Indian J. pure appl. Math., **31**(8) : 933-948, August 2000

OSCILLATION FOR SYSTEMS OF NEUTRAL DELAY HYPERBOLIC DIFFERENTIAL EQUATIONS*

WEI NIAN LI AND BAO TONG CUI

[1]*Department of Mathematics, Qufu Normal University, Shandong*. 273165, *People's Republic of China*
[2]*Department of Automatic Control Engineering, South China University of Technology, Guangzhou* 510641, *People's Republic of China*
[3]*Department of Mathematics, Binzhou Normal College, Shandong* 256604, *People's Republic of China*[1,3 2,3]

(*Received* 12 *October* 1998; *accepted* 25 *August* 1999)

Sufficient conditions are established for the oscillation of systems of neutral delay hyperbolic differential equations. These results are illustrated by some examples.

Key Words : Oscillation; System; Hyperbolic Differential Equation; Delay; Neutral Type

1. INTRODUCTION

In the past decade, the oscillation problem for the partial functional differential equations has been extensively investigated. For instance, see [1-8] and the references therein. However, only [9-11] have been published on the oscillation theory of systems of partial functional differential equations.

In this paper, we study the oscillation of systems of neutral delay hyperbolic differential equations of the form

$$\frac{\partial^2}{\partial t^2}[u_i(x,t)+\mu(t)u_i(x,t-\rho)] = a_i(t)\,\Delta u_i(x,t) + \sum_{k=1}^{m}\sum_{j=1}^{d} a_{ikj}(t)\,\Delta u_k(x,t-\tau_j)$$

$$- p_i(x,t)\,u_i(x,t) - \sum_{k=1}^{m}\sum_{h=1}^{l}\int_a^b q_{ikh}(x,t,\xi)\,u_k(x,g_h(t,\xi))\,d\sigma(\xi), \quad \dots (1)$$

$$(x,t)\in \Omega\times[0,\infty)\equiv G,\ i=1,2,\dots,m,$$

where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, and $\Delta u_i(x,t) = \sum_{r=1}^{n}\frac{\partial^2 u_i(x,t)}{\partial x_r^2}$, $i=1,2,\dots,m$, and the integral in (1) is Stieltjes integral.

*Supported by the Natural Science Foundation of Shandong Province (Q99A06), China.

Suppose that the following conditions hold :

(H1) $\mu \in C^2([0,\infty);[0,\infty))$, $\rho=$ const. >0;

(H2) $a_i \in C([0,\infty);[0,\infty))$, $a_{ikj} \in C([0,\infty);R)$, $a_{iij}(t)>0$, and

$$A_j(t)=\min_{1\le i\le m}\left\{a_{iij}(t)-\sum_{k=1,k\ne 1}^{m}|a_{kij}(t)|\right\}>0,\ i=1,2,\ldots,m;\ j=1,2,\ldots,d;$$

(H3) $\tau_j=$ const. >0, $j=1,2,\ldots,d$;

(H4) $p_i \in C(\overline{G};[0,\infty))$, $p_i(t)=\min_{x\in\Omega} p_i(x,t)$, $p(t)=\min_{1\le i\le m}\{p_i(t)\}$, $i=1,2,\ldots,m$;

(H5) $q_{ikh} \in C(\overline{G}\times[a,b];R)$, $q_{iih}(x,t,\xi)>0$, and

$$q_{iih}(t,\xi)=\min_{x\in\Omega} q_{iih}(x,t,\xi),\ \overline{q}_{ikh}(t,\xi)=\max_{x\in\Omega}|q_{ikh}(x,t,\xi)|,$$

$$Q_h(t,\xi)=\min_{1\le i\le m}\left\{q_{iih}(t,\xi)-\sum_{k=1,k\ne i}^{m}\overline{q}_{kih}(t,\xi)\right\}\ge 0,$$

$i=1,2,\ldots,m$; $k=1,2,\ldots,m$; $h=1,2,\ldots,l$;

(H6) $g_h \in C([0,\infty)\times[a,b];R)$, $g_h(t,\xi)\le t$, $\xi\in[a,b]$,

and $g_h(t,\xi)$ is a nondecreasing function with respect to t and ξ, respectively,

$$\lim_{t\to\infty}\ \min_{\xi\in[a,b]}\{g_h(t,\xi)\}=\infty,\ h=1,2,\ldots,l;$$

(H7) $\sigma\in([a,b];R)$ and $\sigma(\xi)$ is nondecreasing in ξ.

We consider two kinds of boundary conditions:

$$\frac{\partial u_i(x,t)}{\partial N}+f_i(x,t)\,u_i(x,t)=0,\ (x,t)\in\partial\Omega\times[0,\infty),\ i=1,2,\ldots,m, \qquad \ldots (2)$$

where N is the unit exterior normal vector to $\partial\Omega$ and $f_i(x,t)$ is a nonnegative continuous function on $\partial\Omega\times[0,\infty)$, $i=1,2,\ldots,m$, and

$$u_i(x,t)=0,\ (x,t)\in\partial\Omega\times[0,\infty),\ i=1,2,\ldots,m. \qquad \ldots (3)$$

Definition 1.1 — The vector function $u(x, t) = \{u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\}^T$ is said to be a solution of the problem (1), (2) (or (1), (3)) if it satisfies (1) in $G = \Omega \times [0, \infty)$ and boundary condition (2) (or (3)).

Definition 1.2 — The vector solution $u(x, t) = \{u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\}^T$ of the problem (1), (2) (or (1), (3)) is said to be oscillatory in the domain $G = \Omega \times [0, \infty)$ if at least one of its nontrivial component is oscillatory in G. Otherwise, the vector solution $u(x, t)$ is said to be nonoscillatory.

