Indian J. pure appl. Math., **31**(8) : 1025-1043, August 2000

STABILITY OF AN IMPLODING PLASMA SHELL

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(*Received* 22 *March* 1999; *accepted* 19 *May* 1999)

Hydrodynamic stability of a finitely thick cylindrical plasma shell, which is imploded by a magnetic field idealized as a massless fluid, is studied. The plasma shell is assumed to consist of an isentropic, ideal and compressible gas ($p = a\,\rho^{\gamma}$ with $\gamma \neq 1$) which in the unperturbed basic flow undergoes a self-similar motion. The analytic description of this self-similar motion is presented in terms of the incomplete beta functions. Axisymmetric, incompressible disturbances, where velocity perturbations have radial and axial components only, are shown to give rise to an instability. The unstable disturbances, studied in this paper, grow faster than the unstable, two-dimensional, non-axisymmetric disturbances which were studied earlier and in which the velocity perturbations had only radial and azimuthal components. In the special cases of short wavelength disturbances, the perturbation in the radial component of velocity of a fluid particle on the outer surfcace of the plasma shel can be shown to vary as $(k/R_+)^{1/2}$, where k is the wave number of the disturbance and R_+ is the radius of the outer plasma surface.

Key Words : Hydrodynamic Stability; Plasma Shell; Genotypic & Compressible Gas; Asymmetric Disturbances

1. INTRODUCTION

Imploding cylindrical plasma shells may be considered as models for hollow z-pinches which are accelerated radially inward by the Lorentz force $\boldsymbol{J} \times \boldsymbol{B}$. The accelerated plasma is susceptible to the Rayleigh-Taylor instability in which the plasma may be regarded as a heavy fluid and the outside, driving magnetic field as the light fluid.

As pointed out by Han and Suydam[1], experiments (Springfield[2]; Baker[3]) show that the finite thickness of the shell is an important factor in the problem which should be taken into account. Such a finite thickness of the cylindrical plasma shell has been taken into account in the discussion of Han and Suydam[1] and Book and Bernstein[4]. The ratio of the two specific heats γ is taken as unity by Book and Bernstein[4], while Han and Suydam[1] have taken γ to be a number different from unity. In all these models the plasma is taken free from the magnetic field and the magnetic field outside the plasma is modeled to be replaced by a light fluid exerting a pressure on the imploding plasma boundary.

The basic flow of the plasma in the imploding compressible shell problem, before it is subjected to any small perturbations, is given by a self-similar motion (Sedov[5]; Han and Suydam[1]; Book and Bernstein[4]). The stability of the imploding cylindrical plasma shell has been considered by Han and Suydam[1], Book and Bernstein[4] against two-dimensional disturbances in which all quantities are independent of the axial coordinate z (the z axis coinciding with axis of the plasma shell), and the velocity perturbations are in the r-z planes.

The models of the imploding plasma shells considered by Han and Suydam[1], Book and Bernstein[4] are interesting since the inner radius of the plasma shell is taken as nonzero and the effect of compressibility of the plasma in influencing the unperturbed basic self-similar plasma flow and its stability has been considered by the authors in these references. However, as we noted already, their stability analysis is restricted to two-dimensional perturbations that are nonaxisymmetric and independent of the z-coordinate. The purpose of the present paper is to extend the stability analysis of Han and Suydam[1] by considering the stability of the compressible plasma shell against small axisymmetric perturbations in which velocity perturbations are in r-z planes. We have, for analytical tractability, considered the small perturbations, against which instability is being considered, as incompressible, although the basic unperturbed flow is taken as compressible. It is of interest to note that Han and Suydam[1] have also considered similar, incompressible but nonaxisymmetric two-dimensional perturbations and found that these perturbations were more unstable than the corresponding compressible perturbations. We find that axisymmetric, small, incompressible perturbations considered by us are unstable and they grow faster than the non-axisymmetric, two-dimensional, small and incompressible perturbations discussed in Han and Suydam[1].

The plan of the rest of the paper is as follows. We discuss the self-similar motion of the compressible plasma shell in the unperturbed state in section 2 and the analytic solution for the fluid motion is obtained in this section in tems of the incomplete beta functions. We discuss the stability of the problem, when disturbances are incompressible and axisymmetric, in section III. In the special case of disturbances of short wavelengths the perturbations in the radial component of velocity of a fluid particle on the outer surface of the plasma shell takes a simple form.

2. Equations of Motion

The equations for the plasma in the shell (Han and Suydam[1]; Book and Bernstein[4]), are

$$\rho \frac{dv}{dt} = -\nabla p, \qquad \text{... (1)}$$

$$\frac{\partial \rho}{\partial t} + div\,.\,(\rho v) = 0 \qquad \text{... (2)}$$

and

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0, \qquad \text{... (3)}$$

where ρ, p, v and γ are density, pressure, velocity and ratio of specific heats of the plasma, respectively. As in Han and Suydam[1], γ is to be taken different from 1 in our discussion.

We shall use the cylindrical polar system of coordinates (r, θ, z) with the axis of the unperturbed plasma shell taken as the z-axis. A fluid element, which was initially at $r_0\, \boldsymbol{e}_r + z_0\, \boldsymbol{e}_z$, where $\boldsymbol{e}_r$ and $\boldsymbol{e}_z$ are unit vectors along the radial and axial directions, respectively, will have, if there is no disturbance, the position vector given by

$$\boldsymbol{R} = R(r_0, t)\, \boldsymbol{e}_r + z_0\, \boldsymbol{e}_z, \qquad \text{... (4)}$$

To solve eqs. (1)-(3) analytically for describing the basic unperturbed flow we shall follow Han and Suydam[1] and Book and Bernstein[4], and introduce Sedov's similarity hypothesis and

Lagrangian variables taking

$$R = r_0 f(t). \quad \text{... (5)}$$

Eq. (5) implies that a fluid particle, initially at a distance r_0 from the axis, will move radially to a distance R at time t, so that $f(0) = 1$.

