Indian J. pure appl. Math., **31**(8) : 1051-1058, August 2000
© Printed in India.

THE COLLAPSE OF A MODEL OF THE VOID IN A ROBERTSON-WALKER UNIVERSE WITH NONZERO SPATIAL CURVATURE

M. K. RAY*, S. CHAUDHURY** AND S. BANERJI+

**Department of Mathematics, Sarat Centenary College, Dhaniakhali, Hooghly, W. Bengal, India*
***Department of Physics, Gushkara Mahavidyalaya, Gushkara, Burdwan 713 128, India*
+Relativity & Cosmology Research Centre, Physics Department, Jadavpur University, Calcutta 700 032, India

(*Received 29 April* 1999; *accepted* 2 *September* 1999)

We consider the model of a spherical void (or its precursor) with heat conducting perfect fluid in Region I with Maiti metric surrounded by a spherical shell filled with radiation (Region II) having Vaidya metric. The combination is embedded in Robertson-Walker (RW) Universe in Region III. Earlier, Mandal and Banerji[11] took RW Universe with zero spatial curvature and showed that to an observer, a little way from the boundary in Region III, the void appears to collapse provided the arrow of time is future directed in all the regions. In this paper, we show that the same result holds also for the case of non-zero spatial curvature of Region III. The rate of collapse is the fastest for $k = +1$, moderate for $k = 0$, and the slowest for $k = -1$.

Key Words : General Relativity; Cosmology; Voids

1. INTRODUCTION

This paper is a generalisation of a paper by Mandal and Banerji[1] (this will be referred to as paper I) which considered the model of a collapsing void in a Robertson-Walker Universe with flat space sections. Although most cosmologists now believe that there is sufficient dark matter in the universe to make $k = 0$, there is no definite experimental evidence in favour of this belief. Recent experiments suggest[2, 3] that the average density of the Universe is less than critical, which signifies a value $k = -1$. For completeness we shall study here both the cases $k = \pm 1$, corresponding to the closed and open Universes respectively.

In paper I[1], we took the model of a spherical void containing low density conducting fluid (Region I) surrounded by a thick spherical shell of radiation (Region II) embedded in a Robertson-Walker (RW) Universe (Region III). Here we take the same model except that the space sections of the RW Universe are not flat. As before we again assume Region III to be filled with a perfect fluid with a linear equation of state. We showed in [1] that the matching conditions of the second fundamental forms involve two signs, one corresponding to the future directed time line and the other to the past directed one. This point has been emphasized by Goldwirth and Katz[4] and Fayos *et al.*[5]. In the earlier paper we showed that if the time coordinate in each region is future directed then a comoving observer situated in Region III, a little away from the boundary with

Region II finds the void (or its precursor) consisting of Regions I and II to go on contracting as the Universe expands. However, if the pressure in Region III vanishes (approximately the present day condition), the void remains static. We shall show here that the same conclusions hold even for an RW Universe with nonflat space sections. The only difference is that the rate of collapse of the void depends on the spatial curvature. We may choose various values of the constants in themetric in the core (Region I) of the void and its radius to get various relative rates of contraction of the core and the shell (Region II).

We refer the reader to the Introduction of paper[1] for a detailed discussion of the justification for this particular model of the void (or its precursor).