Lemma 1.1[4] — Suppose that $y \in C^2([t_0, \infty), R))$ and that

$$y(t) > 0,\ y'(t) > 0 \ \text{and}\ y''(t) \le 0,\ t \ge t_0 > 0.$$

Then for any $\lambda_0 \in (0, 1)$ there exists a number $t_1 > t_0$ such that

$$y(t) \ge \lambda_0 t y'(t) \ \text{for}\ t \ge t_1. \quad \ldots (4)$$

Lemma 1.2[12] — Suppose that $F \in C([0, \infty) \times [a, b]; [0, \infty))$ and that the following conditions hold:

(A1) there exists a function $h(t, \xi) \in C([0, \infty) \times [a, b]; [0, \infty))$ such that $h(h(t, \xi), \xi) = h(t, \xi)$; $h(t, \xi)$ is a nondecreasing function with respect to t and ξ, respectively; and

$$t \ge h(t, \xi) \ge g(t, \xi), \quad \lim_{t \to \infty} \min_{\xi\ [a, b]} \{h(t, \xi)\} = \infty;$$

(A2)
$$\liminf_{t \to \infty} \int_{g(t, b)}^{t} \int_a^b F(s, \xi)\, d\sigma(\xi)\, ds > \frac{1}{e};$$

(A3)
$$\liminf_{t \to \infty} \int_{h(t, b)}^{t} \int_a^b F(s, \xi)\, d\sigma(\xi)\, ds > 0.$$

Then the differential inequality

$$z'(t) + \int_a^b F(t, \xi)\, z(g(t, \xi))\, d\sigma(\xi) \le 0$$

has no eventually positive solutions.

2. Oscillation of the Problem (1), (2)

***Theorem* 2.1** — *If the neutral differential inequality with continuous distributed deviating arguments*

$$[V(t)+\mu(t)V(t-\rho)]''+p(t)\,V(t)+\sum_{h=1}^{l}\int_a^b Q_h(t,\xi)\,V(g_h(t,\xi))\,d\sigma(\xi)\le 0 \qquad \dots (6)$$

has no eventually positive solutions, then every solution of the problem (1), (2) is oscillatory in G.

PROOF : Suppose to the contrary that there is a nonoscillatory solution $u(x,t) = \{u_1(x,t), u_2(x,t), \dots, u_m(x,t)\}^T$ of the problem (1), (2). We assume that $|u_i(x,t)| > 0$ for $t \ge t_0 \ge 0$, $i = 1, 2, \dots, m$. Let $\delta_i = \operatorname{sgn} u_i(x,t)$, $Z_i(x,t) = \delta_i u_i(x,t)$, then $Z_i(x,t) > 0$, $(x,t) \in \Omega \times [t_0, \infty)$, $i = 1, 2, \dots, m$. From (H3) and (H6) there exists a number $t_1 \ge t_0$ such that $Z_i(x,t) > 0$, $Z_i(x, t-\tau_j) > 0$ and $Z_i(x, g_h(t,\xi)) > 0$ in $\Omega \times [t_1, \infty)$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, d$; $h = 1, 2, \dots, l$.

Integrating (1) with respect to x over the domain Ω, we have

$$\frac{d^2}{dt^2}\left[\int_\Omega u_i(x,t)dx+\mu(t)\int_\Omega u_i(x,t-\rho)dx\right]=a_i(t)\int_\Omega \Delta u_i(x,t)dx$$

$$+\sum_{k=1}^{m}\sum_{j=1}^{d} a_{ikj}(t)\int_\Omega \Delta u_k(x,t-\tau_j)dx-\int_\Omega p_i(x,t)u_i(x,t)dx \qquad \dots (7)$$

$$-\sum_{k=1}^{m}\sum_{h=1}^{l}\int_\Omega\int_a^b q_{ikh}(x,t,\xi)\,u_k(x,g_h(t,\xi))\,d\sigma(\xi)dx,\ t\ge t_1,\ i=1,2,\dots,m.$$

It is easy to see that

$$\int_\Omega\int_a^b q_{ikh}(x,t,\xi)\,u_k(x,g_h(t,\xi))\,d\sigma(\xi)dx$$

$$=\int_a^b\int_\Omega q_{ikh}(x,t,\xi)\,u_k(x,g_h(t,\xi))\,dx\,d\sigma(\xi), \qquad \dots (8)$$

$i, k = 1, 2, \dots, m$; $h = 1, 2, \dots, l$.

Therefore

$$\frac{d^2}{dt^2}\left[\int_\Omega Z_i(x,t)\,dx + \mu(t)\int_\Omega Z^i(x,t-\rho)dx\right] = a_i(t)\int_\Omega \Delta Z_i(x,t)dx$$

$$+\sum_{j=1}^{d} a_{iij}(t)\int_\Omega \Delta Z_i(x,t,-\tau_i)\,dx + \sum_{k=1,k\neq i}^{m}\sum_{j=1}^{d} a_{ikj}(t)\frac{\delta_k}{\delta_i}\int_\Omega \Delta Z_k(x,t-\tau_j)dx$$

$$-\int_\Omega p_i(x,t)\,Z_i(x,t)\,dx - \sum_{h=1}^{l}\int_a^b\int_\Omega q_{iih}(x,t,\xi)\,Z_i(x,g_h(t,\xi))\,dxd\sigma(\xi) \qquad \ldots (9)$$

$$-\sum_{h=1}^{l}\sum_{k=1,k\neq i}^{m}\frac{\delta_k}{\delta_i}\int_a^b\int_\Omega q_{ikh}(x,t,\xi)\,Z_k(x,g_h(t,\xi))\,dxd\sigma(\xi),$$