The density ρ and pressure p can be shown to be given by equations

$$\rho(R, t) = \frac{\rho_0 (r_0)}{f^2(t)}, \quad \text{... (6)}$$

and

$$p(R, t) = p_0 (r_0) f^{-2\gamma}. \quad \text{... (7)}$$

Here $\rho_0 (r_0)$ and $p_0(r_0)$ are density and pressure initially (at time $t = 0$) at a distance r_0 from the axis. We can write, using (3), the relation

$$p = a\rho^{\gamma}, \quad \text{... (8)}$$

where a is a constant.

Making use of (1), (5), (6), (7) and (8), we obtain

$$f''(t) f^{2\gamma-1}(t) = -\frac{a\gamma}{r_0} \rho_0^{\gamma-2} \frac{\partial \rho_0}{\partial r_0} = -1/t_c^2, \quad \text{... (9)}$$

where $-t_c^{-2}$ is the constant of separation of variables and t_c has the dimension of time, the minus sign on the right hand side of (9) corresponds to an imploding process. Solving the spatial part of (9), one obtains

$$\rho_0^{\gamma-1} \gamma a t_c^2/(\gamma-1) = (r_0^2 - r_-^2)/2, \quad \text{... (10)}$$

where r_- is the inner radius of the plasma shell. We have used the boundary condition $\rho(r_-) = 0$ to obtain (10). This boundary condition implies that the plasma density vanishes at the inner surface of the plasma shell both initially and subsequently at any other instant, (cf. (6)). In applying this boundary condition we follow Han and Suydam[1] and as mentioned by them, this mode has some expectation of representing the real physical situation in the stability problem of the imploding plasma shell which was studied in Los Alamos.

Taking ρ_+ as the value of. $\rho(r_0)$ at the outer boundary $(r_0 = r_+)$ of the plasma, we obtain from (10), the equation

$$\frac{\rho_0}{\rho_+} = \frac{(r_0^2 - r_-^2)^{1/\gamma-1}}{(r_+^2 - r_-^2)^{1/\gamma-1}}. \quad \text{... (11)}$$

Eqs. (6)-(8) and give us

$$\frac{p_0}{p_+} = \left\{ \frac{(r_0^2 - r_-^2)^{\frac{\gamma}{\gamma-1}}}{(r_+^2 - r_-^2)^{\frac{\gamma}{\gamma-1}}} \right\}, \quad \text{... (12)}$$

where p_+ is the value of $p_0(r_0)$ at the outer boundary $r_0 = r_+$ of the plasma shell. The results (6), (7), (9), (11) and (12) are given in Han and Suydam[1]. However, the factor a in (9) is missing in their result.

This omission affects neither their results (11) and (12) for the ratios ρ_0/ρ_+ and p_0/p_+ nor the value of the separation constant t_c^{-2} (see (13) below). The omission of a in (9) will have no effect in our analytic results since we finally deal with dimensionless quantities.

In view of (8) and (10), we have

$$2\gamma p_+ t_c^2 = (\gamma - 1)\, \rho_+ (r_+^2 - r_-^2) \quad \text{... (13)}$$

Eq. (13) determines the separation constant t_c^{-2} in terms of the initial value of pressure p_+ and density ρ_+ at the outer boundary of the plasma shell.

We integrate the temporal part of eq. (9) once, making use of the conditions

$$f = 1, \; f' = 0 \quad \text{... (14}a, b\text{)}$$

at $t = 0$, and obtain the result

$$t_c f' = -(\gamma - 1)^{-1/2} f^{1-\gamma} (1 - f^{2(\gamma-1)})^{1/2}. \quad \text{... (15)}$$

The initial condition (14b), in view of (5), merely states that the plasma shell initially starts from rest.

Putting

$$H = f^{2(r-1)}, \quad \text{... (16)}$$

We can integrate eq. (15) and obtain the result

$$\int_H^1 \frac{2-\gamma}{H^{2(\gamma-1)}}\, dH = \int_t^0 \frac{-2(\gamma-1)^{1/2}}{t_c}\, dt, \quad \text{... (17)}$$

where, in view of (14a) and (16), we have used the (initial) condition

$$H = 1 \quad \text{... (18)}$$

at $t = 0$. Since the beta function $B(p, q)$ and the incomplete beta function $B_x(p, q)$ are defined as (Pearson[6])

$$B(p, q) = \int_0^1 x^{p-1} (1 - x)^{q-1}\, dx \quad \text{... (19)}$$

and

$$B_x(p,q) = \int_0^x x^{p-1}(1-x)^{q-1}\,dx \qquad \dots (20)$$

respectively, we can obtain from (17) the relation

$$B\left(\frac{\gamma}{2(\gamma-1)}, \frac{1}{2}\right) - B_H\left(\frac{\gamma}{2(\gamma-1)}, \frac{1}{2}\right) = \frac{2t\,(\gamma-1)^{1/2}}{t_c} \,. \qquad \dots (21)$$

Eq. (21) defines H implicitly as a function of t. In view of (16) and (21), we can finally obtain the relation between f and t thus obtaining a description (cf. (5)) of the basic unperturbed plasma flow with time. Fig. 1 shows the variation of f with t/t_c for $\gamma = 5/3$ which is the value of the ratio of the specific heats taken for a fully ionized gas. Eq. (21) shows that the time for collapse of the plasma shell, t_0, is given by the relation

$$t_0/t_c = \frac{1}{2(\gamma-1)^{1/2}} \cdot B\left(\frac{\gamma}{2(\gamma-1)}, \frac{1}{2}\right). \qquad \dots (22)$$

