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COMMON FIXED POINT THEOREMS IN *D*-METRIC SPACES WITH LOCAL BOUNDEDNESS

JEONG SHEOK UME AND JONG KYU KIM*

Department of Applied Mathematics, Changwon National University, Changwon 641-773, Korea.
E-mail: jsume@sarim.changwon.ac.kr

**Department of Mathematics, Kyungnam University, Masan 631-701, Korea,*
E-mail: jongkyuk@hanma.kyungnam.ac.kr

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In this paper, we improved the result of Rhoades. That is, we proved some common fixed point theorems for more general selfmappings with a condition which is weaker than the boundedness on a *D*-metric space.

Key Words : Common Fixed Point Theorems; *D*-Metric Spaces; Local Boundedness; Banach Contraction Principle; Caristis Condition; *D*-Compatible

1. INTRODUCTION

In 1994, Dien[3] established that a pair of mappings satisfying both the Banach contraction principle and Caristi's condition in a complete metric space has a common fixed point.

Recently, Dhage[2] introduced the concept of the *D*-metric space and obtained beutiful fixed point theorems, and Rhoades[6] improved the result of Dhage[2]. Let X be a bounded complete *D*-metric space and T a selfmap of X satisfying

$$D(Tx, Ty, Tz) \leq q \ \max\ \{D(x, y, z), D(x, Tx, z), D(y, Ty, z), \ D(x, Ty, z), D(y, Tx, z)\}$$

for all $x, y, z \in X, 0 \leq q < 1$. Then T has a unique fixed point p in X and T is continuous at p.

In this paper, we improved the result of Rhoades[6]. That is, we proved some common fixed point theorems for more general selfmappings with a condition (see, (2.2)) which is weaker than the boundedness of X. These results are also generalizations of the well known fixed point theorems which have been established by Ciric[1], Dhage[2], and Dien[3].

2. PRELIMINARIES

Throughout this paper, we denote by N the set of all positive integers and by R the set of all real numbers.

Dhage[2] introduced the following *D*-metric space.

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Definition 2.1 [2] — Let X be a non-empty set. A D-metric on X is a function $D : X \times X \times X \to R$ such that

(i) $D(x, y, z) \geq 0$ for all $x, y, z \in X$, and equality holds if and only if $x = y = z$,

(ii) $D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) = D(z, x, y) = D(z, y, x)$ for all $x, y, z \in X$

and (iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$ (tetrahedral inequality).

If D is a D-metric on X, then the pair (X, D) is called a D-metric space.

Definition 2.2 — Let (X, d) be a metric space and A, B be subsets of X.

$$\delta_d(A, B) = \sup \{d(x, y) \mid x \in A, y \in B\}.$$

We denote $\delta_d(A, A)$ by $\delta_d(A)$. Let X be a D-metric space and A, B, C be subsets of X. Define

$$\delta_D(A, B, C) = \sup \{D(x, y, z) \mid x \in A, y \in B, z \in C\}.$$

We denote $\delta_D(A, A, A)$ by $\delta_D(A)$ and call it the diameter of A (with respect to a D-metric). A subset B of X will be called bounded if $\delta_D(B)$ is finite.

Definition 2.8 — Let X be a D-metric space, S, T mappings from X into itself and let $\{x_n\}_{n=0}^{\infty}$ be a sequence in X such that $Sx_{n-1} = Tx_n$ for all $n \in N$. Then we denote $O_s(Tx_n) = \{Tx_p \mid p \geq n\}$ for each $n \in N$.

Definition 2.4[2] — A sequence $\{x_n\}$ in a D-metric space X converges to $x \in X$ if for an arbitrary $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n > m > n_0$, $D(x_m, x_n, x) < \varepsilon$.

Definition 2.5[2] — A sequence $\{x_n\}$ in a D-metric space X is D-Cauchy sequence if for an arbitrary $\varepsilon > 0$ there exists a positive integer n_0 such that for all $p > n > m \geq n_0$, $D(x_m, x_n, x_p) < \varepsilon$.

Definition 2.6[2] — A D-metric space X is complete if every D-Cauchy sequence $\{x_n\}$ in X converges in X.

