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A FEW NEW IDENTITIES OF THE ROGERS-RAMANUJAN TYPE-I

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We derive Rogers-Ramanujan Type Identities modulo $7s$, $9s$, $11s$, $13s$, $14s$, $21s$ and $27s$ where s is any finite positive integer. We deduce such identities modulo 14, 18, 21, 27, 39, 45, 49 and 54.

We conclude by proving a general result which gives Rogers-Ramanujan Type Identities modulo $7s$, $9s$, $11s$, $13s$, $15s$, ... onwards and thereby derive general results for Rogers-Ramanujan Type Identities modulo $14s$, $18s$, $22s$, $26s$, ... onwards, and modulo $21s$, $27s$, $33s$, ... onwards.

Key Words : Basic Hypergeometric Series; q-analogue of Saalschütz Theorem; Jacbobi's Triple Product Identity

1. INTRODUCTION

For $|q| < 1$, the q-shifted factorial is defined by $(a; q)_0 = 1$,

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k) \text{ for } n \geq 1, \text{ and } (a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

It follows that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

The multiple q-shifted factorial is

$$(a_1, a_2, ..., a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$$

and

$$(a_1, a_2 \ldots, a_m; q)_\infty = (a_1; q)_\infty \cdot (a_2; q)_\infty \ldots. (a_m; q)_\infty$$

The basic hypergeometric series is ${}_{p+1}\Phi_{p+r}\left(\begin{matrix} a_1, a_2, \ldots a_{p+1}; q; x \\ b_1, b_2, \ldots, b_{p+r} \end{matrix}\right)$

$$= \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \ldots (a_{p+1};q)_n x^n (-1)^{nr} q^{\frac{n(n-1)r}{2}}}{(q;q)_n (b_1;q)_n (b_2;q)_n \ldots (b_{p+r};q)_n}$$

The series ${}_{p+1}\Phi_{p+r}$ converges $\forall$ positive integers r and $\forall x$. For $r = 0$, it converges only when $|x| < 1$

1.1 The q-analogue of Saalschutz Theorem is ${}_3\Phi_2\left(\begin{matrix} e, f, q^{-n}; q \\ \frac{aq}{c}, \frac{cefq^{-n}}{a} \end{matrix}\right) = \frac{\left(\frac{aq}{ec}\right)_n \cdot \left(\frac{aq}{cf}\right)_n}{\left(\frac{aq}{c}\right)_n \cdot \left(\frac{aq}{cef}\right)_n}$

1.2 We require the following Jacobi's Triple Product Identity (see Andrews[2], 2.2.10 and 2.2.11)

$$\left(zq^{\frac{1}{2}}, z^{-1} q^{\frac{1}{2}}; q; q\right)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n \cdot z^n \cdot q^{\frac{n^2}{2}}$$

and its corollary

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1)n(n+1)i/2 - in} = \sum_{n=0}^{\infty} (-1)^n q^{\frac{(2k+1)n(n+1)}{2} - in} \cdot (1 - q^{(2n+1)i})$$

$$= \prod_{n=0}^{\infty} (1 - q^{(2k+1)(n+1)}) (1 - q^{(2k+1)n+i}) (1 - q^{(2k+1)(n+1)-i})$$

1.3 The following Lemma is due to Bailey:

If L is a non-negative integer, then

$$(aq;q)_\infty \sum_{n=0}^{\infty} a^n \cdot q^{n^2 - Ln} \cdot \beta_n = \sum_{j=0}^{L} \frac{(q^{-L};q)_j (-a)^j q^{\frac{j(j+1)}{2}}}{(q;q)_j} \sum_{n=0}^{\infty} a^n \cdot q^{n^2 - Ln + 2nj} \cdot \alpha_n$$

where

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q;q)_{n-k} (aq;q)_{n+k}}.$$

(For proof of this Lemma, see Verma and Jain[5], 2.15)

2. We Begin by Introducing the Following Transformation

$$ {}_{10}\Phi_9\left(\begin{matrix} a, q\sqrt{a} - q\sqrt{a}, b, x, -x, y, -y, q^{-n}, -q^{-n}; q; -\frac{a^3 q^{3+2n}}{bx^2y^2} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{x}, \frac{-aq}{x}, \frac{aq}{y}, \frac{-aq}{y}, -aq^{n+1}, aq^{n+1} \end{matrix}\right) $$

$$ = \frac{(a^2q^2;q^2)_n\left(\frac{a^2q^2}{x^2y^2};q^2\right)_n}{\left(\frac{a^2q^2}{x^2};q^2\right)_n\left(\frac{a^2q^2}{y^2};q^2\right)_n} \cdot {}_5\Phi_4\left(\begin{matrix} x^2, y^2, \frac{-aq}{b}, \frac{-aq^2}{b}, q^{-2n}; q^2; q^2 \\ -aq, -aq^2, \frac{a^2q^2}{b^2}, \frac{x^2y^2}{a^2}q^{-2n} \end{matrix}\right) \qquad \ldots (2.1) $$

Proof of 2.1 : For $|q|<1$ and $\left|\frac{aq}{bcd}\right|<1$, we have the following (q-analogue of Dougall's Summation formula) : (see Agarwal[1], 6.10 and Verma and Jain[5], 2.1)

$$ {}_6\Phi_5\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a},, c, d; q\frac{aq}{bcd} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d} \end{matrix}\right) = \prod\left(\begin{matrix} aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{cd}; q \\ \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{bcd} \end{matrix}\right) $$

Putting $c = -q^{-r}, d = q^{-r}$ in this, we get

$$ {}_6\Phi_5\left(\begin{matrix} a, q\sqrt{a}; -q\sqrt{a}, b, -q^{-r}; q^{-r}; q; \frac{-aq^{1+2r}}{b} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, -aq^{1+r}, aq^{1+r} \end{matrix}\right) = \frac{(a^2q^2;q^2)_r\left(-\frac{aq}{b};q\right)_{2r}}{(-aq;q)_{2r}\cdot\left(\frac{a^2q^2}{b^2};q^2\right)_r} $$

Multiplying both sides now by

$$ \frac{(x^2;q^2)_r(y^2;q^2)_r(q^{-2n};q^2)_r q^{2r}}{(q^2;q^2)_r(a^2q^2;q^2)_r\left(x^2y^2\frac{q^{-2n}}{a^2};q^2\right)_r}, $$

and summing for r from 0 to n, and then using the q-analogue of Sallschutz Theorem, we get 2.1.

By induction on p we get the following generalisation of 2.1 (see Verma and Jain[5], 4.1): For $p \geq 3$

$$ {}_{2p+4}\Phi_{2p+3} $$

$$\left(\begin{matrix} a, qa^{1/2}, -qa^{1/2}, b, x, -x, y-y, (c_{p-3}), (d_{p-3}), -q^{-n}, q^{-n}; q; \dfrac{-a^p q^{p+2n}}{bx^2 y^2 c_1 d_1 \dots c_{p-3} d_{p-3}} \\ a^{1/2}, -a^{1/2}, \dfrac{aq}{b}, \dfrac{aq}{x}, \dfrac{aq}{-x}, \dfrac{aq}{y}, \dfrac{-aq}{y}, \dfrac{aq}{(c_{p-3})}, \dfrac{aq}{(d_{p-3})}, -aq^{n+1}, aq^{n+1} \end{matrix}\right)$$

$$= \frac{(a^2q^2;q^2)_n \left(\dfrac{a^2q^2}{x^2y^2};q^2\right)_n}{\left(\dfrac{a^2q^2}{x^2};q^2\right)_n \left(\dfrac{a^2q^2}{y^2};q^2\right)_n} \sum_{r_1, r_2, \dots r_{p-3} \ge 0} \prod_{j=1}^{p-3}$$

