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ON CAUCHY'S BOUND FOR THE MODULI OF ZEROS OF A POLYNOMIAL

V. K. JAIN

Mathematics Department, I.I.T., Kharagpur 721 302, *India*

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In this paper, we have obtained a bound for the moduli of the zeros of a polynomial. In many cases, this bound is better than many known bounds.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let

$$p(z) = z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z + a_0 \qquad \ldots (1.1)$$

be a polynomial of degree n. Then we have the following classical result, due to Cauchy[2] on the location of zeros of a polynomial.

Theorem A — *All zeros of the polynomial p(z) lie in the disc*

$$|z| < 1 + A, \qquad \ldots (1.2)$$

where

$$A = \max_{0 \le k \le n-1} |a_k|. \qquad \ldots (1.3)$$

Joyal, Labelle and Rahman[4] improved Cauchy's bound (1.2) and obtained the following result.

Theorem B — *All zeros of the polynomial p(z) lie in the disc*

$$|z| \le 1 + r, \qquad \ldots (1.4)$$

where

$$r = 1/2\,(-b + \sqrt{b^2 + 4\beta}), \qquad \ldots (1.5)$$

$$\beta = \max_{0 \le j \le n-2} |a_j|, \qquad \ldots (1.6)$$

$$b = 1 - |a_{n-1}|. \qquad \ldots (1.7)$$

Dutt and Govil[3] also improved Cauchy's bound and obtained

Theorem C — *All zeros of the polynomial p(z) lie in the ring shaped region*

$$\frac{|a_0|}{2(1+A)^{n-1}(1+An)} \le |z| \le 1 + A\left(1 - \frac{1}{(1+A)^n}\right) \qquad \text{... (1.8)}$$

There is one disadvantage in all these bounds : they don't reflect the fact that for $A \to 0$, all zeros approach the origin $z = 0$. Accordingly Boese and Luther[1] obtained a bound and proved.

Theorem D — *All zeros of the polynomial p(z) lie in the disc* $|z| < R$, *where*

$$R = \begin{cases} (A[1-nA]/[1-(nA)^{1/n}]), & A < 1/n \\ \text{Min}\,((1+A)(1-A/[(1+A)^{n+1}-nA]), \\ 1+2(nA-1)/(n+1)), & A \ge 1/n. \end{cases}$$

In this paper, we have also obtained a bound, with a similar intention, by using $|a_{n-1}|$ and $\beta = \max\limits_{0 \le j \le n-2} |a_j|$, instead of A. Our bound, in many cases, turns out to be better than many of the known bounds. More precisely we prove.

Theorem 1 — *All zeros of the polynomial p(z) lie in the disc*

$$|z| \le R(|a_{n-1}|, \beta),$$

$$R(|a_{n-1}|, \beta) = \begin{cases} |a_{n-1}|, \quad \text{for } \beta = 0, & \text{...(1.9)} \\[2ex] [(|a_{n-1}| - \beta)(|a_{n-1}| + (n-1)\beta)^{(n-1)/n} + \beta - \\ |a_{n-1}|^2 - (n-1)\beta|a_{n-1}|]^{1/n}[1-(|a_{n-1}| + (n-1)\beta)^{1/n}]^{-1/n}, \\ \quad \text{for } \beta \ne 0 \;\&\; |a_{n-1}| + (n-1)\beta < 1, & \text{...(1.10)} \\[2ex] 1, \quad \text{for } \beta \ne 0 \;\&\; |a_{n-1}| + (n-1)\beta = 1, & \text{...(1.11)} \\[2ex] (|a_{n-1}|\rho^{n-1}(\rho-1) + \beta(\rho^{n-1}-1) + \rho^n)^{1/(n+1)}, \quad \text{for} \\ \quad \beta \ne 0, |a_{n-1}| + (n-1)\beta > 1 \;\&\; |a_{n-1}| < \beta & \text{...(1.12)} \\[2ex] (|a_{n-1}|\lambda^{n-1}(\lambda-1) + \beta(\lambda^{n-1}-1) + \lambda^n)^{1/(n+1)}, \quad \text{for} \\ \quad \beta \ne 0, |a_{n-1}| + (n-1)\beta > 1 \;\&\; |a_{n-1}| \ge \beta, & \text{...(1.13)} \end{cases}$$

