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OSCILLATION OF CERTAIN NEUTRAL HYPERBOLIC EQUATIONS*

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This paper studies a class of delay hyperbolic equations boundary value problem, and obtain oscillatory criteria for the solutions of eq. (*E*) satisfying boundary value condition (*B*). These results extend some of the known theorems in the literature.

Key Words : Deviating Arguments; Hyperbolic Equation; Forced Oscillation

1. INTRODUCTION

Recently, there has been an increasing interest in oscillation theory of hyperbolic partial functional differential equations. We can refer to [1-6] and their cited references. The purpose of this paper is to extend some of the known results in the literature to the more general equations with continuous distributed deviating arguments

$$\frac{\partial^2}{\partial t^2}\left[u+\sum_{i=1}^{n}\lambda_i(t)\,u(x,t-\tau_i)\right]=a(t)\,\Delta u+\sum_{j=1}^{m}a_j(t)\,\Delta u(x,t-\rho_j)$$

$$-c(x,t,u)-\int_a^b q(x,t,\xi)\,u[x,g(t,\xi)]\,d\sigma(\xi)+f(x,t) \qquad \dots (E)$$

with the boundary value condition

$$\frac{\partial u}{\partial n}=\psi(x,t) \text{ on } (x,t)\in\partial\Omega\times R_+ \qquad \dots (B)$$

and to obtain some oscillatory criteria, where τ_i = const. > 0, ρ_j = const. > 0, $\Delta u=\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}$, $(x,t)\in\Omega\times R_+=G$, $R_+=[0,+\infty)$, $u=u(x,t)$. Ω is a bounded domain in R^n with a piecewise smooth

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boundary $\partial\Omega$. $\psi(x, t)$ is a continuous function on $\partial\Omega \times R_+$. n denotes the unit exterior normal vector to $\partial\Omega$.

It is easy to see that eq. (E) includes the following delay hyperbolic equations

$$\frac{\partial^2}{\partial t^2}\left[u + \sum_{i=1}^{n} \lambda_i(t)\, u(x, t-\tau_i)\right] = a(t)\,\Delta u + \sum_{j=1}^{m} a_j(t)\,\Delta u(x, t-\rho_j)$$

$$- c(x, t, u) - \sum_{j=1}^{m} q_j(x, t) u[x, g_j(t)] + f(x, t) \quad \ldots (E')$$

Kreith, Kusono and Yoshida[2]; Chen and Yu[3] concerned the following equations,

$$\frac{\partial^2 u}{\partial t^2} = \Delta u - c(x, t, u) + f(x, t) \quad \ldots (E_1)$$

$$\frac{\partial^2}{\partial t^2}[u + \lambda u(x, t-\tau)] = \Delta u - c(x, t, u) + f(x, t) \quad \ldots (E_2)$$

respectively. Those equations all are special cases of eq. (E').

Suppose that the following conditions (H) hold.

(H_1) $a(t), a_j(t), \lambda_i(t) \in C(R_+, R_+)$, $q(x, t, \xi) \in C(\bar{\Omega} \times R_+ \times [a, b], R_+)$

$i = 1, 2, \ldots, n; j = 1, 2, \ldots m;$

(H_2) $g(t, \xi) \in C(R_+ \times [a, b], R)$ $g(t, \xi) \le t$, $\xi \in [a, b]$; $g(t, \xi)$

are nondecreasing with to t and ξ, respectively; and $\lim_{t \to +\infty} \min_{\xi \in [a, b]} \{g(t, \xi)\} = +\infty$;

(H_3) $c(x, t, u)$ $C(\bar{\Omega} \times R_+ \times R, R)$; $c(x, t, \xi) \ge p(t)\varphi(\xi)$,

in which $p(t) \in (R_+, R_+)$, $\varphi(\xi) \in C([a, b], R)$; $\varphi(\xi)$ is a positive and convex function in $(0, +\infty)$, and $c(x, t, -\xi) = -c(x, t, \xi)$;

(H_4) $f(x, t) \in C(\Omega \times R_+, R)$; $\sigma(\xi) \in ([a, b], R)$

is nondecreasing, integral of eq. (E) is Stieltjes integral.

Definition 1 — A function $u(x, t) \in C^2(G) \cap C^1(\bar{G})$ is called a solution of the problem (E), (B), if it satisfies eq. (E) in the domain G along with the corresponding boundary condition.

Definition 2 — A solution $u(x, t)$ of eq. (E) is called oscillatory in the domain G if for each positive number μ there exists a point $(x_0, t_0) \in \Omega \times [\mu, +\infty)$ such that the condition $u(x_0, t_0) = 0$ holds.