$$\left\{\frac{\left(\dfrac{aq}{c_jd_j};q\right)_{r_j} \cdot (c_j;q)_{M_{j-1}} \cdot (d_j;q)_{M_{j-1}}}{(q;q)_j \left(\dfrac{aq}{c_j};q\right)_{M_j} \cdot \left(\dfrac{aq}{d_j};q\right)_{M_j}}\right\}$$

$$\frac{(b;q)_{M_{p-3}} \cdot (x^2;q^2)_{M_{p-3}} (y^2;q^2)_{M_{p-3}} (q^{-2n};q^2)_{M_{p-3}} \cdot q^{M_{p-3}(M_{p-3}+1)/2}}{(-aq;q)_{2M_{p-3}} \cdot \left(\dfrac{aq}{b;}q\right)_{M_{p-3}} \left(\dfrac{x^2y^2}{a^2} q^{-2n}; q^2\right)_{M_{p-3}}}.$$

$$\times \left(-\frac{a}{b} q^2\right)^{r_{p-3}} \cdot \left(-\frac{a^2q^3}{bc_{p-3}d_{p-3}}\right)^{r_{p-4}} \dots\dots \left(-\frac{a^{p-3}q^{p-2}}{bc_{p-3} \cdot d_{p-3} \dots c_2d_2}\right)^{r_1}$$

$$\times {}_5\Phi_4 \left(\begin{matrix} x^2 q^{2M_{p-3}}; y^2 q^{2M_{p-3}}, -\dfrac{aq^{1+M_{p-3}}}{b}, \dfrac{-aq^{2+M_{p-3}}}{b}, q^{-2n+2M_{p-3}}; q^2; q^2 \\ -aq^{1+2M_{p-3}} - aq^{2+2M_{p-3}}, \dfrac{a^2}{b^2} q^{2+2M_{p-3}}, \dfrac{x^2y^2}{a^2} q^{-2n+2M_{p-3}} \end{matrix}\right) \dots (2.2)$$

where $M_i = r_1 + r_2 + \dots + r_i$ and $M_1 = M_0 = 0$ and $(a_{M,N})$ stands for the $N - M + 1$ symbols $a_M, a_{M+1} \dots, a_N$ (when $M = 1$ we write (a_N) instead of $(a_{1,N})$.

3 ROGERS - RAMANUJAN TYPE IDENTITIES MODULO 13s: (s IS ANY FINITE POSITIVE INTEGER)

In 2.2, with $p = 4$, $c_1 = z, d_1 = -z, q = q^s$ and with $b, x, y, z \to \infty$ we get

$$\sum_{k=0}^{n} \frac{(aq^s;q^s)_{k-1}(1-aq^{2ks})(-1)^k a^{4k} q^{\frac{9k^2s-ks}{2}}}{(q^s;q^s)_k (q^{2s};q^{2s})_{n-k}(a^2q^{2s};q^{2s})_{n+k}}$$

$$= \sum_{r=0}^{\infty}\sum_{k=0}^{n-r} \frac{a^{3r+2k}\cdot q^{3r^2s+4rsk+2k^2s}}{(q^s;q^s)_r(-aq^s;q^s)_{k+2r}(q^{2s};q^{2s})_k(q^{2s};q^{2s})_{n-r-k}} \quad \text{... (3.1)}$$

The L.H.S. of Bailey's Lemma for $(q=q^{2s})$ gives

$$(aq^{2s};q^{2s})_\infty \sum_{n=0}^{\infty} a^n \cdot q^{2n^2s-2lsn} \sum_{k=0}^{n} \frac{\alpha_k}{(q^{2s};q^{2s})_{n-k}(aq^{2s};q^{2s})_{n+k}}$$

$$= (aq^{2s};q^{2s})_\infty \sum_{n=0}^{\infty} a^n \cdot q^{2n^2s-2Lns} \sum_{k=0}^{n} \frac{(aq^s;q^s)_{k-1}(1-aq^{2ks})(-1)^k a^{4k} q^{\frac{9k^2s-ks}{2}}}{(q^s;q^s)_k(q^{2s};q^{2s})_{n-k}(a^2q^{2s};q^{2s})_{n+k}}$$

$$\left(\textit{taking } \alpha_k = \frac{(aq^s;q^s)_{k-1}(1-aq^{2ks})(-1)^k a^{4k}\cdot q^{\frac{9k^2s-ks}{2}}}{(q^2;q^2)_k}, \text{ and } \alpha_0 = 1 \right)$$

$$= (aq^{2s};q^{2s})_\infty \sum_{n=0}^{\infty} a^n \cdot q^{2n^2s-2Lns} \sum_{r=0}^{\infty}\sum_{k=0}^{n-r} \frac{a^{3r+2k}\cdot q^{3r^2s+4rsk+2k^2s}}{(q^s;q^s)_r(-aq^s;q^s)_{k+2r}(q^{2s};q^{2s})_k(q^{2s};q^{2s})_{n-r-k}}$$

(on using 3.1)

$$= (aq^{2s};q^{2s})_\infty \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{a^{n+2n+3k}\cdot q^{2s(n^2+r^2+k^2-2nr+2nk)-2Ls(n-r+k)+s(3r^2+2k^2)}}{(q^s;q^s)_r(-aq^s;q^s)_{k+2r}\cdot(q^{2s};q^{2s})_k(q^{2s};q^{2s})_{n-2r}} \quad \text{... (3.2)}$$

The corresponding RHS of Bailey's Lemma for q^{2s} in place of q gives

$$\sum_{j=0}^{L} \frac{(q^{-2Ls};q^{2s})_j(-1)^j a^j q^{js(j+1)}}{(q^{2s};q^{2s})_j}$$

$$\times \sum_{n=0}^{\infty} \frac{a^{5n}\cdot(-1)^n(aq^s;q^s)_{n-1}(1-aq^{2ns})}{(q^s;q^s)_n}\cdot q^{\frac{13n^2s-ns}{2}-2Lns+4njs} \quad \text{... (3.3)}$$

We equate 3.2 and 3.3 (i.e we equate both sudes of Bailey's Lemma for $q=q^{2s}$ and α_k as above) ... (3.4)

Setting $a=1$, and $L=0, 1, 2$ successively in the identity 3.4 we get respectively, on using Jacobi's Triple Product Identity, the following :

$$\frac{(q^{2s};q^{2s})_\infty}{(q:q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{2s(n^2-2nr+2nk+r^2+k^2)+s(3r^2+2k^2)}}{(q^s;q^s)_r(-q^s;q^s)_{k+2r}(q^{2s};q^{2s})_k(q^{2s};q^{2s})_{n-2r}}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

where $n \not\equiv o, \pm 6s \pmod{13s}$ ... (3.5)

$$\frac{(q^{2s}q^{2s})_\infty}{(q:q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{2s(n^2+r^2+k^2-2nr+2nk)-2s(n-r+k)+s(3r^2+2k^2)}}{(q^s;q^s)_r(-q^s;q^s)_{k+2r}(q^{2s};q^{2s})_k(q^{2s};q^{2s})_{n-2r}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

$n \not\equiv o, \pm 4s \pmod{13s}$ $n \not\equiv o, \pm 3s \pmod{13s}$ $n \not\equiv o, \pm 4s \pmod{13s}$ ... (3.6)

$$\frac{(q^{2s}q^{2s})_\infty}{(q:q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{2s(n^2+r^2+k^2\,2nr+2nk)\,4s(n+r+k)rs\,(3r^2+2k^2)}}{(q^s;q^s)_r(-q^s;q^s)_{k+2r}(q^{2s};q^{2s})_k(q^{2s};q^{2s})_{n-2r}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\frac{1+q^{2s}}{q^{2s}}\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

$n \not\equiv o, \pm 2s \pmod{13s}$ $n \not\equiv o, \pm 3s$ $n \not\equiv o, \pm 4s \pmod{13s}$ ... (3.7)

Taking $s = 1, 2, 3, 4, \ldots.$ successively in the identities 3.5-3.7 we get many identities of the Rogers-Ramanujan Type modulo multiples of 13.