$$\rho = \min\,[\left\{|a_{n-1}| + (n-1)\beta\right\},$$

$$\left\{(1+|a_{n-1}|)(|a_{n-1}| + (n-1)\beta) + \beta - |a_{n-1}|\right\}^{1/2}] \qquad \text{... (1.14)}$$

$$\lambda = \min\,(\rho, \eta), \qquad \text{... (1.15)}$$

$$\eta = (1+|a_{n-1}|)\,[1 - (\beta(1+|a_{n-1}|)^{-n+1} + (|a_{n-1}| - \beta|)(1+|a_{n-1}|)^{-2})\, x$$

$$(1 - n\beta(1+|a_{n-1}|)^{-(n+1)} - (|a_{n-1}| - \beta)(1+|a_{n-1}|)^{-2})^{-1}]. \qquad \text{... (1.16)}$$

Remark : (*i*) For the polynomial

$p_1(z) = z^5 + a_4 z^4 + a_0;\ |a_4| = .3, |a_0| = .15$, (1.10) gives the bound

$|z| \leq R(|a_4|, \beta) \approx .9697,$

better than other known bounds

$$\left.\begin{array}{l} |z| \leq R \approx 1.1667 \text{ ([1])}, \\ |z| \leq R \approx 1.1720 \text{ ([4])}, \\ |z| \leq R \approx 1.2192 \text{ ([3])}, \\ |z| \leq R \approx 1.0223 \text{ ([5])}, \\ |z| \leq R \approx 1.0440 \text{ ([7])}, \\ |z| \leq R, \quad R > 1 \text{ ([6])}. \end{array}\right]$$

(*ii*) For the polynomial

$p_2(z) = z^{11} + a_{10} z^{10} + a_0;\ |a_{10}| = .01, |a_0| = .1$, (1.12) gives the bound

$|z| \leq R(|a_{10}|, \beta) \approx 1.0100,$

better than the bounds

$$\left.\begin{array}{l} |z| \leq 1.0167 \text{ ([1])}, \\ |z| \leq 1.0924 \text{ ([4])}, \\ |z| \leq 1.0650 \text{ ([3])}. \end{array}\right]$$

(*iii*) For the polynomial

$p_3(z) = z^{10} + a_9 z^9 + a_8 z^8 + a_0;\ |a_9| = .3, |a_8| = |a_0| = .29$, (1.13) gives the bound

$|z| \leq R(|a_9|, \beta) \approx 1.2654$

better than the bounds

$$\left.\begin{array}{l} |z| \leq 1.2739 \text{ ([1])}, \\ |z| \leq 1.2923 \text{ ([4])}, \\ |z| \leq 1.2783 \text{ ([3])}. \end{array}\right]$$

The following corollary gives a better location of the zeros at the cost of a higher computational effort.

Corollary 1 — The angle independent bound $R(|a_{n-1}|, \beta)$ for all zeros of $p(z)$ of Theorem 1 can be replaced by $R(|a_{n-1}(\theta)|, \beta(\theta))$, where

$$-a_k = A_k e^{i\alpha_k},\ 0 \leq k \leq (n-1), \qquad \text{... (1.17)}$$

$$\cos_+(t) = \max(0, \cos t), \text{ for real } t, \qquad \text{... (1.18)}$$

$$\left.\begin{aligned} |a_{n-1}(\theta)| &= A_{n-1}\cos_+(\alpha_{n-1}-\theta), \\ \beta(\theta) &= \max_{0\le j\le n-2} A_j\cos_+(\alpha_j-(n-j)\theta), \end{aligned}\right] \theta\in[0,2\pi) \qquad \text{... (1.19)}$$

Using Corollary 1, one easily gets

Corollary 2 — All zeros of the polynomial $p(z)$ lie in the closed bounded region S with boundary

$$T = \{\langle R(|a_{n-1}|(\theta), \beta(\theta)), \theta\rangle : \theta\in[0,2\pi]\}.$$

2. Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1 — Let $z\ (\neq 0) = re^{i\theta}$ be a zero of $p(z)$. We, therefore, have

$$r^n \le |a_{n-1}|r^{n-1} + \beta(r^{n-2}+r^{n-3}+\ldots+1). \qquad \text{... (2.1)}$$

Firstly, we think of the possibility

$$\beta = 0.$$

And so, from (2.1), we have

$$|z| = r \le |a_{n-1}| = R(|a_{n-1}|, \beta), \text{ (by (1.9))}.$$

Now onwards, we will assume

$$\beta\neq 0.$$

Let $X = R$ be the unique positive root of the equation

$$X^n - |a_{n-1}|X^{n-1} - \beta(X^{n-2}+X^{n-3}+\ldots.+1) = 0. \qquad \text{... (2.2)}$$