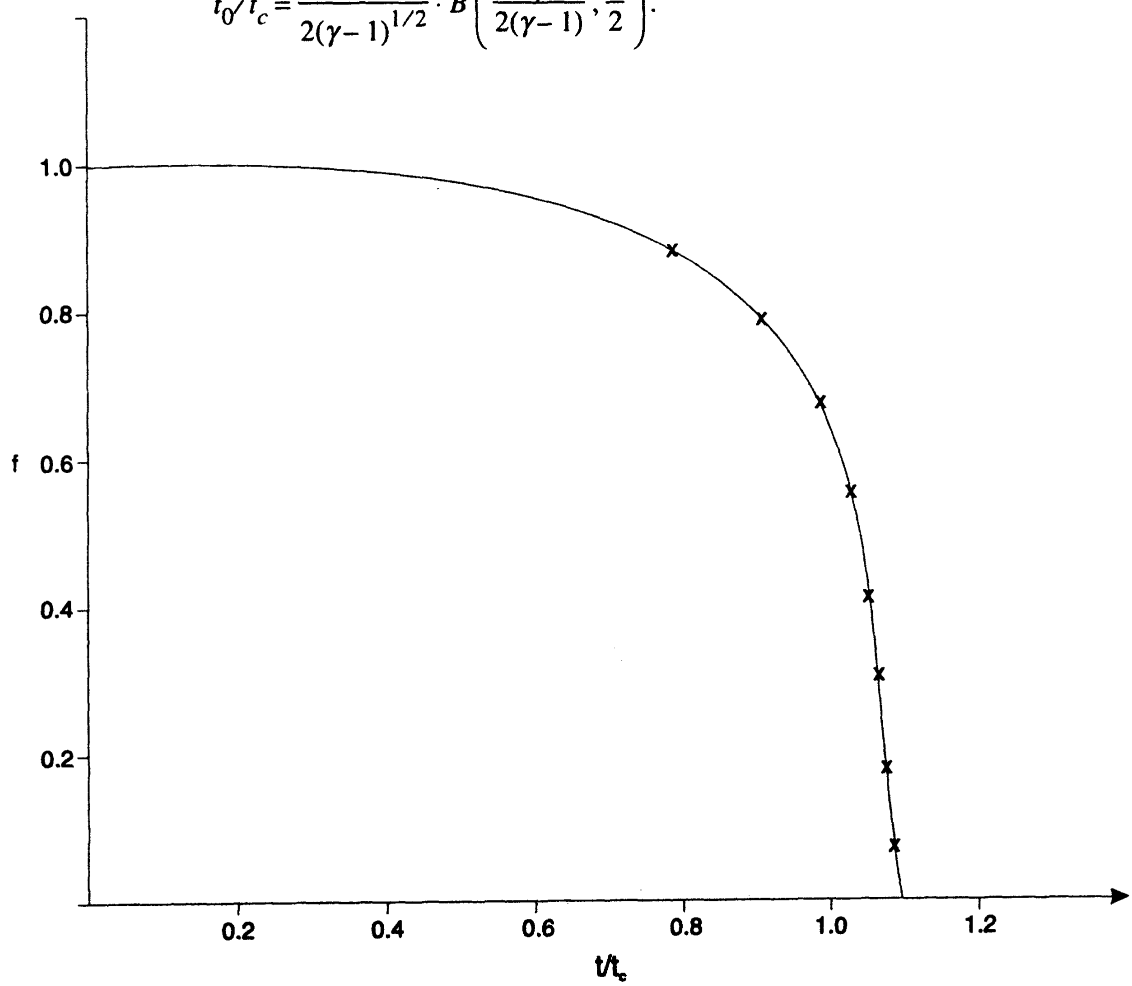

FIG. 1.

We shall use p_+, ρ_+, t_c and L as standard pressure, density time and length, respectively. Using these as standard quantities, we obtain the dimensionless pressure, density, time and position vector as

$$p^* = p/p_+,\ \rho^* = \rho/\rho_+,\ t^* = t/t_c \text{ and } r^* = r/L,$$

respectively. The basic eqs. (1)-(3), (8) and (9) can be written in dimensionless form as

$$\rho^* \frac{dv^*}{dt^*} = -\nabla p^*, \qquad \text{... (23)}$$

$$\frac{\partial \rho^*}{\partial t^*} + div^* (\rho^* V^*) = 0, \qquad \text{... (24)}$$

$$\frac{d}{dt^*}\left(\frac{p^*}{\rho^{*\gamma}}\right) = 0, \qquad \text{... (25)}$$

$$p^* = \rho^{*\gamma} \qquad \text{... (26)}$$

and

$$f^{2\gamma-1}\frac{d^2 f}{dt^{*2}} = -\frac{\gamma}{r^*}\rho^{*r-2}\frac{\partial \rho^*}{\partial r^*} = -1. \qquad \text{... (27)}$$

In deriving eqs. (23) and (27), we have assumed that the standard length L is chosen in such a way that the relation

$$\frac{p + t_c^2}{L^2 \rho_+} = 1$$

holds between p_+, t_c, ρ_+ and L.

In the further analysis, we shall use the dimensionless quantities and the dimensionless eqs. (23)-(27). We shall, however, omit the asterisks. Eqs. (23)-(27), when the asterisks are omitted, are the same in form as eqns (1)-(3), (8) and (9), if in the latter set of equations we take $t_c = 1$ and $a = 1$. Eqs. (4), (5), (8), (11), (12), (14 a, b), (15), (16), (21) and (22), when expressed in dimensionless forms, remain unchanged except that both t_c and a become 1. We shall, therefore, use these equations in further analysis taking $t_c = 1$ and $a = 1$.

3. Stability Analysis

If ξ is the small displacement of the fluid element from its unperturbed position $\boldsymbol{R}(r_0, e_r, z_0\ t)$ and at time t, where $\boldsymbol{R}(r_0, e_r, z_0, t)$ is given by (4), we can linearize eqs. (23)-(25) in small perturbations, and proceeding as in Han and Suydam[1], we obtain the equation

$$\rho(R)\,\xi'' = -(\nabla_R \cdot \xi)\,\nabla_R\,(p(R)) + \gamma\nabla_R\,(p(R)\,\nabla_R \cdot \xi) + \nabla_R\,\xi \cdot \nabla_R\,(p(R)), \qquad \text{... (28)}$$

where ∇_R is defined (cf. 4) as

$$\nabla_R = e_r \frac{\partial}{\partial R} + e_\theta \frac{1}{R}\frac{\partial}{\partial \theta} + e_z \frac{\partial}{\partial z_0}, \qquad \text{... (29)}$$

R being given by (5).