4. Rogers-Ramanujan Type Identities Modulo 14 s

The identity 2.2, for $p = 4$, $c_1 = z$, $d_1 = -z$, $q = q^{2s}$ and with $b, z \to 0, x, y \to \infty$ becomes

$$\sum_{k=0}^{n}\frac{(aq^{2s};q^{2s})_{k-1}(1-aq^{4ks})\,a^2(-1)^k q^{3k^2s-ks}}{(q^{4s};q^{4s})_{n-k}(q^{2s};q^{2s})_k(a^2q^{4s};q^{4s})_{n+k}}$$

$$\sum_{r=0}^{\infty}\sum_{k=0}^{n-r}\frac{(-1)^{r+k}a^{3r-2k}q^{6k^2+8rsk+3r^2s-rs}}{(q^{2s};q^{2s})_r(-aq^{2s};q^{2s})_{2r+2k}(q^{4s};q^{4s})_k(q^{4s};q^{4s})_{n-r-k}} \quad \ldots (4.1)$$

Equating both sides of Bailey's Lemma for

$$q=q^{4s},\ \alpha_k=\frac{(aq^{2s};q^{2s})_{k-1}(1-aq^{4ks})\,a^k(-1)^k q^{3k^2s-ks}}{(q^{2s};q^{2s})_k}$$

and $\alpha_0 = 1$, $a = 1$, we get (on using 4.1),

$$(q^{4s}; q^{4s})_\infty \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{r+k} q^{4s(n^2+r^2+k^2+2nk-2nr)-4Ls(n-r+k)+6k^2s+3r^2s-rs}}{(q^{2s}; q^{2s})_r (-q^{2s}; q^{2s})_{2r+2k} (q^{4s}; q^{4s})_k (q^{4s}; q^{4s})_{n-2r}}$$

$$= \sum_{j=0}^{L} \frac{(q^{-4Ls}; q^{4s})_j (-1)^j q^{2js(j+1)}}{(q^{4s}; q^{4s})_j} \sum_{n=0}^{\infty} (-1)^n (1 = q^{2ns}) q^{7n^2s-ns-4Lns+8njs}. \quad \text{... (4.2)}$$

Setting $L = 0$, 1 successively in the identity 4.2, and using Jacobi's Triple Product Identity, we get respectively,

$$\frac{(q^{4s}; q^{4s})_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{r+k} q^{4s(n^2+r^2+k^2+2nk-2nr)+6k^2s+(3r^2-r)s}}{(q^{2s}; q^{2s})_r (-q^{2s}; q^{2s})_{2r+2k} (q^{4s}; q^{4s})_k (q^{4s}; q^{4s})_{n-2r}} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

when $n \not\equiv 0, \pm 6\ s \pmod{14s}$. ... (4.3)

and

$$\frac{(q^{4s}; q^{4s})_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{r+k} q^{4s(n^2+r^2+k^2+2nk-2nr)-4s(n-r=k)+6k^2s+(3r^2-r)s}}{(q^{2s}; q^{2s})_r (-q^{2s}; q^{2s})_{2r+2k} (q^{4s}; q^{4s})_k (q^{4s}; q^{4s})_{n-2r}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

$n \not\equiv 0, \pm 2s \pmod{14s}$ and $n \not\equiv 0, \pm 4s \pmod{14s}$ respectively.

5. ROGERS-RAMANUJAN TYPE IDENTITIES MODULO 21 s : (WHERE s IS ANY FINITE NATURAL NUMBER)

The identity 2.2 for $p = 4$, $c_1 = z, d_1 = -z, q = q^{3s}$ with $x, y \to \infty$ and $b, z \to 0$

reduces to

$$(q^{6s}; q^{6s})_n (a^2q^{6s}; q^{6s})_n \sum_{k=0}^{n} \frac{(aq^{3s}; q^{3s})_{k-1} (1-aq^{6ks}) a^4 (-1)^k q^{\frac{9k^2s-3ks}{2}}}{(q^{3s}; q^{3s})_k (q^{6s}; q^{6s})_{n-k} (a^2q^{6s}; q^{6s})_{n+k}}.$$

$$= \sum_{r=0}^{\infty} \sum_{k=0}^{n-r} \frac{(-1)^k a^{r+2k} q^{\frac{9r^2s-3rs}{2}+9k^2s+12rsk}}{(q^{3s}; q^{3s})_k (-aq^{3s}; q^{3s})_{2r+2k} (q^{6s}; q^{6s})_k (q^{6s}; q^{6s})_{n-r-k}}. \quad \text{... (5.1)}$$

Equating both sides of Bailey's lemma for $q = q^{6s}$, $\alpha_0 = 1$,

$$\alpha_k = \frac{(aq^{3s}; q^{3s})_{k-1} \cdot (1 - aq^{6ks})\, a^k (-1)^k q^{\frac{9k^2 s - 3ks}{2}}}{(q^{3s}; q^{3s})_k}$$

we get

$$(aq^{6s}; q^{6s})_\infty \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^r a^{n+3k} \cdot q^{6s(n^2 - 2nr + 2nk + r^2 + k^2) - 6Ls(n-r+k) + \frac{9^2 s - 3rs}{2} + 9k^2 s}}{(q^{3s}; q^{3s})_r (-aq^{3s}; q^{3s})_{2r+2k} \cdot (q^{6s}; q^{6s})_k (q^{6s}; q^{6s})_{n-2r}}$$

$$= \sum_{j=0}^{L} \frac{(q^{-6Ls}; q^{6s})_j (-1)^j a^j q^{3j(j+1)^s}}{(q^{6s}; q^{6s})_j} \cdot \sum_{n=0}^{\infty} \frac{a^n (-1)^n (aq^{3s}; q^{3s})_{n-1} (1 - aq^{6ns})}{(q^{3s}; q^{3s})_n} \cdot q^{\frac{21n^2 s - 3ns}{2} - 6Lns + 12njs}$$

... (5.2)

on using 5.1 and after some simplification.

Setting $L = 0$ and $L = 1$ (with $a = 1$), successively in the identity 5.2, we get, upon using Jacobi's Identity

$$\frac{(q^{6s}; q^{6s})}{(q; q)_\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^r q^{6s(n^2 - 2nr + 2nk + k^2 + r^2) + \frac{9r^2 s - 3rs}{2} + 9k^2 s}}{(q^{3s}; q^{3s})_r (-q^{3s}; q^{3s})_{2r+2k} (q^{6s}; q^{6s})_k (q^{6s}; q^{6s})_{n-2r}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1 - q^n}, \; n \not\equiv 0, \pm 9s \pmod{21s} \qquad \text{... (5.3)}$$

and

$$\frac{(q^{6s}; q^{6s})_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^r q^{6s(n^2 - 2nr + 2nk + r^2 + k^2) - 6s(n-r+k) + \frac{9r^2 s - 3rs}{2} + 9k^2 s}}{(q^{3s}; q^{3s})_r (-q^{3s}; q^{3s})_{2r+2k} (q^{6s}; q^{6s})_k (q^{6s}; q^{6s})_{n-2r}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1 - q^n} + \prod_{n=1}^{\infty} \frac{1}{1 - q^n} \qquad \text{... (5.4)}$$

$n \not\equiv 0, \pm 3s \pmod{21s}$, $n \not\equiv 0, \pm 6s \pmod{21s}$ respectively.