Then, we obviously have

$$r \le R \qquad \text{... (2.3)}$$

(i) Now we consider the possibility

$$|a_{n-1}| + (n-1)\beta < 1, \qquad \text{... (2.4)}$$

thereby implying

$$R < 1. \qquad \text{... (2.5)}$$

By (2.3), (2.5) and (2.1), we have

$$r^n \le |a_{n-1}| + \beta(n-1),$$

i.e.,

$$r \leq \left\{ | a_{n-1} | + (n-1)\, \beta \right\}^{1/n}, \qquad \text{... (2.6)}$$

which, by (2.1), implies

$$r^n \leq | a_{n-1} \left\{ | a_{n-1} | + (n-1)\, \beta \right\}^{(n-1)/n} + \beta \;\; [\{| a_{n-1} | + (n-1)\, \beta \;\})^{(n-2)/n} +$$

$$\{| a_{n-1} | + (n-1)\, \beta\}^{(n-3)/n} + ... + 1],$$

$$= | a_{n-1} | \left\{ | a_{n-1} | + (n-1)\beta \right\}^{(n-1)/n} +$$

$$\beta \left[\frac{1 - (| a_{n-1} | + (n-1)\, \beta)^{(n-1)/n}}{1 - (| a_{n-1} | + (n-1)\beta)^{1/n}} \right],$$

thereby giving us

$$| z | = r \leq R\, (| a_{n-1} |, \beta), \qquad \text{(by (1.10))}$$

(*ii*) For the next possibility

$$| a_{n-1} | + (n-1)\, \beta = 1,$$

we obviously have

$$R = 1,$$

thereby implying, by (2.3)

$$| z | = r \leq 1 = R\, (| a_{n-1} |, \beta), \qquad \text{(by (1.11))}$$

(*iii*) Now we consider the last possibility

$$| a_{n-1} | + (n-1)\, \beta > 1. \qquad \text{... (2.7)}$$

By (2.7), we have

$$R > 1. \qquad \text{... (2.8)}$$

Further, R satisfies the relation

$$R^n = | a_{n-1} | R^{n-1} + \beta\, (R^{n-2} + R^{n-3} + \ldots + 1), \qquad \text{... (2.9)}$$

which, by (2.8), implies

$$R^n < | a_{n-1} | R^{n-1} + \beta\, (R^{n-1} + \ldots + R^{n-1}),$$

i.e.,

$$R < |a_{n-1}| + (n-1)\beta. \quad \text{... (2.10)}$$

Again, by (2.8) and (2.9), we have

$$R^n = |a_{n-1}| R^{n-1} + \beta \frac{R^{n-1} - 1}{R - 1}, \quad \text{... (2.11)}$$

$$< |a_{n-1}| R^{n-1} + \beta \frac{R^{n-1}}{R-1},$$

i.e.,

$$R < |a_{n-1}| + \frac{\beta}{R-1},$$

i.e.

$$R^2 < (1 + |a_{n-1}|) R + (\beta - |a_{n-1}|),$$

$$< (1 + |a_{n-1}|)(|a_{n-1}| + (n-1)\beta) + (\beta - |a_{n-1}|), \quad \text{(by (2.10))}$$

i.e.,

$$R < [(1 + |a_{n-1}|)(|a_{n-1}| + (n-1)\beta) + \beta - |a_{n-1}|]^{1/2}, \quad \text{... (2.12)}$$

and therefore, we have

$$R < \min\left(\{|a_{n-1}| + (n-1)\beta\},\ [(1 + |a_{n-1}|)\left\{|a_{n-1}| + (n-1)\beta\right\}\right.$$

$$\left. + (\beta - |a_{n-1}|)]^{1/2}\right), \text{ (by (2.10) \& (2.12))},$$

$$= \rho, \text{ (by (1.14))}. \quad \text{... (2.13)}$$

Further, by (2.11), we have

$$R^{n+1} = |a_{n-1}| R^{n-1}(R-1) + \beta(R^{n-1} - 1) + R^n,$$

i.e.,

$$R = \{|a_{n-1}| R^{n-1}(R-1) + \beta(R^{n-1} - 1) + R^n\}^{1/(n+1)}. \quad \text{... (2.14)}$$

