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MAGIC STRENGTH OF A GRAPH

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A Graph G is said to be magic if there exists a bijection $f: V \cup E \rightarrow \{1, 2, 3, ..., v+\varepsilon\}$ such that for all edges xy, $f(x) + f(y) + f(xy)$ is a constant. Such a bijection is called a magic labeling of G. In this paper, the concept of the magic strength of a graph is introduced. The magic strength of a graph G is denoted by $m(G)$ and is defined as the minimum of all constants where the minimum is taken over all magic labelings of G. The magic strength of some well-known graphs such as $P_n, C_n, (2n+1)P_2$ etc. are obtained in this paper. More over, relationship between k-sequential labeling and magic labeling is established which strengthens the conjecture that any tree is magic.

Key Words : Graph Labeling; Magic Labeling; Magic Strength

1. INTRODUCTION

In this paper, we consider only finite simple undirected graphs. Our notations and terminology are as in [1]. In particular, $\varepsilon(G)$ (or simply ε) denotes the number of edges in G. The graph P_n is the path on n vertices where as C_n is the cycle on n vertices. $B_{n,n}$ is the graph obtained from two copies of $K_{1,n}$ by joining the vertices of maximum degree by an edge which is called as n-bistar.

In 1970, Kotzig and Rosa[5] defined a magic labeling of a graph $G(V, E)$ as a bijection $f: V \cup E \rightarrow \{1, 2, 3, ..., v+\varepsilon\}$ such that for all edges $xy, f(x)+f(y)+f(xy)$ are the same.

For example, the magic labeling of some graphs are shown in Figure 1.

A graph G is said to be magic if it has a magic labeling. Ringel and Llado[6] called this graph edge-magic.

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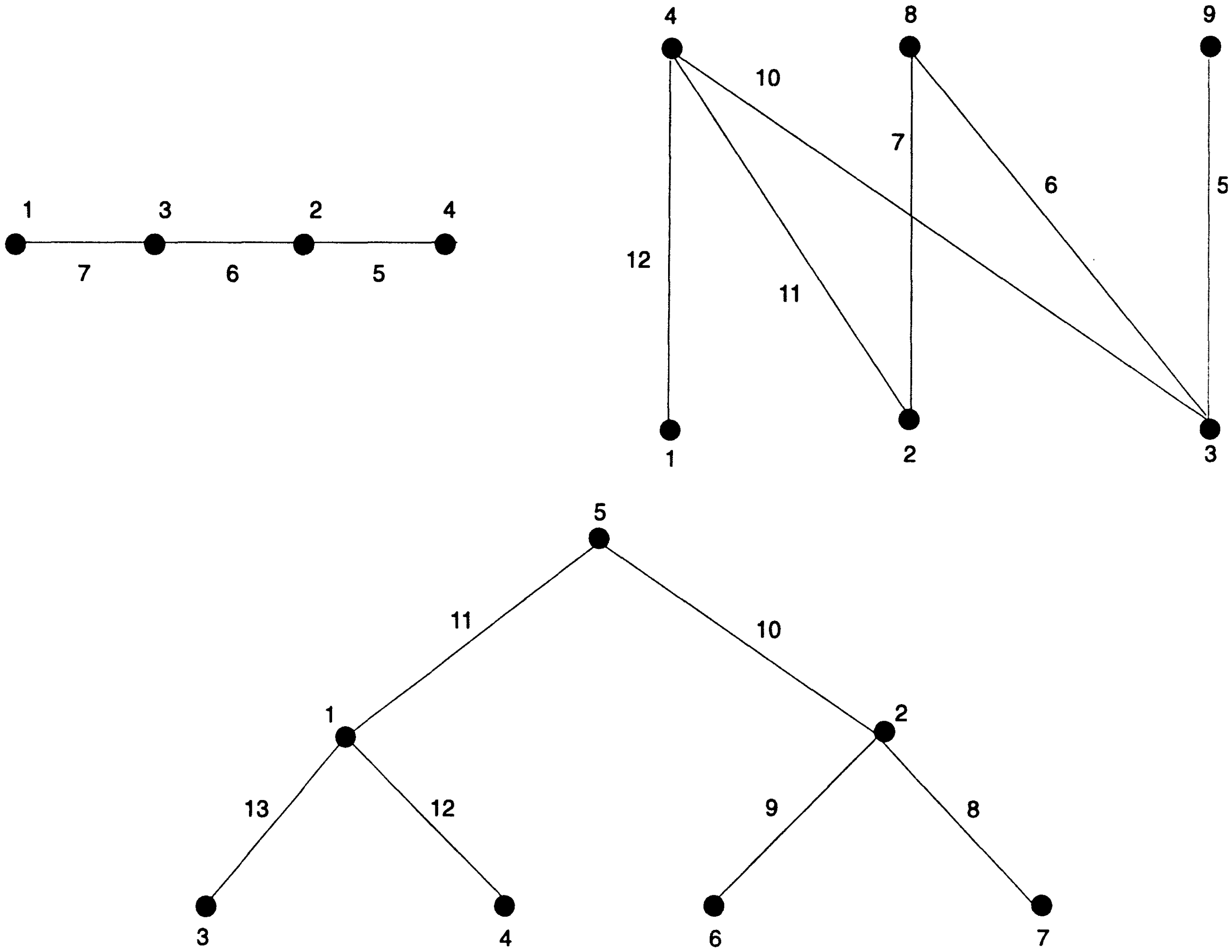