6. Deduction

i)

$$\frac{(q^{2s}; q^{2s})_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{r+k} q^{2s(n^2 + r^2 + k^2 + 2nk - 2nr) + 3k^2 s + (3r^2 - r)\frac{s}{2}}}{(q^s; q^s)_r (-q^s; q^s)_{2r+2k} (q^{2s}; q^{2s})_k (q^{2s}; q^{2s})_{n-2r}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1 - q^n}, \; n \not\equiv 0, \pm 3s \pmod{7s} \qquad \text{... (6.1)}$$

ii)

$$\frac{(q^{2s};q^{2s})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{r+k}q^{2s(n^2+r^2+k^2+2nk-2nr)+3k^2s+\frac{(3r^2-r)s}{2}}}{(q^s;q^s)_r(-q^s;q^s)_{2r+2k}(q^{2s};q^{2s})_k(q^{2s};q^{2s})_{n-2r}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

$n\not\equiv 0,\pm s \pmod{7s}$ $n\not\equiv 0,\pm 2s \pmod{7s}$ respectively.

7. Rogers Ramanujan type Identities Modulo 27s

The identity 2.2 for $p=4$ $c_1=z, d_1=-z, q=q^{3s}$ and $b,x,y\to\infty, z\to 0$ gives

$$\sum_{k=0}^{n}\frac{(aq^{3s};q^{3s})_{k-1}(1-aq^{6ks})(-1)^k a^{2k}\cdot q^{\frac{15k^2s-3ks}{2}}}{(q^{3s};q^{3s})_k(a^2q^{6s};q^{6s})_{n+k}(q^{6s};q^{6s})_{n-k}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=\cdot}^{n-r}\frac{(-1)^r a^{2k}\cdot q^{6k^2s+12rsk+\frac{15r^2s-3rs}{2}}}{(q^{3s};q^{3s})_r(-aq^{3s};q^{3s})_{2r+2k}(q^{6s};q^{6s})_k(q^{6s};q^{6s})_{n-r-k}} \quad \dots (7.1)$$

Equating both sides of Bailey's Lemma for $a=a^2, q=q^{6s}$ and with $\alpha_0=1$,

$$\alpha_k=\frac{(aq^s;q^s)_{k-1}(1-aq^{6ks})(-1)^k a^k q^{\frac{15k^2s-3ks}{2}}}{(q^{3s};q^{3s})_k}$$

we get

$$(a^2q^{6s};q^{6s})_\infty\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{a^{2n-2r+4k}(-1)^r\cdot q^{6s(n^2-2nr+2nk+2r^2+2k^2)-6Ls(n-r+k)+\frac{3rs(r-1)}{2}}}{(q^{3s};q^{3s})_r(-aq^{3s};q^{3s})_{2r+2k}(q^{6s};q^{6s})_k(q^{6s};q^{6s})_{n-2r}}$$

$$=\sum_{j=0}^{L}\frac{(q^{-6Ls};q^{6s})_j(-1)^j a^{2j}q^{3js(j+1)}}{(q^{6s};q^{6s})_j}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n a^{4n}(aq^{3s};q^{3s})_{n-1}(1-aq^{6ns})}{(q^{3s};q^{3s})_n}\cdot q^{\frac{(27n^2-3n)s}{2}-6Lns+12njs}$$

... (7.2)

(on using 7.1)

Applying Jacobi's Triple Product Identity and setting $a=1$ and $L=0, 1, 2$ successively, 7.2 yields

$$\frac{(q^{6s};q^{6s})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^r\cdot q^{6s(n^2-2nr+2nk+2r^2+2k^2)+\frac{3rs(r-1)}{2}}}{(q^{3s};q^{3s})_r(-q^{3s};q^{3s})_{2k+2r}(q^{6s};q^{6s})_k(q^{6s};q^{6s})_{n-2r}}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

where $n \not\equiv 0, \pm 12s \pmod{27s}$ ... (7.3)

$$\frac{(q^{6s};q^{6s})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^r\cdot q^{6s(n^2-2nr+2nk+2r^2+2k^2)-6s(n-r+k)+\frac{3rs(r-1)}{2}}}{(q^{3s};q^{3s})_r(-q^{3s};q^{3s})_{2k+2r}(q^{6s};q^{6s})_k(q^{6s};q^{6s})_{n-2r}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad \text{... (7.4)}$$

$n \not\equiv 0, \pm 6s \pmod{27s}$, $n \not\equiv 0, \pm 9s \pmod{27s}$ respectively

and
$$\frac{(q^{6s};q^{6s})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^r\cdot q^{6s(n^2-2nr+2nk+2r^2+2k^2)-12s(n-r+k)+\frac{3rs(r-1)}{2}}}{(q^{3s};q^{3s})_r(-q^{3s};q^{3s})_{2k+2r}(q^{6s};q^{6s})_k(q^{6s};q^{6s})_{n-2r}}$$

$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{27ns(n+1)}{2}}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\frac{1+q^{6s}}{q^{6s}}\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad \text{... (7.5)}$$

$n \not\equiv 0, \pm 3s \pmod{27s}$, $n \not\equiv \pm 12s \pmod{27s}$ respectively.

8. Deduction : Rogers-Ramanujan Type Identities Modulo 9s

i)
$$\frac{(q^{2s};q^{2s})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^e\cdot q^{2s(n^2-2nr+2nk+2r^2+2k^2)+\frac{rs(r-1)}{2}}}{(q^s;q^s)_r(-q^s;q^s)_{2k+2r}(q^{2s};q^{2s})_k(q^{2s};q^{2s})_{n-2r}}=\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad \text{... (8.1)}$$

$n \not\equiv 0, \pm 4s (\mathrm{mod}\ 9s)$

ii)
$$\frac{(q^{2s};q^{2s})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^r\cdot q^{2s(n^2-2nr+2nk+2r^2+2k^2)+\frac{rs(r-1)}{2}-2s(n-r+k)}}{(q^s;q^s)_r(-q^s;q^s)_{2k+2r}(q^{2s};q^{2s})_k(q^{2s};q^{2s})_{n-2r}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad \text{... (8.2)}$$

$n \not\equiv 0, \pm 2\,s$ (mos $9s$) $n \not\equiv 0, \pm 3\,s \pmod{9s}$

and *iii)*
$$\frac{(q^{2s};q^{2s})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^r\cdot q^{2s(n^2-2nr+2nk+2r^2+2k^2)-4s(n-r+k)+\frac{rs(r-1)}{2}}}{(q^s;q^s)_r(-q^s;q^s)_{2k+2r}(q^{2s};q^{2s})_k(q^{2s};q^{2s})_{n-2r}}$$

$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{9ns(n+1)}{2}}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\frac{1+q^{2s}}{q^{2s}}\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

$n \not\equiv 0, \pm s$ (mos $9s$), $n \not\equiv 0, \pm 4\,s \pmod{9s}$ respectively. ... (8.3)