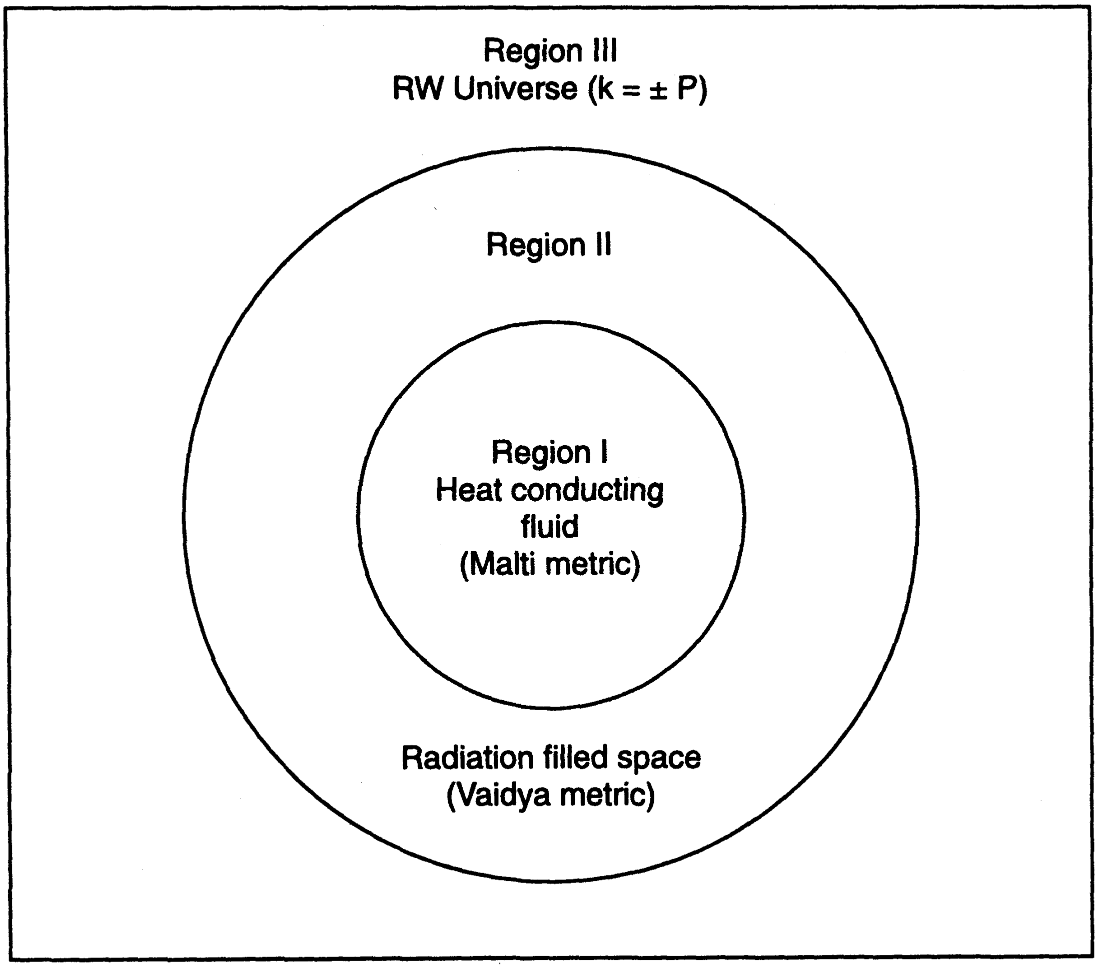

FIG. 1. Three Regions of Space-time

The high degree of isotropy of the microwave background radiation detected by the COBE satellite implies high isotropy of the Universe when matter decoupled from radiation. But we now see a large amount of structure in the Universe. So, many cosmologists have the impression that any small anisotropy present at the time of decoupling may have expanded to its present size. But we have shown in the earlier[1] as well as the present paper that in our model the voids (or their precursors) tend to disappear as the Universe expands. However, a void formed at the present epoch of matter domination or one left over from the past will remain static. This result may appear a bit surprising to the theorists.

In §2, we shall present the metrics in the three different regions and the field equations for Region III together with the solutions corresponding to a perfect fluid with a linear equation of state. In §3, we shall present the matching conditions between Regions I and II in brief as given in the earlier paper [1] and those of Regions II and III in details. In § 4, we shall discuss the main conclusions.

2. MODEL OF THE VOID

The core of the underdense region called void (Region I) has the metric of the form given by Maiti[6] and is supposed to be filled with a perfect fluid with thermal conductivity.

$$ds_I^2 = \left[1 + \frac{a}{1+\xi r_1^2} \right]^2 dt_1^2 - \frac{R^2(t_1)}{(1+\xi r_1^2)^2}(dr_1^2 + r_1^2 d\theta^2 + r_1^2 \sin^2\theta \, d\phi^2), \qquad \text{... (2.1)}$$

where a and ξ are constants.