Han and Suydam have considered two-dimensional, nonaxisymmetric disturbances which are independent of z_0. The operator ∇_R in this case takes the form

$$\nabla_R = e_r \frac{\partial}{\partial R} + e_\theta \frac{1}{R}\frac{\partial}{\partial \theta}, \qquad \text{... (30)}$$

and, in view of (5) and (30), we have

$$\nabla_R = f^{-1}\,\nabla_{r_0}, \qquad \text{... (31}a\text{)}$$

where

$$\nabla_{r_0} = e_r \frac{\partial}{\partial r_0} + e_\theta \frac{1}{r_0}\frac{\partial}{\partial \theta}. \qquad \text{... (31}b\text{)}$$

As shown by Han and Suydam[1], the substitution (31*a*) enables us to solve (28), in the case of disturbances independent of z_0, by separation of dependence of r_0, θ and time t.

In the present problem of instability against axisymmetric disturbances, perturbations depend upon z_0, but not on θ. The operators ∇_R and ∇_{r_0} are now simplified (cf. (4)) to

$$\nabla_R = e_r \frac{\partial}{\partial R} + e_z \frac{\partial}{\partial z_0} \qquad \text{... (32}a\text{)}$$

and

$$\nabla_{r_0} = e_r \frac{\partial}{\partial r_0} + e_z \frac{\partial}{\partial z_0}. \qquad \text{... (32}b\text{)}$$

In view of (5), (32a) canm be reduced to

$$\nabla_R = e_r \frac{1}{f}\frac{\partial}{\partial r_0} + e_z \frac{\partial}{\partial z_0}. \qquad \text{... (33)}$$

Eqs. (32*b*) and (33) show that the relation (31*a*) does not hold now and hence a variable separable solution of (28) in which functions of r_0, z_0 and t separate is not possible.

In discussing instability against axisymmetric disturbances, we shall simplify the analysis considering only incompressible perturbations of the compressible basic flow and obtain a separation of variable solution for perturations. We find the Eulerian description for perturbations in fluid motion convenient and hence we do not adopt the Langrangian description, in the further analysis, for these perturbations.

The incompressible perturbations, against which instability is being considered, do not by themselves alter the density of a fluid particle, although a change in density occurs due to the undisturbed motion. We may note that in discussing the Rayleigh-Taylor instability in their problem, Book and Bernstein[4] and Han and Suydam[1] have considered the stability against these incompressible perturbations as special cases.

We may mention that Rayleigh[7] adopted the Eulerian description to discuss the fluid motion due to the collapse of a spherical cavity in a fluid. The spherical surface of the cavity formed the moving boundary of the fluid. In the present paper also we have fluid motion within the plasma shell, the inner and outer surfaces of which act as moving boundaries of the fluid.

We shall use the Eulerian coordinates (r, θ, z). A fluid particle, which, in the absence of any disturbances, is initially at a distance r_0 from the axis, will be at a distance r from it at time t when (cf. (5))

$$r = R(r_0, t) = r_0 f(t). \quad \text{... (34)}$$

The plasma shell, in the absence of any perturbations, is confined to the region $r_- f(t) < r < r_+ f(t)$ at time t. We shall assume the velocity perturbations to be irrotational and given as

$$\tilde{V} = -\nabla\tilde{\phi} \quad \text{... (35)}$$

where $\tilde{V}$ is small perturbation in the velocity of fluid.

The equation of continuity may be put as (Landau and Lifshitz[8])

$$\frac{d}{dt}(\log\rho) + div\, v = 0. \quad \text{... (36)}$$

Eq. (36) shows that the rate of change of log ρ for a given fluid particle, as it moves about, is negative of the divergence of the fluid velocity v. Since we consider incompressible disturbances which do not by themselves change the density of a fluid particle as it moves about, the perturbation in $\frac{d}{dt}(\log\rho)$ in (36) due to these disturbances vanishes, hence, in view of (36), the perturbation in div v also vanishes due to these disturbances. We shall, therefore, take

$$div\, \tilde{V} = 0, \quad \text{... (37)}$$

for incompressible disturbances. Eqs. (36) and (37) imply

$$\nabla^2\tilde{\phi} = 0, \quad \text{... (38)}$$

where

$$\tilde{\phi} = \hat{\phi}(r)\, T(t) \exp(ikz), \quad \text{... (39)}$$

for axisymmetric disturbances, k being the axial wave number (taken real). Eqs. (38) and (39) give

$$\hat{\phi} = AI_0(kr) + BK_0(kr), \quad \text{... (40)}$$

where A and B are constants and $I_0(kr)$ and $K_0(kr)$ are modified Bessel functions of first and second kinds, respectively.

The equations of motion of the fluid, after a small disturbance is given to the system and linearization in small perturbations is effected, can be put in the form

$$\rho\left\{\frac{\partial \tilde{v}}{\partial t}+\nabla(v\cdot\tilde{v})\right\}+\tilde{\rho}\left\{\frac{\partial v}{\partial t}+\nabla\left(\frac{v^2}{2}\right)\right\}=-\nabla\tilde{p}, \quad \text{... (41)}$$

where $\tilde{\rho}$ and $\tilde{p}$ are perturbations in fluid density and pressure, respectively at (r, θ, z) and at time t. The unperturbed velocity and density at (r, θ, z) and at time t are given by v and ρ, respectively. v has only one component v_r given by cf. (5))

$$v_r = r_0 f'(t) = r_0\frac{df}{dt}. \quad \text{... (42)}$$

v_r and ρ are given in terms of the variable r_0 and time t (cf. (6), (11) and (42)). We can express v and ρ in Eulerian co-ordinates r and time t using (34), and use these expressions in (41), (43) and (45) below.

As v^2 is a function of r and time t only, ∇v^2 is along the radial direction and since v is also in the same direction, the z-component of (41), in view (35) and (39), gives us the equation

$$\tilde{p}=\rho\left(\frac{\partial\tilde{\phi}}{dt}+v_r\frac{\partial\tilde{\phi}}{\partial r}\right). \quad \text{... (43)}$$

In deriving eq. (43), we assume that the z-dependence of $\tilde{p}$ is through the factor exp (ikz), as in the case of $\tilde{\phi}$ (cf. (39)). As we use the Eulerian description, $\tilde{p}$ in equation (43) gives us the local change in pressure due to perturbation at a fixed point (r, θ, z) and at time t, in contrast to the change in pressure δp due to a perturbation, in a fluid particle as it moves about (cf. (45) below).