FIG. 1

They have proved that if G is a graph in which both ν and ε are even such that $\nu + \varepsilon \equiv 2 \pmod 4$ and in which each vertex has odd degree, then G is not magic. Also it was shown in [6] that each caterpillar is magic.

In [5], the following results have been proved:

1. $K_{m,n}$ is magic for all m and n.

2. C_n is magic for all $n \geq 3$.

3. nP_2 (the disjoint union of n copies of P_2) is magic if and only if n is odd.

The following conjecture has been raised in [5]:

Conjecture 1 — Every tree is magic.

If there is a bijection $f: V \cup E \to \{1, 2, 3, ..., \nu + \varepsilon\}$ such that for all edges xy, $f(x) + f(y) + f(xy)$ are all distinct, then G is called anti magic. (or edge anti-magic[6]) Ringel and Llado[6] proved that any graph is anti magic.

Many more results, conjectures and open problems on graph labeling have been discussed in [2] and [3].

In this paper, we introduce the concept of magic strength of a graph. We know that for any magic labeling f of G, there is a constant $c(f)$ such that $f(x) + f(y) + f(xy) = c(f)$ for any edge $xy \in E(G)$. The magic strength of G, $m(G)$, is defined as the minimum of all $c(f)$ where the minimum is taken over all magic labelings f of G. That is, $m(G) = \min \{c(f) : f \text{ is a magic labeling of } G\}$.

One can easily note that, since the labels are from the set $\{1, 2, 3, ..., \nu + \varepsilon\}$, $\nu + \varepsilon + 3 \le m(G) \le 2\nu + 2\varepsilon$.

In this paper, we find the magic strength of some well-known graphs.

Grace[4] defined a graph with ε edges to be k-sequential if there is an injection $f : V(G) \to \{0, 1, 2, ..., \varepsilon - 1\}$ (with the label ε also being allowed if G is a tree) such that when each edge uv is assigned the label $f(u) + f(v)$, the resulting edge labels form a sequence of distinct consecutive integers, say, $k, k+1, k+2, ..., k+\varepsilon-1$ for some k.

Here, we prove that if a tree T is k-sequential, then T is magic.