Definition 2.7[2] — Let $x_0 \in X$ and $\varepsilon > 0$ be given. Then we define the open ball $B(x_0, \varepsilon)$ in X centered at x_0 of radius of ε by

$$B(x_0, \varepsilon) = \{y \in X \mid D(x_0, y, y) < \varepsilon \text{ if } y = x_0 \text{ and } \sup_{z \in X} D(x_0, y, z) < \varepsilon \text{ if } y \neq x_0\}.$$

Then the collection of all open balls $\{B(x, \varepsilon) : x \in X\}$ defines the topology on X denoted by τ. Throughout this paper we assume that the D-metric space X is equipped with the topology τ.

Definition 2.8[5] — Let (X, d) be a metric space and $S, T : X \to X$ are mappings. Then the pair (S, T) is said to be compatible if and only if for every sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = u \text{ for some } u \in X,$$

it implies

$$\lim_{n \to \infty} d(STx_n, TSx_n) = 0.$$

Now we would like to define a concept of compatibility in a D-metric space corresponding to compatibility in a metric space (X, d).

Definition 2.9 — Let X be a D-metric space and $S, T : X \to X$ are mappings. Then the pair (S, T) is said to be D-compatible if and only if

$$D(STx, STy, TSz) \leq \alpha \cdot D(Tx, Ty, Sz)$$

for all $x, y, z \in X$ and some $\alpha \in (0, \infty)$.

Using the definition of a D-metric on X and topology τ on X, we have the following lemma.

Lemma 2.10[2] — The D-metric for X is a continuous function on $X \times X \times X$ in the topology τ on X.

By employing method similar to Lemma 4.1 in [4], we have the following lemma.

Lemma 2.11 — Let X be a D-metric space and S, T be selfmappings of X satisfying there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that

$$Sx_n = Tx_{n+1} \text{ for } n = 0, 1, 2, 3, \ldots, \qquad \ldots (2.1)$$

and

$$\delta_D(O_S(Tx_0)) < \infty. \qquad \ldots (2.2)$$

Then the followings are equivalent:

(2.3) for an arbitrary real number $\varepsilon > 0$, there exist $\varepsilon', \varepsilon''$ such that $0 < \varepsilon' < \varepsilon < \varepsilon''$ and $C(Tx, Ty, Tz) < \varepsilon''$ implies $D(Sx, Sy, Sz) < \varepsilon'$, where

$$C(Tx, Ty, Tz) = \max \{D(Tx, Ty, Tz), D(Tx, Sy, Tz), D(Ty, Sx, Tz), D(Tx, Sx, Tz), D(Ty, Sy, Tz)\}. \qquad \ldots (2.3)$$

(2.4) There exists an increasing upper semicontinuous $\varphi : R_+ \to R_+$ satisfying $\varphi(t) < t$ for each $t > 0$ such that

$$D(Sx, Sy, z) \leq \varphi(C(Tx, Ty, Tz)) \text{ for all } x, y, z \in X, \qquad \ldots (2.4)$$

where

$$R_+ = \{u \in R \mid 0 \leq u\}.$$

PROOF : It is clear that (2.4) implies (2.3). Conversely, suppose that (2.3) holds. Define $\varphi : R_+ \to R_+$ by

$$\varphi(x) = \begin{cases} \frac{1}{9}x & 0 \leq x < 1, \\ \frac{1}{9}(x + [x]) & [x] \leq x < [x] + 1, \ 1 \leq x, \\ \frac{3}{8}(x + [x]) & x = 1 + [x], \ 1 \leq x, \end{cases}$$

where $[x]$ is the greatest integer not exceeding x. Then (2.4) follows easily from (2.3) and definition of φ.

Lemma 2.12 — Let X be a D-metric space and let S, T be selfmappings of X satisfying (2.1), (2.2) and (2.3). Then $\{Tx_n\}_{n=1}^{\infty}$ is a D-Cauchy sequence.