$t \geq t_1,\ i = 1, 2, \ldots, m.$

Green's formula and (2) yield

$$\int_\Omega \Delta Z_i(x,t)\,dx = \int_{\partial\Omega}\frac{\partial Z_i(x\,t)}{\partial N}dS = -\int_{\partial\Omega} f_i(x,t)\,Z_i(x,t)\,dS \leq 0, \qquad \ldots (10)$$

and

$$\int_\Omega \Delta Z_k(x,t-\tau_j)dx = \int_{\partial\Omega}\frac{\partial Z_k(x,t-\tau_j)}{\partial N}dS$$

$$= -\int_{\partial\Omega} f_k(x,t-\tau_j)\,Z_k(x,t-\tau_j)dS, \qquad \ldots (11)$$

$t \geq t_1,\ i, k = 1, 2, \ldots, m;\ j = 1, 2, \ldots, d,$

where dS is the surface element on $\partial\Omega$. Combining (9)-(11), we get

$$\frac{d^2}{dt^2}\left[\int_\Omega Z_i(x,t)\,dx + \mu(t)\int_\Omega Z_i(x,t-\rho)dx\right]$$

$$\leq -\sum_{j=1}^{d} a_{iij}(t)\int_{\partial\Omega} f_i(x,t-\tau_j)\,Z_i(x,t-\tau_j)dS$$

$$+\sum_{j=1}^{d}\sum_{k=1,k\neq i}^{m} |\,a_{ikj}(t)\,|\int_{\partial\Omega} f_k(x,t-\tau_j)\,Z_k(x,t-\tau_j)dS$$

$$- p_i(t) \int_\Omega Z_i(x,t)dx - \sum_{h=1}^{l} \int_a^b q_{iih}(t,\xi) \int_\Omega Z_i(x, g_h(t,\xi))\, dx d\sigma(\xi)$$

$$+ \sum_{h=1}^{l} \sum_{k=1, k\neq i}^{m} \int_a^b \bar{q}_{ikh}(t,\xi) \int_\Omega Z_k(x, g_h(t,\xi))\, dx d\sigma(\xi), \qquad \ldots (12)$$

$t \geq t_1$, $i = 1, 2, \ldots, m$.

Set

$V_i(t) = \int_\Omega Z_i(x,t)dx$, $W_i(t) = \int_{\partial\Omega} f_i(x,t)\, Z_i(x,t)\, dS$, $t \geq t_1$, $i = 1, 2, \ldots, m$, we have

$$[V_i(t) + \mu(t)V_i(t-\rho)]'' + \sum_{j=1}^{d} a_{iij}(t)\, W_i(t-\tau_j) - \sum_{j=1}^{d} \sum_{k=1, k\neq i}^{m} |a_{ikj}(t)|\, W_k(t-\tau_j)$$

$$+ p_i(t)\, V_i(t) + \sum_{h=1}^{l} \int_a^b q_{iih}(t,\xi)\, V_i(g_h(t,\xi))(\, d\sigma(\xi) \qquad \ldots (13)$$

$$- \sum_{h=1}^{l} \sum_{k=1, k\neq i}^{m} \int_a^b \bar{q}_{ikh}(t,\xi)\, V_k(g_h(t,\xi))\, d\sigma(\xi) \leq 0,$$

$t \geq t_1$, $i = 1, 2, \ldots, m$.

Let

$V(t) = \sum_{i=1}^{m} V_i(t)$, $W(t) = \sum_{i=1}^{m} W_i(t)$ for $t \geq t_1$, from (13) we obtain

$$[V(t) + \mu(t)V(t-\rho)]'' + \sum_{j=1}^{d} \left\{ \sum_{i=1}^{m} [a_{iij}(t)\, W_i(t-\tau_j) - \sum_{k=1, k\neq i}^{m} |a_{ikj}(t)|\, W_k(t-\tau_j)] \right\}$$

$$+ p(t)V(t) + \sum_{h=1}^{l} \left[\sum_{i=1}^{m} \int_a^b \left[q_{iih}(t,\xi) V_i(g_h(t,\xi))\, d\sigma(\xi) \right.\right. \qquad \ldots (14)$$

$$\left.\left. - \sum_{k=1, k\neq i}^{m} \int_a^b \bar{q}_{ikh}(t,\xi)\, V_k(g_h(t,\xi))\, d\sigma(\xi) \right]\right] \leq 0, \; t \geq t_1.$$

Noting that

$$\sum_{i=1}^{m} \int_a^b \left[q_{iih}(t,\xi) V_i(g_h(t,\xi)) - \sum_{k=1,k\neq i}^{m} \bar{q}_{ikh}(t,\xi) V_k(g_h(t,\xi)) \right] d\sigma(\xi)$$

$$= \int_a^b \left[q_{11h}(t,\xi) V_1(g_h(t,\xi)) - \sum_{k=1,k\neq 1}^{m} \bar{q}_{1kh}(t,\xi) V_k(g_h(t,\xi)) \right] d\sigma(\xi)$$

$$+ \int_a^b \left[q_{22h}(t,\xi) V_2(g_h(t,\xi)) - \sum_{k=1,k\neq 2}^{m} \bar{q}_{2kh}(t,\xi) V_k(g_h(t,\xi)) \right] d\sigma(\xi)$$