Now we consider a function ψ on $[R, \infty)$ defined by

$$\psi(y) = \left\{|a_{n-1}| y^{n-1}(y-1) + \beta(y^{n-1} - 1) + y^n\right\}^{1/(n+1)}. \quad \text{... (2.15)}$$

$\psi(y)$ is a strictly increasing function on $[R, \infty)$ and so, we have

$$\psi(y) > \psi(R), y > R$$

$$= R,\ y = R, \text{ (by (2.15) \& (2.14))}. \quad \text{... (2.16)}$$

Also, as R is a unique positive root of (2.2), we have for $y > R$

$$y^n > |a_{n-1}| y^{n-1} + \beta (y^{n-2} + y^{n-1} + ... + 1),$$

$$= |a_{n-1}| y^{n-1} + \beta \left(\frac{y^{n-1} - 1}{y-1} \right)$$

i.e.,

$$y^{n+1} > |a_{n-1}| y^{n-1} (y-1) + \beta (y^{n-1} - 1) + y^n,$$

i.e.,

$$y > \psi(y). \qquad ... (2.17)$$

From (2.16), (2.17) and (2.13), we have

$$R < \psi(\rho) < \rho. \qquad ... (2.18)$$

Now for the last possibility specified by (2.7), we consider two mutually exclusive cases :

(A) $|a_{n-1}| \geq \beta$,

(B) $|a_{n-1}| < \beta$.

Firstly, we think of (A). By (2.8) and (2.9), we have

$$R^n > |a_{n-1}| + (n-1)\beta,$$

i.e.,

$$R > \left\{ |a_{n-1}| + (n-1)\beta \right\}^{1/n} > \left\{ |a_{n-1}| + (n-1)\beta \right\}^{1/(n+1)}, \qquad \text{(by (2.7))},$$

thereby implying

$$R = \left\{ |a_{n-1}| + (n-1)\beta \right\}^{1/(n+1)} s, \qquad ... (2.19)$$

for certain

$$s > 1 \qquad ... (2.20)$$

Also, by (2.9) we have

$$R^n = |a_{n-1}| R^{n-1} + \beta \left(\frac{R^{n-1} - 1}{R-1} \right), \qquad ... (2.21)$$

i.e.,

$$R^{n+1} = (|a_{n-1}| + 1) R^n - (|a_{n-1}| - \beta) R^{n-1} - \beta, \qquad ... (2.22)$$

which, by (2.19), implies

$$s+\beta\left\{|a_{n-1}|+(n-1)\beta\right\}^{-1}s^{-n}+(|a_{n-1}|-\beta)\left\{|a_{n-1}|+(n-1)\beta\right\}^{-2/(n+1)}s^{-1}$$

$$=(|a_{n-1}|+1)\left\{|a_{n-1}|+(n-1)\beta\right\}^{-1/(n+1)}. \quad \text{... (2.23)}$$

Therefore, we have

$$s<(|a_{n-1}|+1)\left\{|a_{n-1}|+(n-1)\beta\right\}^{-1/(n+1)}, \text{ (by (2.7) \& (A))},$$

$$= F, \text{ say.} \quad \text{... (2.24)}$$

Now, on the interval $[s, F]$, we consider the function g defined by

$$g(t)=t+\beta\left\{|a_{n-1}|+(n-1)\beta\right\}^{-1}t^{-n}+$$

$$(|a_{n-1}|-\beta)\left\{|a_{n-1}|+(n-1)\beta\right\}^{-2/(n+1)}t^{-1}. \quad \text{... (2.25)}$$

We have

$$g'(t)=1-n\beta\left\{|a_{n-1}|+(n-1)\beta\right\}^{-1}t^{-n-1}-$$

$$(|a_{n-1}|-\beta)\left\{|a_{n-1}|+(n-1)\beta\right\}^{-2/(n+1)}t^{-2}, \quad \text{... (2.26)}$$

a strictly increasing continuous function on $[s, F]$ and so, we have

$$g'(t)<g'(F), \quad s<t<F,$$

thereby implying

$$\int_S^F g'(t)dt<(F-s)\,g'(F),$$

i.e.

$$(s-F)\,g'(F)<g(s)-g(F) = F-g(F), \text{ (by (2.23), (2.24) \& (2.25))}. \quad \text{... (2.27)}$$

Also, by (2.26) and (2.24), we have

$$g'(F)=\left\{|a_{n-1}|+(n-1)\beta\right\}^{-1}F^{-n-1}$$

$$\left\{(1+|a_{n-1}|)^{(n+1)}-n\beta-(|a_{n-1}|-\beta)(1+|a_{n-1}|)^{n-1}\right\},$$

$$=\left\{|a_{n+1}|+(n-1)\beta\right\}^{-1}F^{-n-1}\left\{(1+|a_{n-1}|)^{n}-n\beta\right\}|(1+|a_{n-1}|^{n-1}$$

$(\beta + |a_{n-1}|^2)], > 0$, (by (A) and (2.7)).