To proceed further, we make the following note:

Note 1 — Let f be a magic labeling of G with the constant $c(f)$. Then adding all the constants obtained at each edge. we get

$$\varepsilon\, c(f) = \sum_{v \in V} d(v) f(v) + \sum_{e \in E} f(e).$$

2. Magic Strength of some Trees

In this section, we obtain the magic strength of the path P_n on n vertices, the n-bistar $B_{n,n}$ obtained from two disjoint copies of $K_{1,n}$ by joining the center vertices by an edge and the tree $\langle K_{1,n} : 2 \rangle$ obtained from $B_{n,n}$ by subdividing the middle edge with a new vertex.

Lemma 1 — $m(P_{2n}) \le 5n + 1$ and $m(P_{2n+1}) \le 5n + 3$.

Proof : We prove this lemma by assigning a magic labeling to P_{2n} and P_{2n+1}. Let $v_1, v_2, v_3, \ldots, v_{2n}$ be the consecutive vertices and $e_1, e_2, e_3, \ldots, e_{2n-1}$ be the consecutive edges of P_{2n}. That is, $e_i = v_i v_{i+1}$, for $1 \le i \le 2n - 1$. Then the following labeling f is a magic labeling of P_{2n} :

$$f(v_{2i-1}) = i \text{ for } 1 \le i \le n, f(v_{2i}) = n + i \text{ for } 1 \le i \le n,$$

and

$$f(e_i) = 4n - i \text{ for } 1 \le i \le 2n - 1.$$

Similarly, we define a magic labeling g of P_{2n+1} as follows:

$$g(v_{2i}) = i \text{ for } 1 \le i \le n, g(v_{2i-1}) = n + i \text{ for } 1 \le i \le n + 1$$

and

$$g(e_i) = 4n + 2 - i \text{ for } 1 \le i \le 2n.$$

Thus $m(P_{2n}) \leq 5n+1$ and $m(P_{2n+1}) \leq 5n+3$.

For example, the magic labelings of P_8 and P_9 are shown in Fig. 2. ■

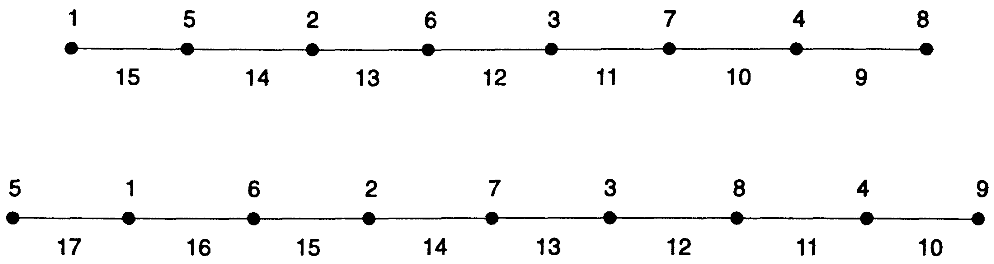

FIG. 2

Lemma 2 — $m(P_{2n}) \geq 5n+1$ and $m(P_{2n+1}) \geq 5n+3$.

PROOF : Here $\varepsilon(P_{2n}) = 2n-1$ and hence $\varepsilon + \nu = 4n-1$. Also by Note 1, if f is a magic labeling of P_{2n} with constant $c(f)$, then

$$\varepsilon c(f) = \sum_{v \in V} d(v) f(v) + \sum_{e \in E} f(e)$$

That is,

$$(2n-1)\, c(f) = \sum_{i=2}^{2n-1} 2f(v_i) + f(v_1) + f(v_{2n}) + \sum_{i=1}^{2n-1} f(e_i)$$

$$= \sum_{i=1}^{2n} f(v_i) + \sum_{i=1}^{2n-1} f(e_i) + \sum_{i=2}^{2n-1} f(v_i)$$

$$= (1+2+\ldots+4n-1) + \sum_{i=2}^{2n-1} f(v_i)$$

$$= (4n-1)\,(2n) + \sum_{i=2}^{2n-1} f(v_i)$$

Therefore,

$$c(f) = (4n-1)\,(2n)/(2n-1) + \sum_{i=2}^{2n-1} f(v_i)/(2n-1)$$

$$> (4n-1) + 2 + (1+2+3+\ldots+(2n-2))/(2n-1)$$

$$= (4n-1) + 2 + (n-1)$$

$$= 5n.$$

Thus $c(f) > 5n$ and hence, $c(f) \geq 5n + 1$ which implies that $m(P_{2n}) \geq 5n + 1$.