$$+ \ldots\ldots\ldots$$

$$+ \int_a^b \left[q_{mmh}(t,\xi) V_m(g_h(t,\xi)) - \sum_{k=1,k\neq m}^{m} \bar{q}_{mkh}(t,\xi) V_k(V_k(g_h(t,\xi)) \right] d\sigma(\xi)$$

$$= \int_a^b \left[q_{11h}(t,\xi) - \sum_{k=1,k\neq 1}^{m} \bar{q}_{k1h}(t,\xi) \right] V_1(g_h(t,\xi))\, d\sigma(\xi)$$

$$+ \int_a^b \left[q_{22h}(t,\xi) - \sum_{k=1,k\neq 2}^{m} \bar{q}_{k2h}(t,\xi) \right] V_2(g_h(t,\xi))\, d\sigma(\xi)$$

$$+ \ldots\ldots$$

$$+ \int_a^b \left[q_{mmh}(t,\xi) - \sum_{k=1,k\neq m}^{m} \bar{q}_{kmh}(t,\xi) \right] V_m(g_h(t,\xi))\, d\sigma(\xi)$$

$$\geq \int_a^b \min_{1\leq i\leq m} \left\{ q_{iih}(t,\xi) - \sum_{k=1,k\neq i}^{m} \bar{q}_{kih}(t,\xi) \right\} \sum_{i=1}^{m} V_i(g_h(t,\xi))\, d\sigma(\xi)$$

$$= \int_a^b Q_h(t,\xi) V(g_h(t,\xi))\, d\sigma(\xi),\; t \geq t_1,\; h = 1, 2, \ldots, l.$$

And, similarly,

$$\sum_{i=1}^{m} \left[a_{iij}(t) W_i(t-\tau_j) - \sum_{k=1,k\neq i}^{m} |a_{ikj}(t)|\, W_k(t-\tau_j) \right]$$

$$\geq \min_{1\leq i\leq m} \left[a_{iij}(t) - \sum_{k=1,k\neq i}^{m} |a_{kij}(t)| \right] \sum_{i=1}^{m} W_i(t-\tau_j)$$

$$= A_j(t) W(t-\tau_j),\; t \geq t_1,\; j = 1, 2, \ldots, d.$$

Then, from (14), we have

$$[V(t)+\mu(t)V(t-\rho)]'' + \sum_{j=1}^{d} A_j(t)W(t-\tau_j) + p(t)V(t)$$

$$+ \sum_{h=1}^{l} \int_a^b Q_h(t,\xi)\, V(g_h(t,\xi))\, d\sigma(\xi) \le 0,\ t \ge t_1. \qquad \text{... (15)}$$

It is easy to see that

$$W(t-\tau_j) = \sum_{i=1}^{m} W_i(t-\tau_j) \ge 0,\ t \ge t_1,\ j = 1, 2, \ldots, d.$$

Therefore,

$$[V(t)+\mu(t)V(t-\rho)]'' + p(t)V(t) + \sum_{h=1}^{l} \int_a^b Q_h(t,\xi)\, V(g_h(t,\xi))\, d\sigma(\xi) \le 0,\ t \ge t_1,$$

which contradicts the assumption that (6) has no eventually positive solutions. This completes the proof.

Theorem 2.2 — *Suppose that* $0 \le \mu(t) \le 1$ *and that for some* $\lambda_0 \in (0, 1)$ *there exists some* $h_0 \in \{1, 2, \ldots, l\}$ *such that*

(B1) *there exists a function* $\eta_{h_0} \in C([0,\infty) \times [a,b]; (0,\infty))$ *such that*

$$\eta_{h_0}(\eta_{h_0}(t,\xi),\xi) = g_{h_0}(t,\xi),\ \eta_{h_0}(t,\xi)$$

is a nondecreasing function with respect to t and ξ, *respectively; and*

$$t \ge \eta_{h_0}(t,\xi) \ge g_{h_0}(t,\xi),\ \lim_{t\to\infty} \min_{\xi\in[a,b]} \eta_{h_0}(t,\xi) = \infty;$$

(B2)
$$\liminf_{t\to\infty} \int_{g_{h_0}(t,b)}^{t} \int_a^b \lambda_0 g_{h_0}(s,\xi)\, Q_{h_0}(s,\xi)\ [1-\mu(g_{h_0}(t,\xi))]\, d\sigma(\xi) ds > \frac{1}{e};$$

(B3)
$$\liminf_{t\to\infty} \int_{\eta_{h_0}(t,b)}^{t} \int_a^b \lambda_0 g_{h_0}(s,\xi)\, Q_{h_0}(s,\xi)\ [1-\mu(g_{h_0}(t,\xi))]\, d\sigma(\xi) ds > 0.$$

Then every solution of the problem (1), (2) is oscillatory in G.

PROOF : We prove that the inequality (6) has no eventually positive solution if the conditions of Theorem 2.2 hold. Suppose that $V(t)$ is an eventually positive solution of the inequality (6), then there exists a number $t_1 \ge t_0$ such that $V(g_h(t,\xi)) > 0,\ h = 1, 2, \ldots, l,$ for $t \ge t_1$. Thus we have

$$[V(t) + \mu(t)\, V(t - \rho)]'' + \int_a^b Q_{h_0}(t, \xi)\, V(g_{h_0}(t, \xi))\, d\sigma(\xi) \le 0, t \ge t_1. \quad \dots (16)$$