PROOF : Let

$$\gamma_n = \delta_D\,(O_S\,(Tx_n)), \quad n = 1, 2, 3, \ldots$$

Then from the Definition 2.3 and (2.2), we obtain $\gamma_1 < \infty$. Since $\{\gamma_n\}$ is a decreasing sequence of nonnegative real numbers, there exists a nonnegative real number ε such that

$$\lim_{n \to \infty} \gamma_n = \varepsilon.$$

By Lemma 2.11, $\gamma_{n+1} \le \varphi(\gamma_n)$. It is enough to prove that $\varepsilon = 0$. If not, then, from Lemma 2.11, we have $\varepsilon \le \varphi(\varepsilon) < \varepsilon$. This is a contradiction. Hence $\varepsilon = 0$. Therefore, $\{Tx_n\}$ is a D-Cauchy sequence.

3. Main Results

Theorem 3.1 — Let X be a complete D-metric space and let S, T be selfmappings of X satisfying (2.2), (2.3),

$SX \subseteq TX$, ... (3.1)

T is continuous, and ... (3.2)

(S, T) is a pair of D-compatible. ... (3.3)

Then S and T have a unique common fixed point in X.

PROOF : From condition (3.1), we obtain condition (2.1), i.e. there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that

$$Sx_n = Tx_{n+1} \quad \text{for } n = 0, 1, 2, \ldots .$$

Then by Lemma 2.12, $\{Tx_n\}_{n=1}^{\infty}$ is a D-Cauchy sequence. Since X is a complete D-metric space, $\{Tx_n\}$ converges to $u \in X$. From the condition (2.1), $\{Sx_n\}$ is also convergent to u. We shall prove that u is a unique common fixed point of S and T. Since T is continuous, we have

$$\lim_{n \to \infty} TSx_n = Tu.$$

Since for all $n, m, r \in N$ with $n < m < r$,

$$D(STx_n, STx_m, TSx_r) \le \alpha \cdot D(Tx_n, Tx_m, Sx_r),$$

from (3.2), (3.3) and Lemma 2.10, we have

$$\lim_{n \to \infty} STx_n = Tu.$$

Suppose that $Tu \neq u$. Let $\varepsilon = D(u, u, Tu)$. Then $0 < \varepsilon$. Now

$$\lim_{n\to\infty} C(Tx_n, Tx_n, TSx_n) = \lim_{n\to\infty} \{\max\, [D(Tx_n, Tx_n, TSx_n),,$$

$$D(Tx_n, Sx_n, TSx_n), D(Tx_n, Sx_n, TSx_n), D(Tx_n, Sx_n, TSx_n), D(Tx_n, Sx_n, TSx_n)]\}$$

$$= D(u, u, Tu).$$

Since

$$\lim_{n\to\infty} C(Tx_n, Tx_n, TSx_n) = D(u, u, Tu) = \varepsilon > 0,$$

by Lemma 2.11, $\varepsilon \leq \varphi(\varepsilon) < \varepsilon$, which is a contradiction. Therefore, $Tu = u$. Suppose that $Su \neq u$. Since

$$\lim_{n\to\infty} C(Tu, Tx_n, TSx_n) = \lim_{n\to\infty} \{\max$$

$$[D(Tu, Tx_n, TSx_n), D(Tu, Sx_n, TSx_n), D(Tx_n, Su, TSx_n),$$

$$D(Tu, Su, TSx_n), D(Tx_n, Sx_n, TSx_n)]\}$$

$$= D(u, u, Su) := \varepsilon > 0,$$

by Lemma 2.11, $\varepsilon \leq \varphi(\varepsilon) < \varepsilon$, which is a contradiction. Hence $Su = u$. To prove uniqueness, let u_1 and u_2 be two common fixed points of S and T, and suppose $u_1 \neq u_2$. Then

$$C(Tu_1, Tu_1, Tu_2) = D(u_1, u_1, u_2) := \varepsilon > 0.$$

Then by Lemma 2.11,

$$\varepsilon = D(Su_1, Su_1, Su_2) = D(u_1, u_1, u_2) \leq \varphi(\varepsilon) < \varepsilon.$$

This is a contradiction. Therefore, S and T have a unique common fixed point in X.

Corollary 3.2 — Let X be a complete D-metric space and let S, T be selfmappings of X satisfying (2.2), (3.1), (3.2), (3.3) and

$$D(Sx, Sy, Sz) \leq q \cdot \max$$

$$\{D(Tx, Ty, Tz), D(Tx, Sy, Tz), D(Ty, Sx, Tz), D(Tx, Sx, Tz), D(Ty, Sy, Tz)\}, \quad \ldots (3.4)$$

for all $x, y, z \in X$ and for some $q \in [0, 1)$. Then S and T have a unique common fixed point in X.

