we have

$$\begin{aligned} T(r, F) &\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - L(z)}\right) + S(r, F) \\ &= \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r, F) \\ &\leq (1 - \Theta(0, F))T(r, F) + (1 - \Theta(0, G))T(r, G) + S(r, F). \qquad \dots (17) \end{aligned}$$

Noting that F is nonlinear, from (16) we have

$$T(r, G) \leq T(r, F) + S(r, F),$$

combining this and (17), we get

$$(\Theta(0, F) + \Theta(0, G) - 1)T(r, F) \leq S(r, F),$$

we arrive at a contradiction since $\Theta(0, F) > \frac{11}{12}$ and $\Theta(0, G) > \frac{11}{12}$. The proof of Lemma 4 is complete.

3. Proof of Theorem 1

Let $F = \frac{f^{n+1}}{a(n+1)}$ and $G = \frac{g^{n+1}}{a(n+1)}$. Then the condition that $f^n f'$ and $g^n g'$ share the value a IM implies that F' and G' share the value 1 IM. Obviously, F and G are nonlinear and

$$\Theta(0, F) = 1 - \varlimsup_{r \to \infty} \frac{\overline{N}\left(r, \frac{1}{F}\right)}{T(r, F)} = 1 - \varlimsup_{r \to \infty} \frac{\overline{N}\left(r, \frac{1}{f^{n+1}}\right)}{T(r, f^{n+1})} \geq \frac{n}{n+1}.$$

Similarly, we have

$$\Theta(0, G) \geq \frac{n}{n+1}.$$

Since $n \geq 12$, then $\Theta(0, F) > \frac{11}{12}$ and $\Theta(0, G) > \frac{11}{12}$. Thus by Lemma 4, we get $F'G' = 1$ or $F = G$. If $F'G' = 1$, i.e., $a^{-2} f^n f' g^n g' = 1$. Set $f_1 = a^{-\frac{1}{n+1}} f$ and $g_1 = a^{-\frac{1}{n+1}} g$, then $f_1^n f_1' g_1^n g_1' = 1$. Using Lemma 3, we get $f(z) = c_1 e^{-cz}$ and $g(z) = c_2 e^{cz}$, where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$. If $F = G$, i.e., $f^{n+1} = g^{n+1}$. Then we have $f = tg$ for some $(n + 1)th$ root of unity t. The proof of Theorem 1 is complete.

ACKNOWLEDGEMENT

The authors are very grateful to Prof. Huaihui Chen and Xinhou Hua for helpful suggestions. The authors also wish to thank the referees for valuable remarks.

REFERENCES

1. J. Clunie, *J. London math. Soc.*, **42**(1967), 389-92.
2. M. L. Fang and X. H. Hua, *J. Nanjing Univ. Math. Biq.*, **13**(1)(1996), 44-48.
3. W. K. Hayman, *Ann. Math.*, **70**(1959), 9-42.
4. W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
5. C. C. Yang and X. H. Hua, *Ann. Acad. Sci. Fenn. ser. A. I. Math.*, **22**(1997), 395-406.
6. L. Yang, *Value Distribution Theory*, Springer-Verlag, Berlin, 1993.
7. H. X. Yi, *Indian J. pure appl. Math.* **28**(4)(1997), 509-19.