Definition 3 — A function $U(t)$ is alled eventually positive (negative) if there exists a number $t_1 \geq 0$ such that $U(t) > 0(<0)$ holds for all $t_1 > 0$.

2. Oscillation Criteria

Lemma — Suppose that (H_1)-(H_4) hold. If u is a positive solution of the problem (E), (B) in $\Omega \times [\mu, +\infty)$, $\mu \geq 0$, then the function

$$U(t) = \frac{\int_\Omega u(x,t)dx}{\int_\Omega dx} \quad \text{... (2.1)}$$

satisfies the following differential inequality

$$\frac{d^2}{dt^2}\left[U(t) + \sum_{i=1}^{n} \lambda_i(t) U(t, \tau_i) \right] + p(t)\varphi(U) + \int_a^b Q(t,\xi)\, U[g(t,\xi)]\, d\sigma(\xi) \leq H(t) \quad \text{... (2.2)}$$

in which $Q(t,\xi) = \min_{x \in \Omega} \{q(x,t,\xi)\}$,

$$H(t) = \left[\int_\Omega dx\right]^{-1} \int_{\partial\Omega} \left[a(t)\,\psi(x,t) + \sum_{j=1}^{m} a_j(t)\,\psi(x, t-\rho_j) \right] d\omega + \int_\Omega f(x,t)\,dx$$

Proof : Let $u(x, t)$ be an positive solution of the problem (E), (B) in $\Omega \times [\mu, +\infty)$, for $\mu \geq 0$. From (H_2), there exist a $t_1 \geq \mu$ such that $u(x, g(t,\xi)) > 0, (x,t,\xi) \in \Omega \times [t_1, +\infty) \times [a,b], u(x, t-\tau_i) > 0, (x,t) \in \Omega \times [t_1, +\infty)$.

Using the Green's formula, we have

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} d\omega = \int_\Omega \psi d\omega,\ t \geq t_1 \quad \text{... (2.3)}$$

$$\int_\Omega \Delta u(x, t-\rho_j)dx = \int_{\partial\Omega} \frac{\partial u(x,t-\rho_j)}{\partial n} d\omega = \int_\Omega \psi(x, t-\rho_j)d\omega,\ t \geq t_1 \quad \text{... (2.4)}$$

Integrating eq. (E) with respect to x over the domain Ω, we have

$$\frac{d^2}{dt^2}\left[\int_\Omega dx + \sum_{i=1}^{n} \lambda_i(t) \int_\Omega u(x, t-\tau_i)\,dx \right]$$

$$= a(t) \int_{\partial\Omega} \psi d\omega + \sum_{j=1}^{m} a_j(t) \int_{\partial\Omega} \psi(x, t-\rho_j)\, d\omega$$

$$-\int_{\Omega} c(x,t,u)dx - \int_{\Omega}\int_{a}^{b} q(x,t,\xi)\, u\,[x, g(t,\xi)]\, d\sigma(\xi)\, dx + \int_{\Omega} f(x,t)\, dx,\ t \ge t_1 \qquad \text{... (2.5)}$$

Using the condition (H_3) and Jensen's inequality, we have

$$\int_{\Omega} c(x,t,u)dx \ge p(t) \int_{\Omega} \varphi(u)dx \ge p(t)\, \varphi\left(\frac{\int_{\Omega} udx}{\int_{\Omega} dx}\right) \int_{\Omega} dx,\ t \ge t_1 \qquad \text{... (2.6)}$$

and noticing that

$$\int_{\Omega} q(x,t,\xi)\, u[x, g(t,\xi)]\, dx \ge Q(t,\xi) \int_{\Omega} u[x, g(t,\xi)]\, dx \qquad \text{... (2.7)}$$

and

$$\int_{\Omega}\int_{a}^{b} q(x,t,\xi)\, u[x, g(t,\xi)]\, d\sigma(\xi)dx = \int_{a}^{b}\int_{\Omega} q(x,t,\xi)u\,[x, g(t,\xi)]\, dx\, d\sigma(\xi) \qquad \text{... (2.8)}$$

we have

$$\frac{d^2}{dt^2}\left[\int_{\Omega} udx + \sum_{i=1}^{n} \lambda_i(t) \int_{\Omega} u(x, t-\tau_i)dx\right] + p(t)\, \varphi(U)$$

$$+ \int_{a}^{b} Q(t,\xi) \int_{\Omega} u[x, g(t,\xi)]\, dx$$

$$\le a(t) \int_{\partial\Omega} \psi d\omega + \sum_{j=1}^{m} a_j(t) \int_{\partial\Omega} \psi(x, t-\rho_j)\, d\omega + \int_{\Omega} f(x,t)\, dx,\ t \ge t_1 \qquad \text{... (2.9)}$$

then $U(t)$ is a solution of the inequality (2.2).