Set $y(t) = V(t) + \mu(t)V(t - \rho)$. It is easy to see that there exists $t_2 \ge t_1$ such that

$$y(t) > 0, y''(t) \le 0 \text{ and } y'(t) > 0, t \ge t_2.$$

Then by Lemma 1.1 we obtain that there exists a number $t_3 \ge t_2$ such that

$$y(t) \ge \lambda_0 t y'(t) \text{ for } t \ge t_3. \quad \dots (17)$$

Noting that $y(t) \ge V(t)$, we obtain

$$y(t) \le V(t) + \mu(t)y(t - \rho) \le V(t) + \mu(t)y(t), t \ge t_3,$$

that is

$$[1 - \mu(t)]\, y(t) \le V(t). \quad \dots (18)$$

Combining (16)-(18), we get

$$0 \ge y''(t) + \int_a^b Q_{h_0}(t, \xi)\, V(g_{h_0}(t, \xi))\, d\sigma(\xi)$$

$$\ge y''(t) + \int_a^b Q_{h_0}(t, \xi)\, [1 - \mu(g_{h_0}(t, \xi))]\, y\, (g_{h_0}(t, \xi))\, d\sigma(\xi) \quad \dots (19)$$

$$\ge y''(t) + \int_a^b \lambda_0\, Q_{h_0}(t, \xi)\, [1 - \mu(g_{h_0}(t, \xi))]\, g_{h_0}(t, \xi)\, y'(g_{h_0}(t, \xi))\, \sigma(\xi), t \ge t_3.$$

Set $U(t) = y'(t)$, from (19) we have

$$U'(t) + \int_a^b \lambda_0 g_{h_0}(t, \xi)\, Q_{h_0}(t, \xi)\, [1 - \mu(g_{h_0}(t, \xi))]\, U\, (g_{h_0}(t, \xi))\, d\sigma(\xi) \le 0. \quad \dots (20)$$

By Lemma 1.2, we obtain that (20) has no eventually positive solutions, which contradicts that $U(t) = y'(t) > 0$ is a solution of (20). The proof is complete.

***Theorem* 2.3** — *Suppose that* $0 \le \mu(t) \le 1$ *and that there exists some* $h_0 \in \{1, 2, ..., l\}$ *such that*

(B4) $$\int^{\infty} \int_a^b Q_{h_0}(s, \xi)\, [1 - \mu(g_{h_0}(s, \xi))]\, d\sigma(\xi)\, ds = +\infty.$$

Then every solution of the problem (1), (2) is oscillatory in G.

PROOF : We prove that the inequality (16) has no eventually positive solutions if the conditions of Theorem 2.3 hold.

As in the proof of Theorem 2.2, we obtain

$$y''(t) + \int_a^b Q_{h_0}(t, \xi)\,[1 - \mu(g_{h_0}(t, \xi))]\, y(g_{h_0}(t, \xi))\, d\sigma(\xi) \le 0,\ t \ge t_1. \quad \text{... (21)}$$

Noting that

$$y(t) > 0,\ y'(t) > 0,\ t \ge t_1,\ \lim_{t \to \infty} \min_{\xi \in [a, b]} g_{h_0}(t, \xi) = \infty,$$

we obtain that there exist $m > t_1$, $t_2 \ge t_1$ such that

$$y(m) > 0,\ g_{h_0}(t, \xi) > m,\ t \ge t_2,\ \xi \in [a, b].$$

Therefore,

$$y(g_{h_0}(t, \xi)) \ge y(m),\ t \ge t_2,\ \xi \in [a, b]. \quad \text{... (22)}$$

Combining (21) and (22), we get

$$y''(t) + y(m) \int_a^b Q_{h_0}(t, \xi)\,[1 - \mu(g_{h_0}(s, \xi))]\, d\sigma(\xi) \le 0,\ t \ge t_2. \quad \text{... (23)}$$

Integrating (23) from t_2 to t, we have

$$y'(t) - y'(t_2) + y(m) \int_{t_2}^{t} \int_a^b Q_{h_0}(s, \xi)\,[1 - \mu(gh_0(s, \xi))]\, d\sigma(\xi) ds \le 0, \quad \text{... (24)}$$

that is

$$\int_{t_2}^{t} \int_a^b Q_{h_0}(s, \xi)\,[1 - \mu(g_{h_0}(s, \xi))]\, d\sigma(\xi)\, ds \le \frac{y'(t_2) - y'(t)}{y(m)},$$

which contradicts the condition (B4). The proof is complete.

3. OSCILLATION OF THE PROBLEM (1), (3)

The following fact will be used :

The smallest eigenvalue α_0 of the Dirichlet problem

$$\begin{cases} \Delta\, \omega(x) + \alpha\, \omega(x) = 0 \text{ in } \Omega, \\ \omega(x) = 0 \text{ on } \partial\Omega, \end{cases}$$

is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω.

***Theorem* 3.1** — *If the differential inequality*

$$[V(t) + \mu(t)V(t-\rho)]'' + \alpha_0 \sum_{j=1}^{d} A_j(t)V(t-\tau_j) + p(t)V(t)$$

$$+ \sum_{h=1}^{l} \int_a^b Q_h(t, \xi)V(g_h(t, \xi))\, d\sigma(\xi) \le 0, \qquad \dots (25)$$

has no eventually positive solution, then every solution of the problem (1), (3) is oscillatory in G.

PROOF : Suppose to the contrary that there is a nonoscillatory solution $u(x, t) = \{u_1(x, t), u_2(x, t), \dots, u_m(x, t)\}^T$ of the problem (1), (3). We assume that $|u_i(x, t)| > 0$ for $t \ge t_0 \ge 0$, $i = 1, 2, \dots, m$. Let $\delta_1 = \operatorname{sgn} u_i(x, t), Z_i(x, t) = \delta_i u_i(x, t)$, then $Z_i(x, t) > 0, (x, t) \in \Omega \times [t_0, \infty)$, $i = 1, 2, \dots, m$. From (H3) and (H6) there exists a number $t_1 \ge t_0$ such that $Z_i(x, t) > 0, Z_i(x, t-\tau_j) > 0$ and $Z_i(x, g_h(t, \xi)) > 0$ in $\Omega \times [t_1, \infty), i = 1, 2, \dots, m;\ j = 1, 2, \dots, d;\ h = 1, 2, \dots, l$.