Similarly, we can prove that $m(P_{2n+1}) \geq 5n + 3$. ■

Combining Lemmas 1 and 2 we can state that

***Theorem* 1** — $m(P_{2n}) = 5n + 1$ *and* $m(P_{2n+1}) = 5n + 3$. ■

***Theorem* 2** — $m(B_{n,n}) = 5n + 6$.

PROOF : Let $V(B_{n,n}) = \{u, v; u_1, u_2, u_3, \ldots, u_n; v_1, v_2, v_3, \ldots, v_n\}$ and $E(B_{n,n}) = \{uu_i, vv_i, uv : 1 \leq i \leq n\}$ First we show that $m(B_{n,n}) \leq 5n + 6$ by giving a magic labeling of $B_{n,n}$. Consider the following labeling f of $B_{n,n}$:

$$f(u_i) = i \text{ for } 1 \leq i \leq n, f(v_i) = 2n + 2 + i \text{ for } 1 \leq i \leq n,\ f(u) = n + 2,\ f(v)$$

$$= n + 1,\ f(uv) = 3n + 3,\ f(uu_i) = 4n + 4 - i \text{ for } 1 \leq i \leq n \text{ and } f(vv_i)$$

$$= 2n + 3 - i \text{ for } 1 \leq i \leq n.$$

One can easily verify that f is a magic labeling of $B_{n,n}$ with $c(f) = 5n + 6$ and hence $m(B_{n,n}) \leq 5n + 6$.

For example, the magic labeling of $B_{5,5}$ is shown in Figure 3.

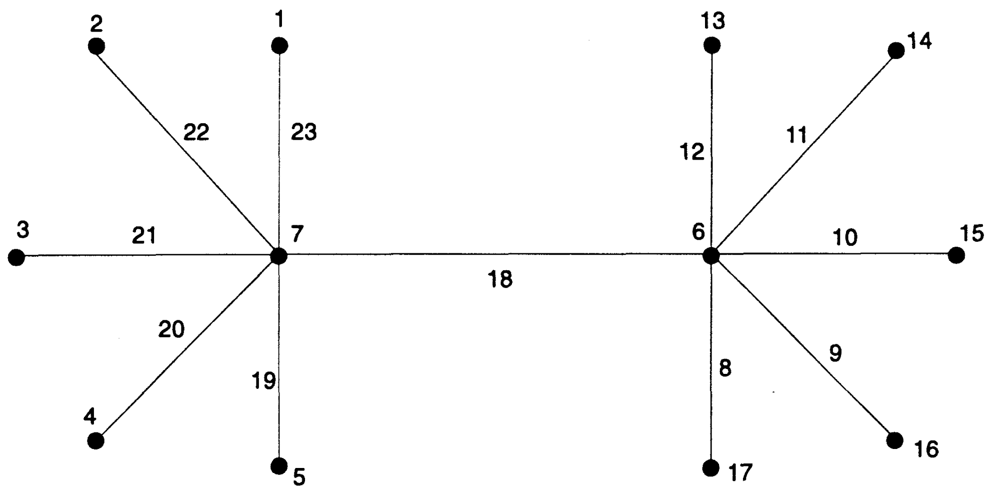

FIG. 3

Now it is enough to show that $m(B_{n,n} \geq 5n + 6$. Let f be a magic labeling of $B_{n,n}$ with constant $c(f)$. Then by Note 1,

$$\varepsilon\, c(f) = \sum_{v \in V} d(v) f(v) + \sum_{e \in E} f(e)$$

$$= \sum_{i=1}^{n} f(v_i) + \sum_{i=1}^{n} f(u_i) + (n+1) f(u) + (n+1) f(v) + \sum_{e \in E} f(e)$$