9. Rogers-Ramanujan Type Identities Modulo $9s$

In 2.1 with $q = q^s$ and letting $b \to 0, x, y \to \infty$ we get

$$\sum_{k=0}^{n} \frac{(aq^s; q^s)_{k-1} (1 - aq^{2ks}) (-1)^k q^{\frac{5k^2 s - ks}{2}}}{(q^s; q^s)_k (q^{2s}; q^{2s})_{n+k} (a^2 q^{2s}; q^{2s})_{n+k}} = \sum_{k=0}^{n} \frac{(-1)^k a^{2k} q^{\frac{3k^2 s}{2}}}{(q^{2s}; q^{2s})_k (-aq^s; q^s)_{2k} (q^{2s}; q^{2s})_{n-k}} \quad \ldots (9.1)$$

Equating both sides of Bailey's Lemma for

$$q = q^{2s}, \alpha_0 = 1 \ \alpha_k = \frac{(aq^s; q^s)_{k-1} (1 - aq^{2ks}) (-1)^k q^{\frac{5k^2 s - ks}{2}}}{(q^s; q^s)_k}, \text{ we get}$$

$$(a^2 q^{2s}; q^{2s})_\infty \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{a^{n+3k} (-1)^k q^{s(2n^2 + 4nk + 5k^2) - 2Ls(n+k)}}{(q^{2s}; q^{2s})_k (-aq^s; q^s)_{2k} (q^{2s}; q^{2s})_n}$$

$$= \sum_{j=0}^{L} \frac{(q^{-2Ls}; q^{2s})_j (-1)^j a^j q^{js(j+1)}}{(q^{2s}; q^{2s})_j} \sum_{n=0}^{\infty} \frac{a^n \cdot (-1)^n (aq^s; q^s)_{n-1} (1 - aq^{2ns}) q^{\frac{9n^2 s - ns}{2} - 2Lns + 4njs}}{(q^s; q^s)_n} \quad \ldots (9.2)$$

For $a = 1$, $L = 0$, $a = 1$, $L = 1$ and $a = 1$, $L = 2$ in succession the identity 9.2, on using Jacobi's identity, yields respectively

$$\frac{(q^{2s}; q^{2s})_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{s(2n^2 + 4nk + 5k^2)}}{(q^{2s}; q^{2s})_k (-q^s; q^s)_{2k} (q^{2s}; q^{2s})_n}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1 - q^n}, n \not\equiv 0, \pm 4s \pmod{9s} \quad \ldots (9.3)$$

$$\frac{(q^{2s}; q^{2s})_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{s(2n^2 + 4nk + 5k^2) - 2s(n+k)}}{(q^{2s}; q^{2s})_k (-q^s; q^s)_{2k} (q^{2s}; q^{2s})_n} = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} + \prod_{n=1}^{\infty} \frac{1}{1 - q^n} \quad \ldots (9.4)$$

$n \not\equiv 0, \pm 2s \pmod{9s}$, $n \not\equiv 0, \pm 3s \pmod{9s}$ respectively

and

$$\frac{(q^{2s}; q^{2s})_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{s(2n^2 + 4nk + 5k^2) - 4s(n+k)}}{(q^{2s}; q^{2s})_k (-q^s; q^s)_{2k} (q^{2s}; q^{2s})_n}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{9ns(n+1)}{2}} + \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \frac{1+q^{2s}}{q^{2s}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} \quad \dots (9.5)$$

$n \not\equiv 0, \pm s \pmod{9s}$, $n \not\equiv 0, \pm 4s \pmod{9s}$ respectively.

10. Rogers-Ramanujan type Identities Modulo 11s

In 2.1 with $q = q^s$ and setting $b, x, y \to \infty$ we get,

$$\sum_{k=0}^{n} \frac{(aq^s;q^s)_{k-1}(1-aq^{2ks})(-1)^k q^{\frac{7k^2s-ks}{2}}}{(q^{2s};q^{2s})_k (q^{2s};q^{2s})_{n-k} (a^2q^{2s};q^{2s})_{n+k}} = \sum_{k=0}^{n} \frac{a^{2k} q^{2k^{2s}}}{(q^{2s};q^{2s})_k (q^{2s};q^{2s})_{n-k} (-aq^s;q^s)_{2k}} \quad \dots (10.1)$$

Placing $a = a^2$ and $q = q^{2s}$ in Bailey's Lemma and using 10.1, we get

$$(a^2q^{2s};q^{2s})_\infty \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a^{2n+4k} q^{2n^2s+4k^2s+4nks-2Ls(n+k)}}{(q^{2s};q^{2s})_k (q^{2s};q^{2s})_n (-aq^s;q^s)_{2k}}$$

$$= \sum_{j=0}^{L} \frac{(q^{-2Ls};q^{2s})_j (-1)^j a^{2j} q^{js(j+1)}}{(q^{2s};q^{2s})_j} \sum_{n=0}^{\infty} \frac{a^{2n}(aq^s;q^s)_{n-1}(1-aq^{2ns})}{(q^s;q^s)_n} (-1)^n q^{\frac{11n^2s-ns}{2} - 2Lns + 4njs}$$

$$\text{(On taking } \alpha_0 = 1, \alpha_K = \frac{(aq^s;q^s)_{k-1}(-1)^k(1-aq^{2ks}) q^{\frac{7k^2s-ks}{2}}}{(q^s;q^s)_k} \quad \dots (10.2)$$

Setting $a = 1$ and then putting $L = 0, 1, 2$ successfively in the identity 10.2 we get respectively

$$(-q^s;q^s)_\infty \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{q^{2n^2s+4nks+4k^s s}}{(q^{2s};q^{2s})_k (q^{2s};q^{2s})_n (-q^s;q^s)_{2k}} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

where $n \not\equiv 0, \pm 5s \pmod{11s}$ ... (10.3)

$$(-q^s;q^s)_\infty \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{q^{2n^2s+4nks+4k^s s-2s(n+k)}}{(q^{2s};q^{2s})_k (q^{2s};q^{2s})_n (-q^s;q^s)_{2k}} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \prod_{n=1}^{\infty} \frac{1}{1-q^n} \quad \dots (10.4)$$

where $n \not\equiv 0, \pm 3s \pmod{11s}$, where $n \not\equiv 0, \pm 4s \pmod{11s}$ respectively

and

$$(-q^s;q^s)_\infty \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{q^{2n^2s+4nks+4k^s s-4s(n+k)}}{(q^{2s};q^{2s})_k (q^{2s};q^{2s})_n (-q^s;q^s)_{2k}} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

where $n \not\equiv 0, \pm s \pmod{11s}$

$$+\prod_{n=1}^{\infty} \frac{1}{1-q^n} + \frac{1+q^2}{q^2} \prod_{n=1}^{\infty} \frac{1}{1-q^n} \qquad \ldots (10.5)$$

where $n \not\equiv 0, \pm 2s \pmod{11s}$, where $n \not\equiv 0, \pm 5s \pmod{11s}$ respectively.

As remarked before, varying s over 1, 2, 3, ..., we get infinitely many identities of the Rogers-Ramanujan Type from the above results.