Theorem 1 — *Suppose that* $(H_1)-(H_4)$ *hold. If the differential inequalities with distributed deviating arguments*

$$\frac{d^2}{dt^2}\left[U(t) + \sum_{i=1}^{n} \lambda_i(t)\, U(t-\tau_i)\right] + p(t)\, \varphi(U)$$

$$+ \int_{a}^{b} Q(t,\xi)\, U[g(t,\xi)]\, d\sigma(\xi) \le H(t) \qquad \text{... (2.10)}$$

$$\frac{d^2}{dt^2}\left[U(t)+\sum_{i=1}^{n}\lambda_i(t)\,U(t-\tau_i)\right]+p(t)\,\varphi(U)$$

$$+\int_a^b Q(t,\xi)\,U[g(t,\xi)]\,d\sigma(\xi)\le -H(t) \qquad \dots (2.11)$$

have no eventually positive solution, then the every solution of eq. (E), (B) is oscillatory in G.

PROOF : Assume to the contrary that there exists a nonoscillatory solution $u(x, t)$ of the problem (*E*), (*B*). If $u(x, t) > 0$, $(x, t) \in \Omega \times [\mu, +\infty)$, $\mu \ge 0$, then from Lemma 1 it follows that $U(t)$ defined by (2.1) is an eventually positive solution of the inequality (2.10), which contradicts the condition of the Theorem.

If $u(x, t) < 0$, $(x, t) \in \Omega \times [\mu, +\infty)$, $\mu \ge 0$, let $v(x, t) = -u(x, t)$, then $v(x, t)$ is a positive solution of the problem

$$\frac{\partial^2}{\partial t^2}\left[u+\sum_{i=1}^{n}\lambda_i(t)u(x,t-\tau_i)\right]=a(t)\,\Delta u+\sum_{j=1}^{m}a_j(t)\,\Delta u(x,t-\rho_j)$$

$$-\,c(x,t,u)-\int_a^b q(x,t,\xi)\,u[x,g(t,\xi)]\,d\sigma(\xi)+f(x,t),\ (x,t)\in G$$

$$\frac{\partial u}{\partial n}=-\,\psi(x,t),\ \text{on}\ (x,t)\in\partial\Omega\times R_+$$

then we can also get

$$V(t)=\frac{\int_\Omega v(x,t)dx}{\int_\Omega dx}$$

be an eventually positive solution of the inequality (2.11), which contradicts the condition of the Theorem as well. This completes the proof of Theorem 1.

Remark : Theorem 1 generalize the Theorem 1 in [2, 3, 5] and Theorem 3 in [6].

Theorem 2 — *Suppose that* $(H_1)-(H_4)$ *hold and*

$$\lim_{t\to+\infty}\inf\int_{t_1}^{t}\left(1-\frac{s}{t}\right)\left\{\int_{\partial\Omega}\left[a(s)\psi(x,s)+\sum_{j=1}^{m}a_j(s)\,\psi(x,s-\rho_j)\right]d\omega\right.$$

$$\left.+\int_\Omega f(x,s)dx\right\}ds=-\infty \qquad \dots (2.12)$$

$$\lim_{t \to +\infty} \inf \int_{t_1}^{t} \left(1 - \frac{s}{t}\right)\left\{ \int_{\partial\Omega} \left[a(s)\psi(x, s) + \sum_{j=1}^{m} a_i(s)\, \psi(x, s - \rho_j) \right] d\omega \right.$$

$$\left. + \int_{\Omega} f(x, s)dx \right\} ds = +\infty \qquad \text{...(2.13)}$$

for sufficiently large t_1, then every solution of eq. (E), (B) is oscillatory in G.