If ξ is the displacement of a fluid particle on a surface of the plasma shell due to perturbations, then its radial component, may be given by

$$\xi_r = \hat{\xi}_r(t)\exp(ikz), \quad \text{... (44)}$$

and the change in pressure due to small perturbations in this fluid particle, as it moves about, is given by

$$\delta p = \tilde{p}+\xi\cdot\nabla p = \tilde{p}+\frac{\partial p}{\partial r}\xi_r, \quad \text{... (45)}$$

where $r = r_{\pm}f(t)$, and the upper or lower sign is taken according as the fluid particle is on the outer or inner surface of the plasma shell, respectively. But $\delta\rho$, the change in density due to incompressible perturbations in a fluid particle as it moves about, vanishes.

Since δp is proportional to $\delta\rho$) cf. (5) and (6) in Han and Suydam[1]), it also vanishes for an incompressible perturbation. (ρ_1 and p_1 in Han and Suydam[1] correspond to $\delta\rho$ and δp, respectively, in the present paper).

As the change in pressure in a fluid particle on the boundaries of the plasma shell ($r = r_{\pm}f(t)$) vanishes due to incompressible disturbances, we adopt

$$\delta p = 0 \qquad \text{... (46)}$$

on $r = r_{\pm}f(t)$ as boundary conditions of the problem.

Eq. (11) shows that $\dot{\rho}_0$ vanishes at the inner boundary $r_0 = r_-$, hence in view of (6) and (43), the density ρ and hence pressure perturbation $\tilde{p}$ vanish at the inner boundary $r_0 = r_-$. Eqs. (6) and (8) show that

$$\frac{\partial p}{\partial r} = a\gamma\rho^{\gamma-1}\frac{\partial \rho}{\partial r}$$

also vanishes, with ρ, at the inner surface $r_0 = r_-$ of the plasma shell. Combining these results, we find that, in view of (45), the boundary condition (46) is automatically satisfied on the inner plasma surface $r_0 = r_-$. The boundary condition (46), when applied to the outer boundary $r_0 = r_+$, will give us differential equation to describe the stability of the plasma shell.

We now apply this boundary condition (46) on the outer boundary $r_0 = r_+$. In view of (45), this boundary condition can be put as

$$\tilde{p} = -\frac{\partial p}{\partial r}\tilde{\xi}_r. \qquad \text{... (47)}$$

Using (43), we can write the equation (47) as

$$\rho\left(\frac{\partial\tilde{\phi}}{\partial t} + v_r\frac{\partial\tilde{\phi}}{\partial r}\right) = -\frac{\partial p}{\partial r}\tilde{\xi}_r. \qquad \text{... (48)}$$

We use (6), (7) and (34) and obtain from (48), the equation

$$\rho_0(r_0)f^{-2}\left(\frac{\partial\tilde{\phi}}{\partial t} + r_0\dot{f}\frac{\partial\tilde{\phi}}{\partial r}\right) = -\frac{\partial p_0(r_0)}{\partial r_0}f^{-2\gamma-1)}\xi_r, \qquad \text{... (49)}$$

holding on the outer surface $r_0 = r_+$ of the plasma shell.

The equation of the perturbed outer boundary surface of the plasma shell can be taken as

$$S = r - r_+f(t) - \tilde{\xi}_r = 0 \qquad \text{... (50)}$$

The general surface equation (Lamb[9]) is that at every point of the boundary surface, we have

$$\frac{\partial S}{\partial t} + v_r\frac{\partial S}{\partial r} + v_z\frac{\partial S}{\partial z} = 0, \qquad \text{... (51)}$$

where v_r and v_z are the r and z components of the fluid velocity $\boldsymbol{v}$. Linearizing eq. (51) in the small perturbations, we have the condition

$$\hat{v}_r = \frac{d\hat{\xi}_r}{dt} \quad \text{... (52)}$$

valid at the outer boundary $r = r_+ f(t)$. In deriving (52), we have used (44) and taken (ef. (35) and (39)

$$\tilde{v}_r = \hat{v}_r \exp(ikz). \quad \text{... (53)}$$

Eqs. (35), (39) and (53) show that

$$\hat{v}_r = -\frac{\partial \hat{\phi}}{\partial r} T(t). \quad \text{.. (54)}$$

Eqs. (52) and (54) show that

$$\frac{d\hat{\xi}_r}{dt} = -T(t)\frac{\partial \hat{\phi}}{\partial r} \quad \text{... (55)}$$

at the outer plasma boundary $r = r_+ f(t)$. Using (39) and (44) we obtain from the equation (49) the relation

$$f^{-2}\rho_0(r_0)\left(\frac{\partial T}{\partial t}\hat{\phi} + r_0 f'(t)\, T\frac{\partial \hat{\phi}}{\partial r}\right) = -\frac{\partial p_0(r_0)}{\partial r_0}\cdot \hat{\xi}_r \cdot f^{-2\gamma-1}$$

valid at

$$r_0 = r_+ f \text{ (i.e. at } r_0 = r_+ \text{ (cf. (34)).} \quad \text{... (56)}$$

Eliminating $\hat{\xi}_r$ between (55) and (56) we shall obtain a differential equation for $T(t)$ which will give us the growth of the disturbance with time. We shall first study the stability against a disturbance for which

$$\hat{\phi} = AI_0(kr) \quad \text{... (57)}$$

(so that we take $B = 0$ in (40)), and later on we shall consider separately the stability in the case when

$$\hat{\phi} = BK_0(kr), \quad \text{... (58)}$$

(so that $A = 0$ in (40)).