11. Particular Cases

(A) Identities Modulo 39 : Putting $s = 3$ in the identities 3.5-3.7 we get

$$\frac{(q^6;q^6)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^{3(2n^2+5r^2+4k^2-4nr+4nk)}}{(q^3;q^3)_r(-q^3;q^3)_{k+2r}(q^6;q^6)_k(q^6;q^6)_{n-2r}} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

where $n \not\equiv 0, \pm 18 \pmod{39}$ ... (11.1)

$$\frac{(q^6;q^6)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^{3(2n^2+5r^2+4k^2-4nr+4nk)-6(n-r+k)}}{(q^3;q^3)_r(-q^3;q^3)_{k+2r}(q^6;q^6)_k(q^6;q^6)_{n-2r}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \prod_{n=1}^{\infty} \frac{1}{1-q^n} \qquad \ldots (11.2)$$

where $n \not\equiv 0, \pm 12 \pmod{39}$, where $n \not\equiv 0, \pm 15 \pmod{39}$ respectively

$$\frac{(q^6;q^6)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^{3(2n^2+5r^2+4k^s-4nr+4nk)-12(n-r+k)}}{(q^3;q^3)_r(-q^3;q^3)_{k+2r}(q^6;q^6)_k(q^6;q^6)_{n-2r}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \frac{1+q^2}{q^2} \prod_{n=1}^{\infty} \frac{1}{1-q^n} \qquad \ldots (11.3)$$

where $n \not\equiv 0, \pm 6 \pmod{39}$ where $n \not\equiv 0, \pm 9$, where $n \not\equiv 0, \pm 12 \pmod{39}$ respectively.

(B) Identities Modulo 14

(*i*) Putting $s = 1$ in the identities 4.3-4.4 we get

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{r+k} q^{4n^2+7r^2+10k^2+8nk-8nk^r-r}}{(q^2;q^2)_r(-q^2;q^2)_{2r+2k} \cdot (q^4;q^4)_k(q^4;q^4)_{n-2r}} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

where $n \not\equiv 0, 6, 8 \pmod{14}$ ... (11.4)

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{r+k}q^{4n^2+7r^2+10k^2+8nk-8nr-4(n+k)+3r}}{(q^2;q^2)_r(-q^2;q^2)_{2r+2k}\cdot(q^4;q^4)_k(q^4;q^4)_{n-2r}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad \ldots (11.5)$$

$n\not\equiv 0, 2, 12 \pmod{14}$, $n\not\equiv 0, 4, 10 \pmod{14}$ respectively.

(C) Identities Modulo 21 and 27

(*i*) Setting $s = 1$ in the identities 5.3 and 5.4 we get respectively,

$$\frac{(q^6;q^6)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\cdot\frac{(-1)^r q^{3(2n^2+5k^2+2nk-6nr)+\frac{3(7r^2-r)}{2}}}{(q^3;q^3)_r(-q^3;q^3)_{2r+2k}\cdot(q^6;q^6)_k(q^6;q^6)_{n-2r}}=\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad \ldots (11.6)$$

where $n\not\equiv 0, \pm 9 \pmod{21}$

and

$$\frac{(q^6;q^6)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\cdot\frac{(-1)^r q^{3(2n^2+5k^2+2nk-6nr)+\frac{3(7r^2-r)}{2}-6(n-r+k)}}{(q^3;q^3)_r(-q^3;q^3)_{2r+2k}\cdot(q^6;q^6)_k(q^6;q^6)_{n-2r}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad \ldots (11.7)$$

$n\not\equiv 0, \pm 3 \pmod{21}$, $n\not\equiv 0, \pm 6 \pmod{21}$ respectively.

(*ii*) Setting $s = 1$ in the identities 7.3-7.5, we get

$$\frac{(q^6;q^6)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\cdot\frac{(-1)^r q^{6(n^2-2nr+2nk+2k^2)+\frac{3r(qr-1)}{2}}}{(q^3;q^3)_r(-q^3;q^3)_{2k+2r}\cdot(q^6;q^6)_k(q^6;q^6)_{n-2r}}=\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad \ldots (11.8)$$

where $n\not\equiv 0, \pm 12 \pmod{27}$

$$\frac{(q^6;q^6)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^r q^{6(n^2-2nr+2nk+2k^2)+\frac{3r}{2}(qr-1)-6(n-r+k)}}{(q^3;q^3)_r(-q^3;q^3)_{2k+2r}\cdot(q^6;q^6)_k(q^6;q^6)_{n-2r}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}. \quad \ldots (11.9)$$

$n\not\equiv 0, \pm 6 \pmod{27}$, $n\not\equiv 0, \pm 9 \pmod{27}$ respectively.

and

$$\frac{(q^6;q^6)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^r q^{6(n^2-2nr+2nk+2k^2)+\frac{3r}{2}(r-1)-12(n-r+k)}}{(q^3;q^3)_r(-q^3;q^3)_{2k+2r}\cdot(q^6;q^6)_k(q^6;q^6)_{n-2r}}$$

$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{27n(n+1)}{2}}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\frac{1+q^6}{q^6}\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

$n\not\equiv 0,\pm 3 \pmod{27}$, $n\not\equiv 0,\pm 12 \pmod{27}$ respectively

(D) Identities Modulo 18

Settting $s=2$ in 9.3-9.5 we get

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{4n^2+8nk+10k^2}}{(q^4;q^4)_k(-q^2;q^2)_{2k}\cdot(q^4;q^4)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

where $n\not\equiv 0,\pm 8 \pmod{18}$

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{4n^2+8nk+10k^2-4(n+k)}}{(q^4;q^4)_k(-q^2;q^2)_{2k}\cdot(q^4;q^4)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^3}+\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad \ldots (11.11)$$

$n\not\equiv 0,\pm 4 \pmod{18}$, $n\not\equiv 0,\pm 6 \pmod{18}$ respectively

and

$$\frac{(q^4;q^4)}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{4n^2+8nk+10k^2-8(n+k)}}{(q^4;q^4)_k(-q^2;q^2)_{2k}\cdot(q^4;q^4)_n}$$

$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{9n(n+1)}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\frac{1+q^4}{q^4}\prod_{n=1}^{\infty}\frac{1}{1-q^n} \ldots (11.12)$$

$n\not\equiv 0,\pm 2 \pmod{18}$, $n\not\equiv 0,\pm 8 \pmod{18}$ respectively.

(E) Identities Modulo 45

Putting $s=5$ in the identities 9.3-9.5, we get

$$\frac{(q^{10};q^{10})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{10n^2+20nk+25k^2}}{(q^{10};q^{10})_k(-q^5;q^5)_{2k}\cdot(q^{10};q^{10})_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad \ldots (11.13)$$

where $n\not\equiv 0,\pm 20 \pmod{45}$

$$\frac{(q^{10};q^{10})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{10n^2+20nk+25k^2-10(n+k)}}{(q^{10};q^{10})_k(-q^5;q^5)_{2k}\cdot(q^{10};q^{10})_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

... (11.14)

$n \not\equiv 0 \pm 10 \pmod{45}$, $n \not\equiv 0 \pm 15 \pmod{45}$ respectively

and

$$\frac{(q^{10};q^{10})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{10n^2+20nk+25k^2-20(n+k)}}{(q^{10};q^{10})_k(-q^5;q^5)_{2k}\cdot(q^{10};q^{10})_n}$$

$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{45n(n+1)}{2}}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\frac{1+q^{10}}{q^{10}}\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad \ldots (11.15)$$

$n \not\equiv 0 \pm 5 \pmod{45}$, $n \not\equiv 0 \pm 20 \pmod{45}$ respectively.