Multiplying both sides of (1) by $\varphi(x)$ and integrating with respect to x over the domain Ω, we have

$$\frac{d^2}{dt^2}\left[\int_\Omega u_i(x, t)\, \varphi(x)dx + \mu(t) \int_\Omega u_i(x, t-\rho)\, \varphi(x)dx\right] = a_i(t) \int_\Omega \Delta u_i(x, t)\, \varphi(x)dx$$

$$+ \sum_{k=1}^{m} \sum_{j=1}^{d} a_{ikj}(t) \int_\Omega \Delta u_k(x, t-\tau_j)\, \varphi(x)dx - \int_\Omega p_i(x, t)\, u_i(x, t)\varphi(x)\, dx \qquad \dots (26)$$

$$- \sum_{k=1}^{m} \sum_{h=1}^{l} \int_\Omega \int_a^b q_{ikh}(x, t, \xi)\, u_k(x, g_h(t, \xi))\, \varphi(x)d\sigma(\xi)dx,$$

$t \ge t_1, i = 1, 2, \dots, m$.

Therefore, we have

$$\frac{d^2}{dt^2}\left[\int_\Omega Z_i(x, t)\varphi(x)dx + \mu(t) \int_\Omega Z_i(x, t-\rho)\, \varphi(x)dx\right] = a_i(t) \int_\Omega \Delta Z_i(x, t)\varphi(x)dx$$

$$+ \sum_{j=1}^{d} a_{iij}(t) \int_\Omega \Delta Z_i(x, t-\tau_j)\varphi(x)dx$$

$$+ \sum_{k=1, k\ne i}^{m} \sum_{j=1}^{d} a_{ikj}(t) \frac{\delta_k}{\delta_i} \int_\Omega \Delta Z_k(x, t-\tau_j)\, \varphi(x)dx$$

$$- \int_{\Omega} p_i(x,t)\, Z_i(x,t)\, \varphi(x)dx - \sum_{h=1}^{l} \int_a^b \int_{\Omega} q_{iih}(x,t,\xi)\, Z_i(x, g_h(t,\xi))\, \varphi(x)dx\, d\sigma(\xi) \quad \text{... (27)}$$

$$- \sum_{h=1}^{l} \sum_{k=1, k\neq i}^{m} \frac{\delta_k}{\delta_i} \int_a^b \int_{\Omega} q_{ikh}(x,t,\xi)\, Z_k(x, g_h(t,\xi))\, \varphi(x)dx\, d\sigma(\xi),$$

$t \geq t_1,\ i = 1, 2, \ldots, m.$

Using Green's formula and (3), we have

$$\int_{\Omega} \Delta Z_i(x,t)\varphi(x)dx = \int_{\Omega} Z_i(x,t)\Delta\varphi(x)dx = -\alpha_0 \int_{\Omega} Z_i(x,t)\varphi(x)dx \leq 0, \quad \text{... (28)}$$

and

$$\int_{\Omega} \Delta Z_k(x, t-\tau_j)\varphi(x)dx = \int_{\Omega} Z_k(x, t-\tau_j)\Delta\, \varphi(x)dx$$

$$= -\alpha_0 \int_{\Omega} Z_k(x, t-\tau_j)\, \varphi(x)dx, \quad \text{... (29)}$$

$t \geq t_1,\ i, k = 1, 2, \ldots, m;\ j = 1, 2, \ldots, d.$

Now from (27)-(29), we obtain

$$\frac{d^2}{dt^2}\left[\int_{\Omega} Z_i(x,t)\varphi(x)dx + \mu(t) \int_{\Omega} Z_i(x, t-\rho)\varphi(x)dx \right]$$

$$\leq -\alpha_0 \sum_{j=1}^{d} a_{iij}(t) \int_{\Omega} Z_i(x, t-\tau_j)\varphi(x)dx$$

$$+ \alpha_0 \sum_{j=1}^{d} \sum_{k=1, k\neq i}^{m} |a_{ikj}(t)| \int_{\Omega} Z_k(x, t-\tau_j)\, \varphi(x)dx$$

$$- p_i(t) \int_{\Omega} Z_i(x,t)\varphi(x)dx - \sum_{h=1}^{l} \int_a^b q_{iih}(t,\xi) \int_{\Omega} Z_i(x, g_h(t,\xi))\, \varphi(x)dx\, d\sigma(\xi) \quad \text{... (30)}$$

$$+ \sum_{h=1}^{l} \sum_{k=1, k\neq i}^{m} \int_a^b \bar{q}_{ikh}(t,\xi) \int_{\Omega} Z_k(x, g_h(t,\xi))\, \varphi(x)dx d\sigma(\xi),$$