Case I : $\hat{\phi} = AI_0(kr)$

Using (57) in (56) and remembering that $r = r_+ f(t)$ (so that $r_0 = r_+$) at the outer boundary, we have

$$A\rho_0(r_+) f^{-2}\{kr_+ f'(t)\, TI_0'(kr_+ f) + I_0(kr_+ f)\, T'(t)\}$$

$$= -\left(\frac{\partial p_0(r_0)}{\partial r_0}\right)_{r_0=r_+} f^{-2\gamma-1}\hat{\xi}_r. \qquad \dots (59)$$

In view of (57), eq. (55), valid at the outer boundary $r = r_+ f(t)$, gives us the equation

$$\frac{d\hat{\xi}_r}{dt} = -AT(t)\, kI_0'(kr_+f). \qquad \dots (60)$$

Eliminating $\hat{\xi}_r$ from (59) and (60), we have

$$I_0'(kr_+f)\, kT = \frac{\rho_0(r_+)}{\left(\frac{\partial p_0(r_0)}{\partial r_0}\right)_{r_0=r_+}} f^{2\gamma-2}\ \{(2\gamma-1)\, I_0'(kr_+f)\, kr_+ f'^2 T$$

$$+ (2\gamma-1) f' I_0(kr_+f)\, T' + ff''\, I_0'(kr_+f) T + f(f' kr_+)^2 I_0''(kr_+f)\, T$$

$$+ ff' kr_+ I_0'(kr_+f)\, T' + ff'(kr_+)\, I_0'(kr_+f)\, T' + fI_0(kr_+f)\, T''(t)\} \qquad \dots (61)$$

We shall change the independent variable from t to f. We can thus write

$$\frac{dT}{dt} = \frac{dT}{df} f' = \frac{-1}{(\gamma-1)^{1/2}} f^{1-\gamma}(1-f^{2(\gamma-1)})^{1/2}\frac{dT}{df} \qquad \dots (62)$$

and

$$\frac{d^2T}{dt^2} = \frac{d^2T}{df^2} f'^2 + \frac{dT}{df} f'' = \frac{1}{(\gamma-1)}\frac{d^2T}{df^2}(f^{2(1-\gamma)}-1) - \frac{dT}{df} f^{1-2\gamma}, \qquad \dots (63)$$

where we have used the expressions for f'' and f' as given in (9) and (15), respectively. (we have taken $t_c = 1$ since we use dimensionless quantities).

Using (9), (15), (62) and (63), we obtain from (61) the relation

$$I_0'(kr_+f)\, kT = \frac{\rho_0(r_+)}{\left(\frac{\partial p_0(r_0)}{\partial r_0}\right)_{r=r_+}} \{[(2\gamma-1)\, I_0'(kr_+f)\,(kr_+)$$

$$+ (kr_+)^2 fI_0''(kr_+f)]\frac{T}{(\gamma-1)}(1-f^{2\gamma-2})$$

$$- (kr_+)\, I_0'(kr_+f)\, T + (2\gamma-1)\, I_0(kr_+f) + 2ff' kr_+ I_0'(kr_+f)$$

$$\left[\frac{1}{(\gamma-1)}(1-f^{2\gamma-2})\frac{dT}{df} - I_0(kr_+f)\frac{dT}{df} + \frac{I_0(kr_+f)}{(\gamma-1)} f(1-f^{2\gamma-2})\frac{d^2T}{df^2}\right] \qquad \dots (64)$$

As the plasma shell collapses, $t \to t_0$ and $f \to 0$, and the behaviour of T as $f \to 0$ will describe the stability of the plasma shell.

Before discussing the stability against disturbances of any general axial wave number k, we shall first consider the case of a disturbance with a large wave number k, so that, as the plasma shell collapses, although f becomes small, kr_+f is still large enough so that we can replace $I_0(kr_+f)$ by its asymptotic value (Copson[10]) given by

$$I_0(kr_+f) \sim \exp(kr_+f)/(2\pi kr_+f)^{1/2}. \qquad \text{... (65)}$$

In view of the last equation (65), eq. (64) can be reduced to

$$\frac{d^2T}{df^2} + 2kr_+\frac{dT}{df} + (kr_+)^2\, T = 0,$$

where f is small so that the solution for T can be put as

$$T = (C + Df)\exp(-kr_+f), \qquad \text{... (66)}$$

where C and D are arbitrary constants. Eqs. (32) and (40) show that, since $\tilde{v}_r = -\frac{\partial\tilde{\phi}}{\partial r}$ and (39) and (57) are true,

$$\tilde{v}_r = -Ak\, I_1(kr_+f)\, T\exp(ikz). \qquad \text{... (67)}$$

Using (65) and (66) in (67), we find that

$$\tilde{v}_r\, \alpha k^{1/2}(2\pi r_+f)^{-1/2}. \qquad \text{... (68)}$$

But R_+, the outer radius of the plasma shell, is given by (cf. (34))

$$R_+ = r_+f. \qquad \text{... (69)}$$

As the plasma shell collapses, (68) and (69) show that the variation of $\tilde{v}_r$, the perturbation in radial component of velocity of a fluid particle on the outer surface of the plasma shell, is given by the relation

$$\hat{v}_r\, \alpha\, (k/R_+)^{1/2} \qquad \text{... (70)}$$

In considering a disturbance with a wave number k not very large, we note that (Watson[11])

$$I_0(kr_+f) = 1 + \frac{(kr_+f)^2}{2^2(1\,!)^2} + \frac{(kr_+f)^4}{2^4(2\,!)^2} + \ldots \qquad \text{... (71)}$$

Using (71) and the infinite series expressions for $I_0'(kr_+f)$ and $I_0''(kr_+f)$ derived from (71), and retainng the dominant terms as $f \to 0$, we obtain from (64) the relation

$$\frac{d^2 T}{df^2} + \frac{\gamma}{f}\frac{dT}{df} + T(kr_+)^2 \frac{(\gamma+1)}{2} - \frac{k^2 r_+}{2P}(\gamma-1)\,T = 0 \qquad \text{... (72)}$$

valid for small f, where

$$P = \frac{\rho_0(r_+)}{\left(\dfrac{\partial p_0(r_0)}{\partial r_0}\right)_{r_0 = r_+}}. \qquad \text{... (73)}$$