(*F*) Identities Modulo 49

Putting $s = 7$ in 6.1 and 6.2 we get

$$\frac{(q^{14};q^{14})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{r+k}q^{14(n^2+r^2+k^2+2nk-2nr)+21k^2+\frac{7r}{2}(3r-1)}}{(q^7;q^7)_r(-q^7;q^7)_{k+2r}(q^{14};q^{14})_k(q^{14};q^{14})_{n-2r}}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

$n \not\equiv 0, \pm 21 \pmod{49}$

and

$$\frac{(q^{14};q^{14})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{r+k}q^{14(n^2+r^2+k^2+2nk-2nr)-14(n-r+k)+21k^2+\frac{7r}{2}(3r-1)}}{(q^7;q^7)_r(-q^7;q^7)_{k+2r}(q^{14};q^{14})_k(q^{14};q^{14})_{n-2r}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

$n \not\equiv 0, \pm 7 \pmod{49}$, $n \not\equiv 0, \pm 14 \pmod{49}$ respectively:

(*G*) Identities Modulo 54

(*i*) Setting $s = 6$ in 9.3-9.5, we get

$$\frac{(q^{12};q^{12})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{12n^2+24nk+30k^2}}{(q^{12};q^{12})_k(-q^6;q^6)_{2k}\cdot(q^{12};q^{12})_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

where $n \not\equiv 0, \pm 24 \pmod{54}$

$$\frac{(q^{12};q^{12})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{12n^2+24nk+30k^2-12(n+k)}}{(q^{12};q^{12})_k(-q^6;q^6)_{2k}\cdot(q^{12};q^{12})_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

$n \not\equiv 0, \pm 12 \pmod{54}$, $n \not\equiv 0, \pm 18 \pmod{54}$ respectively

$$\frac{(q^{12};q^{12})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{12n^2+24nk+30k^2-24(n+k)}}{(q^{12};q^{12})_k(-q^6;q^6)_{2k}\cdot(q^{12};q^{12})_n}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{27n(n+1)} + \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \frac{1+q^{12}}{q^{12}} \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

$n \not\equiv 0, \pm 6 \pmod{54}$, $n \not\equiv 0, \pm 24 \pmod{54}$ respectively

12. A General Result which gives Rogers-Ramanujan type Identities Modulo 7s, 9s, 11s, 13s, Onwards, where s is any Finite Positive Integer

***Theorem* 12.1** — *For any integer $p \geq 4$ and $s = 1, 2, 3...$*

$$\frac{(q^{2s};q^{2s})_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r_1=0}^{\infty} \cdots \sum_{r_{p-3}=0}^{\infty} \sum_{k=0}^{\infty}$$

$$\frac{(-1)^{r_{p-3}+D\Delta} q^{2s(n^2+k^2+2nk\,2nM_{p-3}+M_{p-3}^2)+2sM_{p-3}^2-2sM_{p-3}+s[2r_{p-3}+3r_{p-4}+\ldots+(p-2)r_1]+3k^2s+\frac{r_1^2 s+r_1 s}{2}+s(\Delta_1-\Delta)}}{(q^s;q^s)_{r_1}(q^s;q^s)_{r_2}\ldots(q^s;q^s)_{r_{p-3}}\cdot(-q^s;q^s)_{2k+2M_{p-3}}\cdot(q^{2s};q^{2s})_k(q^2s;q^{2s})_{n-2M_{p-3}}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n} \text{ where } n \not\equiv 0,\ \text{ps},\ (P-1)s \bmod (2p-1)s \qquad \ldots (12.2)$$

$$\frac{(q^{2s};q^{2s})_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r_1=0}^{\infty} \cdots \sum_{r_{p-3}=0}^{\infty} \sum_{k=0}^{\infty}$$

$$\frac{(-1)^{r_{p-3}+\Delta} q^{2s(n^2+k^2+2nk-2nM_{p-3}+M_{p-3}^2)-2s(n-M_{p-3}+k)+2sM_{p-3}^2-2sM_{p-3}+s[2r_{p-3}+3r_{p-4}+\ldots+(p-2)r_1]+3k^2s-\frac{r_1^2 s+r_1 s}{2}+s(\Delta_1-\Delta)}}{(q^s;q^s)_{r_1}(q^s;q^s)_{r_2}\ldots(q^s;q^s)_{r_{p-3}}\cdot(-q^s;q^s)_{2k+2M_{p-3}}\cdot(q^{2s};q^{2s})_k(q^{2s};q^2s)_{n-2M_{p-3}}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \prod_{n=1}^{\infty} \frac{1}{1-q^n} \qquad \ldots (12.3)$$

$n \neq 0, (p-2)s, (p+1)$ smod $(2p-1)s$ $n \neq 0, (p-3)s, (p-2)s \bmod (2p-1)s$

where

$$\Delta_1 = M_1^2 + M_2^2 + \ldots + M_{p-4}^2,\ \Delta = M_1 + M_2 + \ldots + M_{p-4},\ M_i = r_1 + r_2 + \ldots r_i$$

$$M_{-1} = M_0 = 0.$$

PROOF : In 2.2 with $q = q^s, b \to 0, c_1 \to 0, c_2, c_3, \ldots c_{p-3} \to \infty,\ x, y \to \infty$ (where $d_j = -c_j\ \forall j$), we get

$$\sum_{k=0}^{n} \frac{(aq^s; q^s)_{k-1} (1 - aq^{2ks}) (-1)^k a^{(p-3)k} q^{\frac{(2p-5)k^2 s - ks}{2}}}{(q^s; q^s)_k (q^{2s}; q^{2s})_{n-k} (a^2 q^{2s}; q^{2s})_{n+k}} = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_{p-3}=0}^{\infty} \sum_{k=0}^{n-M_{p-3}}$$

$$\frac{(-1)^{r_{p-3}+\Delta} \cdot a^H \cdot q^{2sM_{p-3}(M_{p-3}-1) - \frac{r_1 s(r_1+1)}{2} + s[2r_{p-3} + 3r_{p-4} + \ldots + (p-2)r_1] + 4ksM_{p-3} + \frac{5k^2 s + ks}{2} + s(\Delta_1 - \Delta)}}{(q^s; q^s)_{r_1} \cdots (q^s; q^s)_{r_{p-3}} (-aq^s; q^s)_{2k+2M_{p-3}} (q^{2s}; q^{2s})_k (q^{2s}; q^{2s})_{n-M_{p-3}-k}} \quad (12.4)$$

(where H is the index of a)

The L.H.S of Bailey's Lemma for

$$a = a^2, q = q^{2s}, \alpha_0 = 1, \alpha_k = \frac{(aq^s; q^s)_{k-1} (1 - aq^{2ks}) (-1)^k a^{(p-3)k} q^{\frac{k^2 s(2p-5) - ks}{2}}}{(q^s; q^s)_k}$$

gives

$$(a^2 q^{2s}; q^{2s})_\infty \sum_{n=0}^{\infty} a^{2n} \cdot q^{2n^2 s - 2Lns} \sum_{r_1=0}^{\infty} \cdots \sum_{r_{p-3}=0}^{\infty} \sum_{k=0}^{n-M_{p-3}}$$

$$\frac{(-1)^{r_{p-3}+\Delta} \cdot a^H \cdot q^{2sM_{p-3}(M_{p-3}-1) - \frac{r_1 s(r_1+1)}{2} + s[2r_{p-3} + 3r_{p-4} + \ldots + (p-2)r_1] + 4ksM_{p-3} + \frac{5k^2 s + ks}{2} + s(\Delta_1 - \Delta)}}{(q^s; q^s)_{r_1} \cdots (q^s; q^s)_{r_{p-3}} (-aq^s; q^s)_{2k+2M_{p-3}} (q^{2s}; q^{2s})_k (q^{2s}; q^{2s})_{n-M_{p-3}-k}}$$

(upon using 12.4), ... (12.5)

we set $a = 1$ in 12.5. ... (12.6)

The corresponding R.H.S. of Bailey's Lemma for $a = 1$, $q = q^{2s}$ and α_k as above, yields

$$\sum_{j=0}^{L} \frac{(q^{-2Ls}; q^{2s})_j (-1)^j q^{js(j+1)}}{(q^{2s}; q^{2s})_j} \sum_{n=0}^{\infty} (-1)^n (1 + q^{ns}) q^{\frac{n^2 s(2p-1) - ns}{2} - 3Lns + 4njs} \quad \ldots (12.7)$$

We equate 12.6 and 12.7. ... (12.8)

The identity 12.8, on putting $L = 0$ and 1 in succession, yield respectively the results 12.2 and 12.3, on using Jacobi's Triple Product Identity.