That is,

$$(2n+1)\,c(f) = \sum_{v \in V} f(v) + \sum_{e \in E} f(e) + nf(u) + nf(v)$$

$$= 1 + 2 + 3 + \ldots + (4n+3) + nf(u) + nf(v)$$

Therefore,

$$c(f) = (4n+3)(4n+4)/2(2n+1) + (n(fu) + f(v))/2n+1))$$

$$= (4n+5) + (1/2n+1)) + n(f(u)) + f(v))/(2n+1)$$

$$= 4n + 5 + (n(f(u)) + f(v)) + 1)/(2n+1).$$

Since $c(f)$ is an integer, $(n(f(u)+f(v))+1)/(2n+1)$ is also an integer. That is, $n(f(u)+f(v))+1 \equiv 0 \mod (2n+1)$ and hence $n(f(u)+f(v)) \equiv 2n \; mod\,(2n+1)$ which implies that $f(u)+f(v) \equiv 2 \mod (2n+1)$. But $f(u)+f(v) \geq 3$ and thus $f(u)+f(v) \geq 2n+3$.

Therefore, $c(f) \geq 4n+5+(n(2n+3)+1/(2n+1) = 5n+6$. This shows that $m(B_{n,n}) \geq 5n+6$ and hence we have $m(B_{n,n}) = 5n+6$. ■

Recall that the tree $\langle K_{1,n} : 2 \rangle$ is obtained from the n-bistar $B_{n,n}$ by subdividing the middle edge uv with a new vertex w. The magic strength of $\langle K_{1,n} : 2 \rangle$ is established in the following theorem.

Theorem 3 — $m(\langle K_{1,n} : 2 \rangle) = 4n+9$.

PROOF : First, we prove that $\langle K_{1,n} : 2 \rangle$ is magic. Define the labeling f on $\langle K_{1,n} : 2 \rangle$ as follows:

$$f(u) = 1, f(v) = 2, f(w) = n+3, f(u_i) = i+2 \text{ for } 1 \leq i \leq n, f(v_i) = n+3+i \text{ for } 1 \leq i \leq n,$$

$$f(uw) = 3n+5, f(vw) = 3n+4, f(uu_i) = 4n+6-i \text{ for } 1 \leq i \leq n, f(vv_i) = 3n+4-i$$

for $1 \leq i \leq n$.

One can check that the above labeling f is a magic labeling of $\langle K_{1,n} : 2 \rangle$ with $c(f) = 4n+9$. Thus $m(\langle K_{1,n} : 2 \rangle \leq 4n+9$.

For example, a magic labeling of $\langle K_{1,4} : 2 \rangle$ is shown in Figure 4.

Now we show that $m(\langle K_{1,n} : 2 \rangle \leq 4n+9$. Let f be a magic labeling of $\langle K_{1,n} : 2 \rangle$ with constant $c(f)$. Then by Note 1,

$$\varepsilon c(f) = \sum_{v \in V} d(v) f(v) + \sum_{e \in E} f(e)$$

That is,

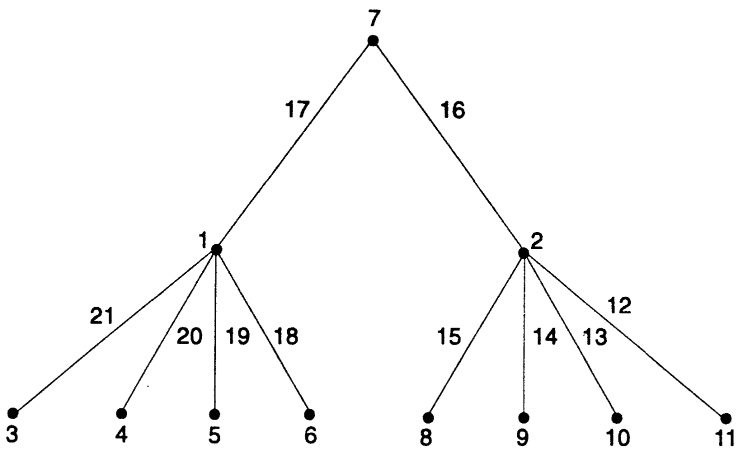

FIG. 4

$$(2n+2)\,c(f) = \sum_{i=1}^{n} f(u_i) + \sum_{i=1}^{n} f(v_i) + (n+1)f(u) + (n+1)f(v) + 2f(w) + \sum_{e \in E} f(e)$$