The differential eq. (72) for T has a regular singularity at $f = 0$ (Ince[12]). Assuming a series solution for T in the form

$$T = \sum_{r=0}^{\infty} c_r f^{r+\sigma}, \qquad \text{... (74)}$$

where c_r's are constant coefficients and σ is the index, we find that the indicial equation for (72) becomes

$$\sigma(\sigma - 1 + \gamma) = 0 \qquad \text{... (75)}$$

giving two distinct indices $\sigma = 0,\ 1-\gamma$. As the unperturbed plasma shell is in motion and is accelerated towards the axis, we follow Book and Bernstein[4] and Han and Suydam[1], and consider the plasma shell as unstable if $\tilde{v}_r/v_r$ is unbounded as $f \to 0$. Using (35), (39) and (57), we find perturbations $\tilde{v}_r$ in the radial component of velocity at the outer boundary $r = r_+ f(t)$ is given by

$$\tilde{v}_r = -Ak\, I_0'(kr_+ f)\, T \exp(ikz). \qquad \text{... (76)}$$

In view of (15) and (42), the radial component of unperturbed velocity on the outer boundary surface $(r_0 + r_+)$ of the plasma shell is given by

$$v_r = -r_+ f^{1-\gamma}(1 - f^{2(\gamma-1)})^{1/2}(\gamma-1)^{-1/2}. \qquad \text{... (77)}$$

(As we use dimensionless quantities, we take $t_c = 1$ in (15).) As the plasma shell collapses $\tilde{v}_r/v_r$ takes the limiting value

$$\lim_{f\to 0} \frac{\tilde{v}_r}{v_r} = \lim_{f\to 0} A(\gamma-1)^{1/2} k f^{\gamma-1} I_0'(kr_+ f) f^{\sigma} \exp(ikz) = 0 \qquad \text{... (78)}$$

since σ is given by (75), and (71) is true. The plasma shell is thus stable against disturbances in which $\hat{\phi}$ is given by (57).

Case II : $\hat{\phi} = BK_0(kr)$

Using (58) in (56), we obtain

$$B\rho_0(r_+) f^{-2}\left[kr_+ f' K_0'(kr_+ f)\, T(t) + K_0(kr_+ f)\, T'(t)\right.$$

$$= -\left(\frac{\partial p_0(r_0)}{\partial r_0}\right)_{r_0=r_+} f^{-2\gamma-1}\hat{\xi}_r. \quad \text{... (79)}$$

In view of (58), eq. (55), valid at the outer surface of the plasma shell gives us the equation

$$d\hat{\xi}_r/dt = -BTkK_0'(kr_+f). \quad \text{... (80)}$$

Eliminating $\hat{\xi}_r$ from (70) and (80), we obtain the equation

$$K_0'(kr_+f)\,kT = \frac{\rho_0(r_+)}{\left(\frac{\partial p_0(r_0)}{\partial r_0}\right)_{r_0=r_+}} f^{2\gamma-2)}\{(2\gamma-1)(kr_+)f'^2 K_0'(kr_+f)\,T$$

$$+ (2\gamma-1)f'K_0(kr_+f)\,T' + ff''(kr_+)K_0'(kr_+f)\,T + ff'^2 kr_+)^2 K_0''(kr_+f)\,T$$

$$+ 2ff'(kr_+)K_0'(kr_+f)\,T' + fK_0(kr_+f)\,T''\}. \quad \text{... (81)}$$

Using (9) and (15) (with $t_c = 1$, since dimensionless quantities are used) and in view of (62) and (63), we obtain from (81) the equation

$$K_0'(kr_+f)\,kT$$

$$= \frac{\rho_0(r_+)}{\left(\frac{\partial p_0(r_0)}{\partial r_0}\right)_{r_0=r_+}} \cdot \Bigg[[(2\gamma-1)K_0'(kr_+f)\,kr_+ + (kr_+)^2 fK_0''(kr+f)]\frac{T}{\gamma-1}(1-f^{2\gamma-2})$$

$$-(kr_+)K_0'(kr_+f)\,T + [(2\gamma-1)K_0(kr_+f) + 2ff'(kr_+)K_0'(kr_+f)]$$

$$\times\frac{1}{(\gamma-1)}(1-f^{2\gamma-2})\frac{dT}{df} - K_0(kr_+f)\frac{dT}{df} + \frac{K_0(kr_+f)}{(\gamma-1)}f(1-f^{2\gamma-2}) - \frac{d^2T}{df^2}\Bigg] \quad \text{... (82)}$$

As the plasma shell collapses $t \to t_0$, the behaviour of T as $f \to 0$ will therefore describe the stability of the plasma shell.

Before describing the stability against disturbances of any general axial wave number k, we first consider the special case of a disturbance with large wave number k, so that, although, as the plasma shall collapses, f becomes small but kr_+f is still large enough so that we can approximate $K_0(kr_+f)$ by its approximate value (Copson[10]) given by

$$K_0(kr_+f) \sim \left(\frac{\pi}{2kr_+f}\right)^{1/2} \exp(-kr_+f). \quad \text{... (83)}$$

In view of the last result, eq. (82) can be reduced to

$$\frac{d^2T}{df^2} - 2\,kr_+\frac{dT}{df} + (kr_+)^2\,T = 0, \quad \text{... (84)}$$

so that the solution for T can be put as

$$T = (C' + D'f)\exp(kr_+f) \quad \text{... (85)}$$

where C' and D' are arbitrary constants. Arguing as in *case I* when $\hat{\phi}$ was given by (57), we find that in the present case also as the plasma shell collapses for a disturbance with a large wave number of k, $\tilde{v}_r$, the perturbations in the radial component of velocity of a fluidparticle on the outer surface of the plasma is given by the relation (70).