In Region II, the metric is taken in the form of Vaidya[7] :

$$ds_{II}^2 = \left(1 - \frac{2m(v)}{r_2} \right) dv^2 + 2dv\, dr_2 - r_2^2 (d\theta^2 + \sin^2\theta \, d\phi^2), \qquad \text{... (2.2)}$$

In Region III, we have the RW metric with the spatial curvature $k = 1$ filled with a perfect fluid with $p = \gamma\rho$, $0 \le \gamma \le 1/3$:

$$ds_{III}^2 = dt_3^2 - \frac{S^2(t_3)}{(1+kr_3^2/4)^2}(dr_3^2 + r_3^2 d\theta^2 + r_3^2 \sin^2\theta \, d\phi^2). \qquad \text{... (2.3)}$$

The corresponding field equations are :

$$\frac{2\ddot{S}}{S} + \frac{\dot{S}^2}{S^2} + \frac{k}{S^2} = -8\pi p \qquad \text{... (2.4)}$$

and

$$\frac{\dot{S}^2}{S^2} + \frac{k}{S^2} = 8\pi\rho/3. \qquad \text{... (2.5)}$$

Here dot represents derivative with respect to t_3.

Putting $p = \gamma\rho$ in (2.4) and eliminating ρ between (2.4) and (2.5) we obtain

$$\frac{2\ddot{S}}{S} + (1+3\gamma)\frac{\dot{S}^2}{S^2} + \frac{k(1+3\gamma)}{S^2} = 0 \qquad \text{... (2.6)}$$

Case I — $k = +1$ we obtain after some manipulations

$$\int (\sin\theta)^{2/(1+3\gamma)} d\theta = g_1 t_3, \qquad \text{... (2.7)}$$

where g_1 is a constant of integration and

$$\sin \theta = h_1 S^{(3\gamma+1)/2} \qquad \text{... (2.7a)}$$

h_1 being a constant.

The integration in (2.7) cannot be performed for a general value of γ. We consider two of the cases where such integration is possible.

Case Ia — Matter dominated Universe where $\gamma = 0$. We may write[8]

$$S = L_1 \sin^2 \alpha_1/2, \qquad \text{... (2.8)}$$

where

$$t_3 = \frac{L_1}{2}(\alpha_1 - \sin \alpha_1). \qquad \text{... (2.8a)}$$

L_1 being a constant.

Case Ib — Radiation dominated Universe $\gamma = 1/3$. After a suitable scaling of t_3 we have

$$S = t_3^{1/2}(2 - t_3)^{1/2}, \qquad \text{... (2.9)}$$

$t_3 = 0, 2$ correspond respectively to the instants of big bang and recollapse (big crunch).

Case II — $k = -1$. We obtain as before

$$\int (\sinh \theta)^{2/(1+3\gamma)} d\theta = g_2 t_3, \qquad \text{... (2.10)}$$

g_2 being a constant and

$$\sinh \theta = h_2 S^{(3\gamma+1)/2}, \qquad \text{... (2.10a)}$$

h_2 being a constant.

As before the integration in (2.10) cannot be obtained in closed form for a general value of γ. We consider again two cases :

Case II a — $\gamma = 0$, we obtain [8] :

$$S = L_2 \sinh^2 \alpha_2/2 \qquad \text{... (2.11)}$$

$$t_3 = \frac{L_2}{2}(\sinh \alpha_2 - \alpha_2) \qquad \text{... (2.11a)}$$

Case II b — $\gamma = 1/3$:

$$S = t_3^{1/2}(t_3 + 2)^{1/2} \qquad \text{... (2.12)}$$

Here $t_3 = 0$ is the instant of the big bang.

3. MATCHING CONDITIONS

We shall equate the first and second fundamental forms at the boundary between Regions II and III while the matching at the other boundary between Regions I and II has already been done in paper I[1]. We shall simply quote the results of solutions obtained in the latter case. The boundary between I and II is assumed to be constant $r_1 = r_0$ in the comoving coordinates of Maiti metric[4] in Region . With this assumption we obtain [1] :

$$R = bt_1, \; b \text{ being a constant} \qquad \text{... (3.1)}$$

Further

$$m = cR = bct_1, \; c \text{ being another constant} \qquad \text{... (3.2)}$$

We take the intrinsic metric of the boundary surface Σ between Regions II and III in the form :

$$ds_\Sigma^2 = d\tau^2 - S^2(\tau)\,(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad \text{... (3.3)}$$

Matching the first fundamental forms we obtain

$$r_2 = \frac{S(t_s)r}{(1 + k/4\, r_3^2)} \qquad \text{... (3.4)}$$

$$\dot{t}_3^2 - \frac{S^2(t_3)}{(1 + k/4\, r_3^2)^2}\dot{r}_3^2 = 1 \qquad \text{... (3.5)}$$

and

$$\left[1 - \frac{2m(v)}{r_2}\right]\dot{v}^2 + 2\dot{r}_2\dot{v} = 1. \qquad \text{... (3.6)}$$

Here dot denotes derivative with respect to τ. It may be easily verified that putting $k = 0$ and $S(t_3) = t_3^n$ we obtain the eqs. (3.17), (3.18) and (3.10) of the earlier paper[1].