$$= \sum_{v \in V} f(v) + \sum_{e \in E} f(e) + n(f(u) + f(v)) + f(w)$$

$$= (1 + 2 + 3 + ... + 4n + 5) + n(f(u) + f(v)) + f(w)$$

$$= (4n+5)(4n+6)/2 + n(f(u) + f(v)) + f(w)$$

Therefore,

$$c(f) = (4n+5)(4n+6)/2(2n+2) + (n(f(u)) + f(v)) + f(w))/(2n+2)$$

$$= 4n + 7 + (n(f(u) + f(v)) + f(w) + 1)/(2n+2)$$

$$\geq 4n + 9 \text{ (since } f(u) + f(v) \geq 3 \text{ and since } c(f) \text{ is an integer).}$$

Thus $m(\langle K_{1,n} : 2 \rangle) \geq 4n+9$ which gives that $m(\langle K_{1,n} : 2 \rangle) = 4n+9$. ■

3. MAGIC STRENGTH OF nP_2

The graph nP_2 is defined as the disjoint union of n copies of P_2. In this section, we find the magic strength of nP_2 when n is odd. (Note that nP_2 is magic if and only if n is odd.)

***Theorem* 4** — *$m((2n+1)P_2) = 9n+6$, for any $n \geq 0$.*

PROOF : Let the vertices of $(2n+1)P_2$ be $u_1, u_2, u_3, \ldots, u_{2n+1}; v_1, v_2, v_3, \ldots, v_{2n+1}$ and let the edge set be $\{u_iv_i : 1 \leq i \leq 2n+1\}$. Define $f : V \cup E \rightarrow \{1, 2, 3, ..., 6n+3\}$ in such a way that $f(u_i) = i$ for $1 \leq i \leq 2n+1, f(v_i) = 6n+4-2i$ for $1 \leq i \leq n, f(v_{n+i}) = 6n+5-2i$ for $1 \leq i \leq n+1, f(u_iv_i) =$

$3n+2+i$ for $1 \le i \le n$ and $f(u_{n+i} v_{n+i}) = 2n+1+i$ for $1 \le i \le n+1$.

It is easy to check that f is a magic labeling of $(2n+1)P_2$ with $c(f) = 9n+6$. Thus $(2n+1)P_2$ is magic. For example, a magic labeling of $7P_2$ is shown in Figure 5.

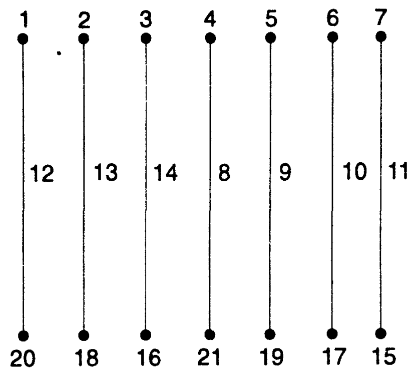

FIG. 5

It remains to show that $m((2n+1)P_2) = 9n+6$. Let f be a magic labeling of $(2n+1)P_2$ with constant $c(f)$. Then by Note 1, we have

$$\varepsilon c(f) = \sum_{v \in V} d(v) f(v) + \sum_{e \in E} f(e)$$

That is,

$$(2n+1)\, c(f) = \sum_{v \in V} f(v) + \sum_{e \in E} f(e)$$

$$= 1 + 2 + 3 + \ldots + (6n+3)$$

$$= (6n+3)(6n+4)/2$$

and hence

$$c(f) = 9n+6.$$

This is true for any magic labeling f of $(2n+1)P_2$. Therefore, $m((2n+1)P_2) = 9n+6$. ■

4 MAGIC STRENGTH OF C_n

In this section, we obtain the magic strength of C_n.