13. Two More Results

The proofs of the following two theorems willnow be obvious. The first, (Theorem 13.1) gives Rogers Ramanujan Type identities modulo 14*s*, 18*s*, 22*s*, ... onwards, and the second, (Theorem 13.2), gives Rogers-Ramanujan Type Identities modulo 21*s*, 27*s*, 33*s*, ... onwards. (where *s* is any finite natural number).

Theorem 13.1 — *For any integer* $p \geq 4$ *and for any* $s = 1, 2, 3, \ldots$

$$\frac{(q^{4s};q^{4s})_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r_1=0}^{\infty} \cdots \sum_{r_{p-3}=0}^{\infty} \sum_{k=0}^{\infty}$$

$$\frac{(-1)^{k+r_{p-3}+\Delta} q^{4s(n^2+k^2+2nk-2nM_{p-3}+M_{p-3}^2)-4s(n-M_{p-3}+k)+4sM_{p-3}^2-4s(M_{p-3}-r_1^2 s-r_1 s+2s[2r_{p-3}+3r_{p-4}+\ldots+(p-2)r_1]+6k^2 s+2s(\Delta_1-}}{(q^{2s};q^{2s})_{r_1}(q^{2s};q^{2s})_{r_2}\ldots(q^{2s};q^{2s})_{r_{p-3}}(-q^{2s};q^{2s})_{2k+2M_{p-3}}\cdot(q^{4s};q^{4s})_k(q^{4s};q^{4s})_{n-2M_{p-3}}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

where $n \not\equiv 0$, $(2p-2)s$, $2ps$, mod $(4p-2)s$

and

$$\frac{(q^{4s};q^{4s})_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r_1=0}^{\infty} \cdots \sum_{r_{p-3}=0}^{\infty} \sum_{k=0}^{\infty}$$

$$\frac{(-1)^{k+r_{p-3}+\Delta} q^{4s(n^2+k^2+2nk-2nM_{p-3}+M_{p-3}^2)+4sM_{p-3}^2)+4sM_{p-3}^2-4sM_{p-3}-r_1^2 s-r_1 s+2s[2r_{p-3}+3r_{p-4}+\ldots+(p-2)r_1]+6k^2 s+2s(\Delta_1-\Delta)}}{(q^{2s};q^{2s})_{r_1}(q^{2s};q^{2s})_{r_2}\ldots(q^{2s};q^{2s})_{r_{p-3}}(-q^{2s};q^{2s})_{2k+2M_{p-3}}\cdot(q^{4s};q^{4s})_k(q^{4s};q^{4s})_{n-2M_{p-3}}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

$n \not\equiv 0$, $(2p-6)s$, $(2p+4)s$ mod $(4p-2)s$ $n \not\equiv 0$, $(2p-4)s$, $(2p+2)s$ mod $(4p-2)s$

and for $p \geq 6$,

$$\frac{(q^{4s};q^{4s})_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r_1=0}^{\infty} \cdots \sum_{r_{p-1}=0}^{\infty} \sum_{k=0}^{\infty}$$

$$\frac{(-1)^{k+r_{p-1}+\Delta^1} q^{4s(n^2+k^2+2nk-2nM_{p-1}+M_{p-1}^2)-8s(n-M_{p-1}+k)+4sM_{p-1}^2-4sM_{p-1}-r_1^2 s-r_1 s+2s[2r_{p-1}+3r_{p-2}+\ldots+pr_1]+6k^2 s+2s(\Delta_1^1-\Delta^1)}}{(q^{2s};q^{2s})_{r_1}(q^{2s};q^{2s})_{r_2}\ldots(q^{2s};q^{2s})_{r_{p-3}}(-q^{2s};q^{2s})_{2k+2M_{p-1}}\cdot(q^{4s};q^{4s})_k(q^{4s};q^{4s})_{n-2M_{p-1}}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{1-q^n} + \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \frac{1+q^{4s}}{q^{4s}} \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

$n \not\equiv 0$, $(2p-6)s$, $(2p+2)s$ mod $(4p+6)s$ $n \not\equiv 0$,$(2p-4)s$, $(2p+10)s$ mod $(4p+6)s$ $n \not\equiv 0$, $(2p+2)s$, $(2p+4)s$, mod $(4p+6)s$

where

$$\Delta_1 = M_1^2 + M_2^2 + ... + M_{p-4}^2, \ \Delta = M_1 + M_2 + ... + M_{p-4}$$

and $\Delta_l^l = M_1^2 + M_2^2 + ... + M_{p-2}^2$, $\Delta^1 = M_1 + M_2 + \ldots + M_{p-2}$ (and where as usual $M_j = r_1 + r_2 + \ldots + r_j$ with $M_{-1} = M_0 = 0$)

Theorem 13.2 — *For any integer $p \geq 4$ and any s = 1, 2, 3, ...*

$$\frac{(q^{6s}; q^{6s})_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r_1=0}^{\infty} \cdots \sum_{r_{p-3}=0}^{\infty} \sum_{k=0}^{\infty}.$$

$$\frac{(-1)^{r_{p-3}+\Delta} q^{6s(n^2+k^2+2nk+M_{p-3}^2-2nM_{p-3})+6sM_{p-3}+3s[2r_{p-3}+3r_{p-4}+\ldots+(p-2)r_1]+9k^2s-\frac{3r_1^2s+3r_1s}{2}+3s(\Delta_1-\Delta)}}{(q^{3s};q^{3s})_{r_1}(q^{3s};q^{3s})_{r_2}\ldots(q^{3s};q^{3s})_{r_{p-3}}(-q^{3s};q^{3s})_{2k+2M_{p-3}}\cdot(q^{6s};q^{6s})_k(q^{6s}q^{6s})_{n-2M_{p-1}}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

when $n \not\equiv 0$, $3ps$, $(3p-3)s$ mod $(6p-3)s$

and

$$\frac{(q^{6s}; q^{6s})_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r_i=0}^{\infty} \cdots \sum_{r_{p-3}=0}^{\infty} \sum_{k=0}^{\infty}$$

$$\frac{(-1)^{r_{p-3}+\Delta} q^{6s(n^2+k^2+2nk+M_{p-3}^2-2nM_{p-3})-6s(n-M_{p-3}+k)+6sM_{p-3}^2-6sM_{p-3}+3s[2r_{p-3}+3r_{p-4}+\ldots+(p-2)r_1]+9k^2s+3s(\Delta_1-\Delta)-\frac{3r_1^2s+3r_is}{2}}}{(q^{3s};q^{3s})_{r_1}(q^{3s};q^{3s})_{r_2}\ldots(q^{3s};q^{3s})_{r_{p-3}}(-q^{3s};q^{3s})_{2k+2M_{p-3}}\cdot(q^{6s};q^{6s})_k(q^{6s};q^{6s})_{n-2M_{p-3}}}$$

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} + \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

$n \not\equiv 0$, $(3p-6)s$, $(2p+3)s$ mod $(6p-3)s$ $n \not\equiv 0$, $(3p-9)s$, $(3p+6)s$ mod $(6p-3)s$

where $\Delta_1 = M_1^2 + M_2^2 + \ldots. + M_{p-4}^2$ and $\Delta_1 = M_1 + M_2 + \ldots.. + M_{p-4}$.

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