In discussing the stability of the plasma shell against a disturbance with a wave number k which is not large, we note that as the plasma shell collapses $f \rightarrow 0$ so that the argument kr_+f of $K_0(kr_+f)$, $K_0'(kr_+f)$ and $K_0''(kr_+f)$ is small and we can use the expressions (Watson[11]) of K_0, K_0' and K_0'' for small arguments. We find that eq. (82) finally takes the form

$$f^2\frac{d^2T}{df^2} + \gamma f\frac{dT}{df} + \left(1 - \frac{1}{r_+P}\right)\frac{(\gamma-1)}{\log f}\,T = 0, \quad \text{... (86)}$$

where P is defined by (73). Putting

$$F = \log f, \quad \text{... (87)}$$

we can write (86) as

$$\frac{d^2T}{dF^2} + (\gamma-1)\frac{dT}{dF} + \left(1 - \frac{1}{r_+P}\right)\frac{(\gamma-1)}{F}\,T = 0. \quad \text{... (88)}$$

We can express (88) as

$$\frac{d^2T}{dF^2} + (a_0 + a_1/F + a_2/F^2 + \ldots)\,\frac{dT}{dF} + (b_0 + b_1/F + b_2/F^2 + \ldots)\,T = 0, \quad \text{... (89)}$$

where

$$a_0 = \gamma - 1, a_1 = a_2 = a_3 = a_4 = \ldots = 0,$$

$$b_0 = 0, b_1 = (1 - 1/r_+P)\,\gamma - 1), b_2 = b_3 = \ldots = 0. \quad \text{... (90)}$$

The asymptotic solution of (89) can be given (Tricomi[13]; Nayfeh[14]) as

$$T \sim e^{\alpha F} \cdot F^{\beta}, \quad \text{... (91)}$$

where

$$\beta = \frac{-(a_1 \alpha + b_1)}{2\alpha + a_0}, \qquad \text{... (92)}$$

and where α is one of the roots of

$$\alpha^2 + a_0\alpha + b_0 = 0. \qquad \text{... (93)}$$

Two values of α, obtained as roots of (93), give two values of β from (92), and we have from (91), in view of (90), the following asymptotic solutions for T,

$$T \sim (\log f)^{-(1-1/r_+P)} \qquad \text{... (94)}$$

and

$$T \sim f^{1-\gamma}(\log f)^{(1-1/r_+P)}, \qquad \text{... (95)}$$

where $f \rightarrow 0$.

The ratio of the asymptotic value of T given by (95) to that given by (94) is $f^{1-\gamma}(\log f)^{2(1-1/pr_+)}$ and, since $1 - \gamma < 0$, this ratio tends to be unbounded as $f \rightarrow 0$ irrespective of the value of $1 - 1/(r_+ P)$. The asymptotic solution (95) is thus dominent as $f \rightarrow 0$ and we shall consider this solution to represent T as the plasma shell collapses ($f \rightarrow 0$).

Proceeding as in case I, we find that in the present case since $\hat{\phi}$ is given by (58), and (77) is true, the value of the ratio $\tilde{v}_r/v_r$ in the limiting case becomes

$$\lim_{f \rightarrow 0} \frac{\tilde{v}_r}{v_r} = \lim_{f \rightarrow 0} \frac{BkK_0'(kr_+f)\, T(\gamma - 1)^{1/2} \exp(ikz)}{r_+ f^{1-\gamma}(1 - f^{2(\gamma-1)})^{1/2}}. \qquad \text{... (96)}$$

Using the asymptotic solution (95) for T and the expression for $K_0(kr_+f)$ for the small value (Copson[10]) of the argument kr_+f, we find that, for a fluid particle on the outer surface of the plasma shell,

$$\lim_{f \rightarrow 0} \frac{\hat{v}_r}{v_r} = \lim_{f \rightarrow 0} \frac{-B(\gamma - 1)^{1/2}(\log f)^{(1-1/r_+P)}}{r_+^2 f} \qquad \text{... (97)}$$

which is unbounded. Thus, we find that the imploding plasma shell is unstable in case II, when $\hat{\phi}$ is given by (58).

A comparison for the growth rate of instability for the axisymmetric disturbances studied here with that of instability of two-dimensional non-axisymmetric disturbances considered in Han and Suydam[1] is interesting. They find the non-axisymmetric disturbances to be unstable, and as the shell collapses, $f \rightarrow 0$ and $\hat{\xi}_r/f$ diverges as $f^{(\gamma-2)/2}$ (cf. (66, 67) in their paper) so that $\hat{\xi}_r \propto f^{\gamma/2}$. This result also implies that, since $\frac{d\hat{\xi}_r}{dt} = \hat{v}_r$ (cf. (52)) and $\hat{v}_r = r_+ f'$ (cf. (42)) at the outer surface of the plasma shell,

$$\lim_{f \to 0} \left| \frac{\hat{v}_r}{v_r} \right| \propto f^{(\gamma - 2)/2} \qquad \ldots (98)$$

in Han and Suydam[1]. From (97) and (98) we find that, for a fluid particle on the outer surface of the plasma shell, the ratio

$$\frac{\hat{v}_r/v_r \text{ for unstable axisymmetric disturbances}}{\hat{v}_r/v_r \text{ for unstable non-axisymmetric disturbances}}$$

diverges as $(\log f)^{(1 - 1/r_+ P)} f^{-\gamma/2}$ when $f \to 0$ (cf. discussion after eq. (95)). We therefore conclude that as the plasma shell collapses, instability against axisymmetric disturbances studied by us grows faster than the instability against two-dimensional non-axisymmetric disturbances studied by Han and Suydam[1].

4. Concluding Remarks

In discussing the instability of an imploding plasma shell, we have considered axisymmetric, incompressible disturbances in which velocity perturbations have radial and axial components only ($\tilde{v}_z \neq 0$). Han and Suydam[1] have considered, among other cases, the case of non-axisymmetric, incompressible disturbances for which perturbations in velocity has no axial components ($\tilde{v}_z = 0$). Finally, we find that the perturbations considered by us are not only unstable but also grow faster with time than those considered in Han and Suydam[1].

We have shown that the ratio $\tilde{v}_r/v_r$ of the perturbations $\tilde{v}_r$ in the radial component of fluid velocity to v_r, its perturbed value, for a fluid particle situated on the outer surface $r = r_+ f(t)$ of the plasma shell tends to be unbounded as the plasma shell collapses ($f \to 0$). The instability discussed by us is thus located on the outer surface of the plasma shell and is a Rayleigh-Taylor instability since the heavy plasma at the outer plasma surface is accelerated towards the plasma axis by the pressure of the magnetic field outside the plasma, identified as a light (massless) fluid.

We have also found that (cf. (70)) $\tilde{v}_r$ varies as $(k/R_+)^{1/2}$ for disturbances with a large wave number k (short wavelength), R_+ being the outer radius of the plasma shell, as the shell collapses.

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