***Theorem* 5** — $m(C_{2n+1}) = 5n+4$.

PROOF : Let C_{2n+1} be $v_1 v_2 \ldots v_{2n+1} v_1$. Then the following labeling f is a magic labeling of C_{2n+1}:

$f(v_{2i+1}) = i + 1$ for $0 \le i \le n, f(v_{2i}) = n + i + 1$

for $1 \le i \le n, f(v_i v_{i+1}) = 4n + 2 - i$ for $1 \le i \le 2n$ and $f(v_{2n+1}\, v_1) = 4n + 2$.

For example, a magic labeling of C_7 is shown in Fig. 6.

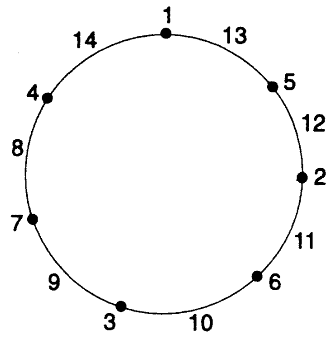

FIG. 6

Thus

$$m(C_{2n+1}) \le 5n + 4.$$

Now by Note 1, if f is a magic labeling of C_{2n+1}, then

$$(2n+1)c(f) = \sum_{v \in V} f(v) + \sum_{e \in E} f(e)$$

$$= \sum_{v \in V} f(v) + \sum_{e \in E} f(e) + \sum_{v \in V} f(v)$$

$$= 1 + 2 + \ldots + (4n+2) + \sum_{v \in V} f(v)$$

$$\ge (4n+2)\,(4n+3)/2 + (1 + 2 + \ldots + (2n+1))$$

$$= 5n + 4.$$

Therefore, $c(f) \ge 5n + 4$. This implies that $m(C_{2n+1}) \ge 5n + 4$ and hence $m(C_{2n+1}) = 5n + 4$.

■

Similar we can prove that $m(C_{2n}) = 5n + 2$. Here to prove that $m(C_{2n}) \le 5n + 2$ we use the labels 1, 2, 3,..., $2n - 1$; $3n$ to label the vertices and the remaining labels $2n$, $2n + 1$, ..., $3n - 1$; $3n + 1$, $3n + 2$, ..., $4n$ are used to label the edges so that the sum at each edge is $5n + 2$. For example the magic strength of C_8 is shown in Fig. 7.

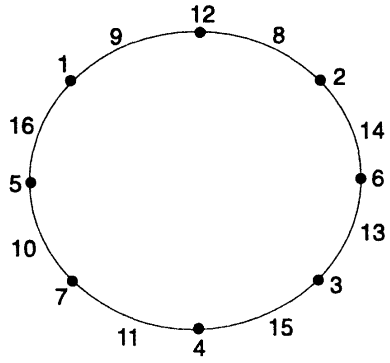

FIG. 7

5. SOME OBSERVATIONS

We conclude this paper with some observations.

First we observe an interesting result connecting the k-sequential and magic labelings.

Observation 1 — If there is a k-sequential labeling for a graph G using the labels $\{0, 1, 2, ..., \nu - 1\}$, then G is magic.

For, let $f: V(G) \to \{0, 1, 2, 3, ..., \nu - 1\}$ be a k-sequential labeling of G and let e_i be the edge xy for which $f(x) + f(y) = k - 1 + i$ for $1 \le i \le \varepsilon$. Now define a new labeling $g : V(G) \cup E(G) \to \{0, 1, 2, 3, ..., \nu + \varepsilon\}$ such that $g(v) = f(v) + 1$ for all $v \in V(G)$ and $g(e_i) = \nu + \varepsilon + 1 - i$ for $1 \le i \le \varepsilon$. Then g is a magic labeling of G with $c(f) = k + \nu + \varepsilon + 2$ and hence G is magic.