$t \geq t_1,\ i = 1, 2, \ldots, m.$

Set

$V_i(t) = \int_\Omega Z_i(x, t)\varphi(x)dx$ for $t \geq t_1$, $i = 1, 2, \ldots, m$, we have

$$[V_i(t) + \mu(t)V_i(t-\rho)]'' + \alpha_0 \sum_{j=1}^{d} a_{iij}(t)V_i(t-\tau_j) - \alpha_0 \sum_{j=1}^{d} \sum_{k=1, k\neq i}^{m} |a_{ikj}(t)| V_k(t-\tau_j)$$

$$+ p_i(t)V_i(t) + \sum_{h=1}^{l} \int_a^b q_{iih}(t, \xi)\, V_i(g_h(t, \xi))d\sigma(\xi) \qquad \ldots (31)$$

$$-\sum_{h=1}^{l} \sum_{k=1, k\neq i}^{m} \int_a^b \bar{q}_{ikh}(t, \xi)V_k(g_h(t, \xi))\, d\sigma(\xi) \leq 0,$$

$$t \geq t_1, i = 1, 2, \ldots, m.$$

Let

$V(t) = \sum_{i=1}^{m} V_i(t)$ for $t \geq t_1$, from (31) we obtain

$$[V(t) + \mu(t)V(t-\rho)]'' + \alpha_0 \sum_{j=1}^{d} \left\{ \sum_{i=1}^{m} \left[a_{iij}(t)\, V_i(t-\tau_j) - \sum_{k=1, k\neq i}^{m} |a_{ikj}(t)|\, V_k(t-\tau_j) \right] \right\}$$

$$+ p(t)V(t) + \sum_{h=1}^{l} \left\{ \sum_{i=1}^{m} \int_a^b \left[q_{iih}(t, \xi)\, V_i(g_h(t, \xi))\, d\sigma(\xi) \right.\right. \qquad \ldots (32)$$

$$\left.\left. - \sum_{k=1, k\neq i}^{m} \int_a^b \bar{q}_{ikh}(t, \xi)V_k(g_h(t, \xi))\, d\sigma(\xi) \right] \right\} \leq 0, t \geq t_1.$$

As in the proof of Theorem 2.1, from (32), we have

$$V(t) + \mu(t)\, V(t-\rho)]'' + \alpha_0 \sum_{j=1}^{d} A_j(t)\, V(t-\tau_j) + p(t)V(t)$$

$$+ \sum_{h=1}^{l} \int_a^b Q_h(t, \xi)\, V(g_h(t, \xi))\, d\sigma(\xi) \leq 0, t \geq t_1,$$

which shows that $V(t) = \sum_{i=1}^{m} V_i(t) > 0$ is a positive solution of the inequality (25). This is a contradiction.

By using Theorem 3.1 we have the following theorems :

Theorem 3.2 — *If all conditions of Theorem 2.1 hold, then every solution of the problem (1), (3) is oscillatory in G.*

Theorem 3.3 — *If all conditions of Theorem 2.2 hold, then every solution of the problem (1), (3) is oscillatory in G.*

Theorem 3.4 — *If all conditions of Theorem 2.3 hold, then every solution of the problem (1), (3) is oscillatory in G.*

4. Examples

Following are illustrative examples :

Example 4.1 — Consider the system of neutral delay hyperbolic equations

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\left[u_1(x,t)+\dfrac{1}{2}u_1(x,t-\pi)\right]=3\Delta u_1(x,t)+4\Delta u_1\left(x,t-\dfrac{3\pi}{2}\right) \\ \qquad +\Delta u_2\left(x,t-\dfrac{3\pi}{2}\right)-\dfrac{1}{2}u_1(x,t) \\ \qquad -\displaystyle\int_{-\pi}^{-\frac{\pi}{2}} 3u_1(x,t+\xi)\,d\xi-\int_{-\pi}^{-\frac{\pi}{2}} u_2(x,t+\xi)d\xi, \\ \dfrac{\partial^2}{\partial t^2}\left[u_2(x,t)+\dfrac{1}{2}u_2(x,t-\pi)\right]=\dfrac{1}{2}\Delta u_2(x,t)+3\Delta u_1\left(x,t-\dfrac{3\pi}{2}\right) \\ \qquad +2\Delta u_2\left(x,t-\dfrac{3\pi}{2}\right)-u_2(x,t) \\ \qquad -\displaystyle\int_{-\pi}^{-\frac{\pi}{2}} u_1(x,t+\xi)\,d\xi-\int_{-\pi}^{-\frac{\pi}{2}} 3u_2(x,t+\xi)d\xi, \\ \qquad (x,t)\in(0,\pi)\times[0,\infty), \end{cases} \qquad \ldots (33)$$

with boundary condition

$$\frac{\partial}{\partial x}u_i(0,t)=\frac{\partial}{\partial x}u_i(\pi,t)=0,\ t\geq 0,\ i=1,2. \qquad \ldots (34)$$

Here $n = 1$, $m = 2$, $d = 1$, $l = 1$, $a_1(t)=3$, $\mu(t)=\frac{1}{2}$, $\rho=\pi$, $a_{111}(t)=4$, $a_{121}(t)=1$, $\tau_1=\frac{3\pi}{2}$, $p_1(x,t)=\frac{1}{2}$, $q_{111}(x,t,\xi) = 3$, $q_{121}(x,t,\xi) = 1$, $g_1(t,\xi)=t+\xi$, $a_2(t) = \frac{1}{2}$, $a_{211}(t) = 3$, $a_{221}(t) = 2$, $p_2(x,t)=1$, $q_{211}(x,t,\xi)=1$, $q_{221}(x,t,\xi)=3$, $a=-\pi$, $b=-\frac{\pi}{2}$. It is easy to see that $Q_1(t,\xi)$ 2, $\eta_1(t,\xi) = t+\frac{\xi}{2}$, $t>\frac{3\pi}{2}$,