Therefore, we have by (2.27)

$$s < F + \frac{F - q(F)}{g'(F)}, \qquad \dots (2.28)$$

$$= F + [F - F - \beta\left\{|a_{n-1}| + (n-1)\beta\right\}^{-1} F^{n} -$$

$$(|a_{n-1}| - \beta)\left\{|a_{n-1}| + (n-1)\beta\right\}^{-2/(n+1)} F^{1}] \times$$

$$[1 - n\beta\left\{|a_{n-1}| + (n-1)\beta\right\}^{-1} F^{n-1} -$$

$$(|a_{n-1}| - \beta)\left\{|a_{n-1}| + (n-1)\beta\right\}^{-2/(n+1)} F^{2}]^{-1}, \text{(by (2.25) and (2.26))},$$

$$= F(1 - [\beta\left\{|a_{n-1}| + (n-1)\beta\right\}^{-1} F^{n-1} -$$

$$(|a_{n-1}| - \beta)\left\{|a_{n-1}| + (n-1)\beta\right\}^{-2/(n+1)} F^{2}] \times$$

$$[1 - n\beta\left\{|a_{n-1}| + (n-1)\beta\right\}^{-1} F^{n-1} -$$

$$(|a_{n-1}| - \beta)\left\{|a_{n-1}| + (n-1)\beta\right\}^{-2/(n+1)} F^{2}]^{-1},$$

$$= (1 + |a_{n-1}|)\left\{|a_{n-1}| + (n-1)\beta\right\}^{-1/(n+1)} [1 -$$

$$\left\{\beta(1 + |a_{n-1}|)^{-(n+1)} + (|a_{n-1}| - \beta)(1 + |a_{n-1}|)^{-2}\right\} \times$$

$$\left\{1 - n\beta(1 + |a_{n-1}|)^{-(n+1)} - (|a_{n-1}| - \beta)(1 + |a_{n-1}|)^{-2}\right\}^{-1}], \text{(by (2.24))},$$

which, by (2.19) and (1.16), implies

$$R < \eta. \qquad \dots (2.29)$$

By (2.13) and (2.29), we have

$$R < \min(\rho, \eta) = \lambda, \text{ (by (1.15))}. \qquad \dots (2.30)$$

and therefore,

$$R < \psi(\lambda) < \lambda, \qquad \text{(by (2.16) and (2.17))},$$

implying thereby, by (2.3)

$$|z| = r < \psi(\lambda) = R(|a_{n-1}|, \beta), \qquad \text{(by (1.13))}.$$

Now we think of (B). Here we can't obtain (2.24) from (2.23), as in case (A). By (2.18) and (2.3), we obtain

$$|U| = r < \psi(\rho) = R(|a_{n-1}|, \beta), \qquad \text{(by (1.12))}$$

This completes the proof of Theorem 1.

PROOF OF COROLLARY 1 — For the zero $z\ (\neq 0) = re^{i\theta}$, of the polynomial $p(z)$, we have

$$z^n = -a_{n-1} z^{n-1} - a_{n-2} z^{n-2} \ldots - a_1 z - a_0$$

i.e.,

$$r^n = \sum_{k=0}^{n-1} A_k r^k e^{i(\alpha_k - (n-k)\theta)}, \qquad \text{(by (1.17))}$$

i.e.,

$$r^n = \sum_{k=0}^{n-1} A_k r^k \cos\{\alpha_k - (n-k)\theta\}$$

$$\leq \sum_{k=0}^{n-1} A_k r^k \cos_+\{\alpha_k - (n-k)\theta\}, \qquad \text{(by (1.18))}$$

$$= |a_{n-1}(\theta)| r^{n-1} + \beta(\theta)\{r^{n-2} + r^{n-3} + \ldots + 1\}, \qquad \text{(by (1.19))}.$$

Now, by following the line of proof of Theorem 1, Corollary 1 follows.

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