Matching the second fundamental forms we obtain further two equations :

$$\frac{r_3 S\dot{t}_3}{(1 + k/4\, r_3^2)} - \frac{k r_3^3 S\dot{t}_3}{2(1 + k/4\, r_3^2)^2} + \frac{S^2 r_3^2 \dot{r}_3 S^1}{(1 + k/4\, r_3^2)^3} = \left[1 - \frac{2m(v)}{r_2}\right]\dot{v}\, r_2 + \dot{r}_2 r_2 \qquad \text{... (3.7)}$$

and

$$\frac{m(v)\dot{v}}{r_2^2}-\frac{\ddot{v}}{\dot{v}}=\frac{\dot{S}t_3\ddot{r}_3}{1+k/4\,r_3^2}-\frac{kr_3r_3^2S\,\dot{t}_3}{2(1+k/4\,r_3^2)^2}+\frac{2\dot{r}_3\,\dot{t}_2^2S'}{1+k/4\,r_3^2}-\frac{S\dot{r}_3\,\ddot{t}_3}{1+k/4\,r_3^2}-\frac{S^2\dot{r}_3^3\,S'}{(1+k/4\,r_3^2)^3} \quad \ldots (3.8)$$

A dash represents derivative with respect to t_3.

We have retained only the first sign of the paper I[1], which was shown to correspond to future directed time lines. If $k = 0$ and $S = t_3^n$ eqs. (3.7) and (3.8) reduce to eqs. (3.19) and (3.20) of the paper I[1] with the first sign.

The solutions of these matching conditions are given at the boundary between Regions II and III by [9] (after a suitable scaling of t_3) :

$$r_3 = u_3 = 2\tan\left[\alpha_0 - \gamma/2\sin^{-1} S^{(1+3\gamma)/2}\right] \text{ for } k = +1 \quad \ldots (3.9a)$$

$$= u_3 = 2\tanh\left[\alpha_0 - \gamma/2\sinh^{-1} S^{(1+3\gamma)/2}\right] \text{ for } k = -1 \quad \ldots (3.9b)$$

$$= u_3 = 2\left[\alpha_0 - \gamma/2\,\frac{3(\gamma+1)}{3\gamma+1}\,S^{(1+3\gamma)/2}\right] \text{ for } k = 0. \quad \ldots (3.9c)$$

The last result was given in paper I[1]. Here $\alpha_0 > 0$.

S can, in principle, be determined as a function of t_3 using eqs. (2.7) and (2.7a) for $k = +1$, (2.10) and (2.10a) for $k = -1$. Further,

$S = t_3^n \left(\text{where } n = \dfrac{2}{3(\gamma+1)}\right)$ for $k = 0$ [1]. Hence, $S' = \dfrac{dS}{dt_3}$ can also be determined. The value $r_3 = u_3$ on the boundary is given as a function of t_3 by eqs. (3.9a) for $k = +1$ and (3.9b) for $k = -1$. Hence, for $k = +1$:

$$\dot{u}_3 = u'_3\,\dot{t}_3 = -\frac{\gamma(3\gamma+1)\,S'\,S^{(3\gamma-1)/2}\,\dot{t}_3}{2(1-S^{(1+3\gamma)})^{1/2}}\sec^2\phi,$$

where

$$\phi = \alpha_0 - \gamma/2\sin^{-1} S^{(1+3\gamma)/2}. \quad \ldots (3.10a)$$

For $k = -1$:

$$\dot{u}_3 = u'_3\,\dot{t}_3 = -\frac{\gamma(3\gamma+1)\,S'\,S^{(3\gamma-1)/2}\,\dot{t}_3}{2(1+S^{1+3\gamma})^{1/2}}\operatorname{sech}^2\phi,$$

where

$$\phi = \alpha_0 - \gamma/2\sinh^{-1} S^{(1+3\gamma)/2}. \quad \ldots (3.10b)$$

From (3.4)

$$r_2 = \frac{Su_3}{p}, \text{ where } p = (1 + k_{/4}\, u_3^2). \qquad \ldots (3.11a)$$