PROOF : Since the condition (3.4) implies (2.3), the result follows from Theorem 3.1.

Remark : In Corollary 3.2, letting $T = I_X$ (where I_X is an identity mapping on X), we can easily prove the Theorem 1 of Rhoades[6].

In Theorem 3.1, letting $T = I_X$ (where I_X is an identity mapping on X), we have the following corollary.

Corollary 3.3 — Let X be a complete D-metric space and S be a selfmapping of X satisfying (2.2) and for an arbitrary real number $\varepsilon > 0$, there exist $\varepsilon', \varepsilon''$ such that $0 < \varepsilon' < \varepsilon < \varepsilon''$ and $C(x, y, z) < \varepsilon''$ implies $D(Sx, Sy, Sz) < \varepsilon'$ where

$$C(x, y, z) = \max \{D(x, y, z), D(x, Sy, z), D(y, Sx, z), D(x, Sx, z), D(y, Sy, z)\}. \quad \ldots (3.5)$$

Then S has a unique fixed point in X.

Theorem 3.4 — *Let X be a complete D-metric space and S, T be selfmappings of X satisfying (2.3), (3.2) and (3.3) there exists a nonempty closed bounded subset A of X such that $SA \subseteq TA \subseteq A$.*

Then S and T have a unique common fixed point in X. ... (3.6)

PROOF : By method similar to Theorem 3.1, the result follows.

Theorem 3.5 — *Let X be a metric space with a metric d and S, T be selfmappings of X satisfying $SX \subseteq TX$ and for an arbitrary real number $\varepsilon > 0$, there exists $\varepsilon', \varepsilon''$ such that $0 < \varepsilon' < \varepsilon < \varepsilon''$ and $C(Tx, Ty) < \varepsilon''$ implies $d(Sx, Sy) < \varepsilon'$, where*

$$C(Tx, Ty) = \max d(Tx, Ty), d(Tx, Sy), d(Ty, Sx)\ d(Tx, Sx), d(Ty, Sy)\} \quad \ldots (3.7)$$

$$\delta_d (O_S (Tx_0)) < \infty. \quad \ldots (3.8)$$

Then $\left\{ Tx_n \right\}_{n=1}^{\infty}$ is a Cauchy sequence.

PROOF : By method similar to Lemma 2.12, the result follows.

Corollary 3.6[3] — Let (X, d) be a complete metric space and S, T be selfmappings of X satisfying (3.1), (3.2), (3.7), (3.8) and

(S, T) is a pair of compatible. ... (3.9)

Then S and T have a unique common fixed point in X.

PROOF : Using Theorem 3.5, the result follows by method similar to Theorem 3.1.

Corollary 3.7 — Let (X, d) be a complete metric space S, T be selfmappings of X satisfying (3.1), (3.2), (3.8), (3.9) and

$$d(Sx, Sy) \leq q \cdot \max \{d(Tx, Ty), d(Tx, Sy), d(Ty, Sx)\ \ d(Tx, Sx), d(Ty, Sy)\}$$

for all $x, y \in X$ and for some $q \in [0, 1)$. Then S and T have a unique common fixed point in X.

PROOF : Since the condition (3.10) implies (3.7), the result follows from Corollary 3.6.

In Corollary 3.6, letting $T = I_X$ we have the following corollary.

Corollary 3.8 — Let (X, d) be a complete metric space and let S be a selfmapping of X satisfying (3.8) and for an arbitrary real number $\varepsilon > 0$, there exists $\varepsilon', \varepsilon''$ such that $0 < \varepsilon' < \varepsilon < \varepsilon''$ and $C(x, y) < \varepsilon''$ implies $d(Sx, Sy) < \varepsilon'$, where

$$C(x, y) = \max \{d(x, y), d(x, Sy), d(y, Sx)\ d(x, Sx), d(y, Sy)\}.$$

Then S has a unique fixed point in X.

Remark : Finally, we also obtain Theorem 1 of Ciric [1] from Corollary 3.8. Thus, our main results are extensions of some well-known theorems[1, 3&6].

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