Observation 2 — Let G be graph with $\nu = \varepsilon$. If G is k-sequential, then G is magic.

Observation 3 — Let G be a 2-regular graph. If G is k-sequential, then G is magic.

Observation 4 — If a tree T is k-sequential, then T is magic.

For, as any k-sequential labeling of a tree uses the labels $\{0, 1, 2, 3, ..., \varepsilon\}$ and as $\varepsilon = \nu - 1$, it now follows from Observation 1.

Since many number of trees are shown to be k-sequential, the above observation strengthens Conjecture 1 which states that any tree is magic.

Observation 5 — $m(K_{n,m}) \le (m + 2)(n + 1)$ where $1 \le n \le m$. For, let (X, Y) be the vertex bipartition of $K_{n,m}$ where $X = \{u_1, u_2, u_3, \ldots, u_n\}$ and $Y = \{v_1, v_2, v_3, \ldots, v_m\}$. Denote the edge $u_i v_j$ by e_{ij} where $1 \le i \le n$ and $1 \le j \le m$. Then $\nu = m + n$ and $\varepsilon = mn$. Now the following labeling is a magic labeling of $K_{n,m}$ where $n \le m$:

$$f(u_i) = i \text{ for } 1 \le i \le n, f(v_j) = j(n + 1) \text{ for } 1 \le j \le m, f(e_{ij}) = (m - j + 2)(n + 1) - i$$

for $1 \le i \le n$ and $1 \le j \le m$.

Thus $m(K_{n,m}) \le (m + 2)(n + 1)$ where $n \le m$. For example a magic labeling of $K_{2,3}$ is shown in Fig. 8.

It seems that $m(K_{n,m}) = (m + 2)(n + 1)$ where $1 \le n \le m$.

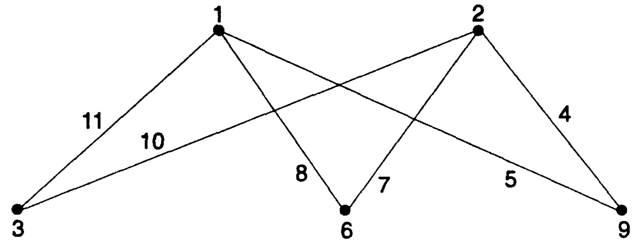

FIG. 8

Observation 6 — $m(K_{1,n}) = 2n + 4$.

Observation 7 — $m(P_n^2) = 3n$.

For, let $v_1, v_2, .., v_n$ be the vertices of P_n^2. Then $E(P_n^2) = \{v_i v_{i+1} : 1 \le i \le n-1;\ v_i v_{i+2} : 1 \le i \le n-2\}$. A magic labeling of P_n^2 is given below.

$$f(v_i) = i \text{ for } 1 \le i \le n; f(v_i v_{i+1}) = 3n - (2i+1) \text{ for } 1 \le i \le n-1;$$

and

$$f(v_i v_{i+2}) = 3n - (2i+2) \text{ for } 1 \le i \le n-2.$$

Therefore, $m(P_n^2) \le 3n$. But since $\varepsilon(P_n^2) = 3n - 3$, we have $m(P_n^2) \ge 3n$. Thus $m(P_n^2) = 3n$. For example a magic labeling of P_5^2 is shown in Fig. 9.

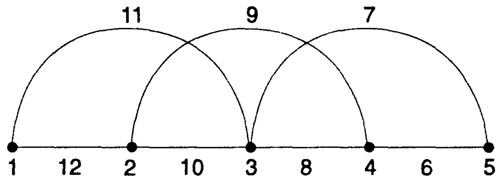

FIG. 9

Observation 8 — If a tree T is 1-sequential, then $m(T) = 2\varepsilon + 4$.

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