PROOF : Assume to there exist a nonoscillatory solution. If $u(x, t) > 0$, $(x, t) \in \Omega \times [\mu, +\infty)$, for $\mu \geq 0$, then from Lemma 1 it follows that the function $U(t)$ defined in (2.1) is an eventually positive solution of the inequality (2.1). Then

$$\frac{d^2}{dt^2}\left[U(t) + \sum_{i=1}^{n} \lambda_i(t)\, U(t - \tau_i) \right] \leq H(t), \; t \geq t_1 \geq \mu. \qquad \text{... (2.14)}$$

Integrating the above inequality twice in the segment $[t_1, t]$, we get

$$U(t) + \sum_{i=1}^{n} \lambda_i(t)\, U(t - \tau_i) \leq c_1 + c_2 (t - t_1) + \int_{t_1}^{t} \int_{t_1}^{\eta} H(s)ds\, d\omega, \; t \geq t_1 \geq \mu \qquad \text{... (2.15)}$$

in which c_1, c_2 are constants. Noticing that

$$\int_{t_1}^{t} \int_{t_1}^{\eta} H(s)dsd\eta = \int_{t_1}^{t} (t - s)H(s)ds$$

Dividing both sides of the last inequality by t, we have

$$\frac{1}{t}\left[U(t) + \sum_{i=1}^{n} \lambda_i(t)\, U(t - \tau_i) \right] \leq \frac{c_1}{t} + c_2\left(1 - \frac{t_1}{t}\right) + \int_{t_1}^{t} \left(1 - \frac{s}{t}\right) H(s)ds. \qquad \text{... (2.16)}$$

From (2.12), we have

$$\lim_{t \to +\infty} \inf \frac{1}{t}\left[U(t) + \sum_{i=1}^{n} \lambda_i(t)\, U(t - \tau_i) \right] = -\infty$$

which contradicts the assumption that $U(t > 0)$.

If

$u(x, t) < 0$, $(x, t) \in \Omega \times [\mu, +\infty)$, $\mu \geq 0$, let $v(x, t) = -u(x, t)$, then

$$V(t) = \frac{\int_{\Omega} \upsilon(x, t)dx}{\int_{\Omega} ds}$$

is an eventually positive solution of the inequality (2.11).

Using (2.13), we get

$$\liminf_{t \to +\infty} \int_{t_1}^{t} \left(1 - \frac{s}{t}\right)(-H(s))\, ds = -\limsup_{t \to +\infty} \int_{t_1}^{t} \left(1 - \frac{s}{t}\right) H(s) ds = -\infty$$

thus, it can be proved by using the above-mentioned similar method. This completes the proof of Theorem 2.

Remark : Theorem 2 generalize the Theorem 2 in [2, 3, 5] and Theorem 4 in [6].

Example — Consider the following equation

$$\frac{\partial^2}{\partial t^2}[u + u(x, t - \pi)] = \frac{1}{2}\Delta u - \frac{1}{2}u - 4\int_{-\pi}^{0} u(x, t + \xi)d\xi$$

$$+ e^t \cos x \sin t + e^{t-\pi} \cos x \cos t \qquad \ldots (E^*)$$

with the boundary condition

$$\frac{\partial u}{\partial n}(0, t) = 0, \frac{\partial u}{\partial n}\left(\frac{\pi}{2}, t\right) = -e^{-t} \sin t \qquad \ldots (B^*)$$

where

$$n = 1,\ \Omega = \left(0, \frac{\pi}{2}\right), a(t) = \frac{1}{2}, q(x, t, \xi) = 4, c(x, t, u) = \frac{1}{2}u, g(t, \xi) = t + \xi, g(t, \xi)$$

$$= t + \xi, f(x, t) = e^t \cos x \sin t + e^{t-\pi} \cos x \cos t,\ \psi(0, t) = 0,\ \psi\left(\frac{\pi}{2}, t\right) = -e^t \sin t.$$

It is to easy see the condition (H) hold, and

$$\int_{t_1}^{t} \left(1 - \frac{s}{t}\right)\left\{\int_{\partial\Omega}\left[a(s)\psi(x, s) + \sum_{j=1}^{m} a_j(s)\, \psi(x, s - \rho_j)\right]d\omega + \int_{\Omega} f(x, s)dx\right\} ds$$

$$= e^t\left\{\left(\frac{1}{2} - \frac{e^{-\pi}}{2} + \frac{1}{2}t\right) - \left(\frac{1}{4}\frac{e^{-\pi}}{2} + \frac{e^{-\pi}}{t}\right)cost\right\} + \frac{c}{t}$$

in which c is a constant. Hence all the conditions of Theorem 2 hold. It follows from Theorem 2 that every solution of the problem (E^*), (B^*) is oscillatory in $\left(0, \frac{\pi}{2}\right) \times [0, +\infty)$. In fact $u(x, t) = e^t \cos x \sin t$ is one such solution.

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