$$\liminf_{t\to\infty} \int_{g_1(t,b)}^{t} \int_a^b \lambda_0 g_1(s,\xi)\, Q_1(s,\xi)\, [1-\mu(g_1(t,\xi))]\, d\xi\, ds$$

$$= \liminf_{t\to\infty} \int_{t-\frac{\pi}{2}}^{t} \int_{-\pi}^{-\frac{\pi}{2}} \lambda_0 (s+\xi)\, d\xi ds = \liminf_{t\to\infty} \frac{\lambda_0 \pi^2}{4}\left(t-\frac{\pi}{2}\right) = +\infty > \frac{1}{e},$$

and

$$\liminf_{t\to\infty} \int_{\eta_1(t,b)}^{t} \int_a^b \lambda_0 g_1(s,\xi)\, Q_1(s,\xi)\, [1-\mu(g_1(t,\xi))]\, d\xi\, ds$$

$$= \liminf_{t\to\infty} \int_{t-\frac{\pi}{4}}^{t} \int_{-\pi}^{-\frac{\pi}{2}} \lambda_0 (s+\xi)\, d\xi ds = \liminf_{t\to\infty} \frac{\lambda_0 \pi^2}{64}(8t-3\pi) = +\infty > 0.$$

Hence, all the conditions of Theorem 2.2 are fulfilled. Then every solution of the problem (33), (34) is oscillatory in $(0,\pi)\times[0,\infty)$. In fact, such a solution is $u_1(x,t)=\cos x \sin t$, $u_2(x,t) = \cos x \cos t$.

Example 4.2 — Consider the system of neutral delay hyperbolic equations

$$\begin{cases} \frac{\partial^2}{\partial t^2}\left[u_1(x,t)+\frac{1}{3}u_1(x,t-\pi)\right] = 3\Delta u_1(x,t) + 2\Delta u_1\left(x,t-\frac{3\pi}{2}\right) \\ \qquad + \Delta u_2\left(x,t-\frac{3\pi}{2}\right) - \frac{2}{3}u_1(x,t) \\ \qquad - \int_{-\pi}^{-\frac{\pi}{2}} 3u_1(x,t+\xi)\, d\xi - \int_{-\pi}^{-\frac{\pi}{2}} u_2(x,t+\xi) d\xi, \\ \frac{\partial^2}{\partial t^2}\left[u_2(x,t)+\frac{1}{3}u_2(x,t-\pi)\right] = \frac{2}{3}\Delta u_2(x,t) + \Delta u_1\left(x,t-\frac{3\pi}{2}\right) \\ \qquad + 4\Delta u_2\left(x,t-\frac{3\pi}{2}\right) - 3u_2(x,t) \\ \qquad - \int_{-\pi}^{-\frac{\pi}{2}} u_1(x,t+\xi)\, d\xi - \int_{-\pi}^{-\frac{\pi}{2}} 3u_2(x,t+\xi) d\xi, \\ \qquad (x,t)\in(0,\pi)\times[0,\infty), \end{cases} \quad \ldots (35)$$

with boundary condition

$$u_i(0, t) = u_i(\pi, t) = 0, t \geq 0, i = 1, 2. \qquad \text{... (36)}$$

Here $n = 1$, $m = 2$, $d = 1$, $l = 1$, $a_1(t) = 3, \mu(t) = \frac{1}{3}, \rho = \pi, a_{11}(t) = 2, a_{121}(t) = 1$, $\tau_1 = \frac{3\pi}{2}, p_1(x, t) = \frac{2}{3}, q_{111}(x, t, \xi) = 3, q_{121}(x, t, \xi) = 1, g_1(t, \xi) = t + \xi, a_2(t) = \frac{2}{3}$, $a_{211}(t) = 1, a_{221}(t) = 4$, $p_2(x, t) = 3$, $q_{211}(x, t, \xi) = 1, q_{221}(x, t, \xi) = 3, a = -\pi, b = -\frac{\pi}{2}$. It is easy to see that all the conditions of Theorem 3.3 are fulfilled. Thus all solutions of the problem (35), (36) are oscillatory in $(0, \pi) \times [0, \infty)$. In fact, such a solution is $u_1(x, t) = \sin x \cos t, u_2(x, t) = \sin x \sin t$.

REFERENCES

1. D. P. Mishev and D. D. Bainov, *Appl. Math. Comput.* **28** (1988), 97-111.
2. X. L. Fu and W. Zhuang, *J. math. Anal. Appl.* **191** (1995), 473-489.
3. B. T. Cui, *Demonstratio Math.* **29** (1996), 61-68.
4. B. S. Lalli, Y. H. Yu and B. T. Cui, *Indian J. pure appl. Math.* **25** (1994), 387-397.
5. B. T. Cui and W. N. Li, *J. comput. appl. Math.* **95** (1998), 155-56.
6. B. S. Lalli, Y. H. Yu and B. T. Cui, *Appl. Math. Comput.* **53** (1993), 97-110.
7. B. T. Cui, Y. H. Yu and S. Z. Lin, *Acta Math. Appl. Sinica* **19** (1996), 80-88. [*In Chinese*]
8. D. Bainov, B. T. Cui and E. Minchev, *J. Comput. Appl. Math.* **72** (1996), 309-318.
9. Y. K. Li, *Acta Math. Sinica* **40** (1997), 100-105. [*In Chinese*]
10. W. N. Li and B. T. Cui, *Demonstratio Math.* **31** (1998), No. 5. 813-24.
11. W. N. Li, *J. Chongqing Teachers College* **15** (1998), **2**: 45-49. [*In Chinese*]
12. J. Ruan, *Acta math. Sinica* **30** (1987), 661-670. [*In Chinese*].