Hence

$$\dot{r}_2 = \frac{\dot{t}_3}{2p^2}\,[2p\,(S' u_3 + S u_3') - kSu_3^2\, u_3'], \qquad \ldots (3.11b)$$

where $\dot{t}_3$ obtained from (3.5) is

$$\dot{t}_3 = \left[1 - \frac{S^2\, u_3'^2}{p^2} \right]^{-1/2} \qquad \ldots (3.12)$$

From (3.6) we obtain

$$\dot{v} = v'\,\dot{t}_3 = \frac{-\,\dot{r}_2 \pm (\dot{r}_2^2 + (1 - 2m/r_2))^{1/2}}{(1 - 2m/r_2)}. \qquad \ldots (3.13)$$

Eliminating $\dot{v}$ from (3.6) using (3.13) we obtain

$$2m = \dot{t}_3^2\, S \left(\frac{u_3}{p} \right)^3 (k + S'^2) \qquad \ldots (3.14)$$

Since we have determined all the quantities from the first four eqs. (3.4) to (3.7) and eq. (3.9), eq. (3.8) serves as the consistency condition. In general, the various quantities $\dot{r}_2$, S, $\dot{v}$, $\ddot{v}$ etc. cannot be written in closed form. However, in the special case when $\gamma = 1/3$ we have verified by substitution of the various quantities derived from (3.4)-(3.7) and (3.9 *a*, *b*) that (3.8) is actually satisfied. It is obvious from eqs. (3.9) that for matter dominated Universe, i.e. $\gamma = 0$ the boundary is static (u_3 is constant).

4. CONCLUSION

As in paper I [1], for this model of the void or its precursor (a combination of Regions I and II), if formed in the early universe (γ non zero) would go on collapsing while the Universe expands even if the spatial curvature is nonzero. But the rate of collapse depends upon the spatial curvature. The rate is fastest for $k = +1$, medium for $k = 0$ and slowest for $k = -1$.

In a nutshell, we may say that if an inhomogeneity occurs in a homogeneous and isotropic Universe having considerable amount of radiation, the Universe with higher average density is more efficient in removing the inhomogeneity than the one with lower.

At present our Universe has become matter dominated ($\gamma = 0$). Hence, if a void is formed now or one is left over from the past then its size will remain unaltered as seen by a comoving observer in Region III.

Our thanks are due to the Department of Science & Technology, Government of India and the University Grants Commission for financial support.

REFERENCES

1. D. R. Mandal and S. Banerji, *Pramana* **51** (1998) 673.
2. S. Perlmutter, G. Aldering, M. Della Valle, S. Deustua, R. S. Ellis, S. Fabbro, A. Fruchter, G. Goldhaber, D. E. Groom, I. M. Hook, A. G. Kim, M. Y. Kim, R. A. Knop, C. Lidman, R. G. McMahon, P. Nugent, R. Pain, N. Panagia, C. R. Pennypacker, P. Ruiz-Lapuente, B. Schaefer and N. Walton, *Nature* **391** (1998) 51.
3. P. M. Garnavich, R. P. Krishner, P. Challis, J. Tonry, R. L. Gilliland, R. C. Smith, A. Clocchiatti, A. Diercks, A. V. Filippenko, M. Hamuy, C. J. Hogan, B. Leibundgut, M. M. Philips, D. Reiss, A. G. Riess, B. P. Schmidt, R. A. Schommer, J. Spyromilio, C. Stubbs, N. B. Suntzeff, and L. Wells, *Astrophys. J. Lett.* **493** (1998) L53.
4. D. S. Goldwirth and J. Katz, *Class. Quantum Grav.* **12** (1995) 769.
5. F. Fayos, J. M. M. Senovilla and R. Torres, *Phys. Rev.* **54** D, (1996) 4862.
6. S. R. Maiti, *Phys. Rev.* **25D** (1982) 2518.
7. P. C. Vaidya, *Nature* **171** (1953) 260.
8. T. K. Dey and S. Banerji, *Phys. Rev.* **48D** (1993) 3478.
9. F. Fayos, X. Jaen, E. Llanta and J. M. M. Senovilla, *Class. Quantum Grav.* **8** (1991) 2057.