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SOME NEW SEQUENCE SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS

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In this paper we introduce and examine some properties of three sequence spaces defined by using a sequence of Orlicz functions.

Key Words : Sequence Spaces; Almost Convergence; Orlicz Function

1. INTRODUCTION

Let l_∞ and c denote the Banach spaces of real bounded and convergent sequences $x = (x_n)$ normed by $\|x\| = \sup_n |x_n|$ respectively. Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n)), m = 1, 2, 3, \dots$. A continuous linear functional φ on l_∞ is said to be an invariant mean or a σ-mean if and only if

1) $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

2) $\varphi(e) = 1$, where $e = (1, 1, \dots)$ and

3) $\varphi(\{x_{\sigma(n)}\}) = \varphi(\{x_n\})$ for all $x \in l_\infty$.

For certain kinds of mappings σ, every invariant mean φ extends the limit functional on the space c, in the sense that $\varphi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$, where V_σ is the set of bounded sequences all of whose σ-means are equal.

If $x = (x_n)$, set $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown that

$$V_\sigma = \{x \in l_\infty : \lim_m t_{mn}(x) = Le \text{ uniformly in } n,\ L = \sigma\text{-}\lim x\}, \qquad \dots (1.1)$$

where

$$t_{mn}(x) = (x_n + Tx_n + \dots + T^m x_n)/m + 1 \text{ (see Schafer}^{12}\text{)}.$$

The special case of (1.1) in which $\sigma(n) = n + 1$ was given by Lorentz[9] (Theorem 1) and the general result can be proved in a similar way.

A bounded sequence $x=(x_n)$ is said to be strongly σ-convergent to a number L if and only if

$$\lim_m \frac{1}{m}\sum_{k=1}^{m} |x^k_{\sigma(n)}-L| \to 0 \text{ uniformly in } n.$$

$[V_\sigma]$ denotes the set of all strongly σ-convergent sequences. (See Mursaleen[10]). For $\sigma(n) = n+1$, the space $[V_\sigma]$ is the space of strongly almost convergent sequences.

An Orlicz function is a function $M:[0,\infty)\to[0,\infty)$, which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x)\to\infty$ as $x\to\infty$. If convexity of Orlicz function M is replaced by $M(x+y)\le M(x)+M(y)$ then this function is called modulus function, defined and discussed by Ruckle[13] and Maddox[5].

Lindenstrauss and Tzafriri[3] used the idea of Orlicz function to construct the sequene space

$$l_M=\left\{x\in w:\ \sum_k M\left(\frac{|x_k|}{r}\right)<\infty \text{ for some } r>0\right\}.$$

The space l_M with the norm

$$\|x\| = \operatorname{Inf}\left\{r>0:\ \sum_k M\left(\frac{|x_k|}{r}\right)\le 1\right\}$$

becomes a Banach space which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x)=x^p$ for $1\le p<\infty$.

A generalization of Orlicz sequence spaces is due to Woo[6]. Let (M_k) be a sequence of Orlicz functions. Define the sequence space $l(M_k)$ by

$$l(M_k)=\left\{x\in w:\ \sum_k M_k\left(\frac{|x_k|}{r}\right)<\infty \text{ for some } r>0\right\}$$

and equip this space with the norm $\|\cdot\|$, where

$$\|x\| = \operatorname{Inf}\left\{r>0:\ \sum_k M_k\left(\frac{|x_k|}{r}\right)\le 1\right\}.$$

The space $l(M_k)$ is a Banach space and is called a modular sequence space. The space $l(M_k)$ also generalizes the concept of modulared sequence space introduced earlier by Nakano[11], who considered the space $l(M_k)$ when $M_k(x)=x^{\alpha_k}$, where $1\le\alpha_k<\infty$ for $k\ge 1$.

In the present note, we introduce and examine some properties of three sequence spaces defined by a sequence of Orlicz functions, which generalize well-known strongly invariant A-summable sequences $[A_\sigma,p]_0$, $[A_\sigma,p]$ and $[A_\sigma,p]_\infty$.

It was quite natural to expect the σ-convergence must give rise to a new type of convergence, namely strong σ-convergence just as $(C, 1)$ mean gives rise to the concept of strong Cesaro summability and this was introduced and discussed by Maddox[4]. If $A = (a_{mk})$ is a nonnegative matrix and $p = (p_k)$ be a sequence of real numbers such that $p_k > 0$ for all k and $\sup_k p_k = H < \infty$, then Savas[2] defined

$$[A_\sigma, p]_0 = \left\{ x \in w : \lim_m \sum_k a_{mk} |x_\sigma^k(n)|^{p_k} = 0 \text{ uniformly in } n \right\},$$

$$[A_\sigma, p] = \left\{ x \in w : \lim_m \sum_k a_{mk} |x_\sigma^k(n) - L|^{p_k} = 0, \text{ for some L, uniformly in } n \right\},$$

$$[A_\sigma, p]_\infty = \left\{ x \in w : \sup_{m,n} \sum_k a_{mk} |X_\sigma^k(n)|^{p_k} = \infty \right\}.$$

2. Main Results

Let $M = (M_k)$ be a sequence of Orlicz functions and suppose that $A = (a_{mk})$ be a regular matrix. Then we define

$$W_0(A_\sigma, M, p, s) = \left\{ x \in w : \lim_m \sum_k a_{mk} k^{-s} [M_k(|x_\sigma^k(n)|/r)]^{p_k} = 0 \right.$$

uniformly in n, for some $r > 0$ and $s \geq 0\}$,

$$W(A_\sigma, M, p, s) = \left\{ x \in w : \lim_m \sum_k a_{mk} k^{-s} [M_k(|x_\sigma^k(n) - L|/r)]^{p_k} = 0 \right.$$

uniformly in n, for some $r > 0$ and $s \geq 0\}$,

$$W_\infty(A_\sigma, M, p, s) = \left\{ x \in w : \sup_{m,n} \sum_k a_{mk} k^{-s} [M_k(|x_\sigma^k(n)|/r)]^{p_k} < \infty \right.$$

for some $r > 0$ and $s \geq 0\}$.

When $M_k(x) = x$ for all k and $s = 0$ then the family of sequences defined above becomes $[A_\sigma, p]_o$, $[A_\sigma, p]$ and $[A_\sigma, p]_\infty$ respectively.

When $M_k(x) = x$ for all k, $s = 0$ and $A = (C, 1)$ Cesaro matrix, we have the following sequence spaces which are generalization of the sequence spaces $[V_\sigma]^o_{(p_k)}$, $[V_\sigma]_{(p_k)}$ and $[V_\sigma]^\infty_{(p_k)}$ defined by Nuray and Gülcü[7] as

$$[V_\sigma, M]_{(p_k)} = \left\{ x \in w : \lim_m \frac{1}{m} \sum_{k=1}^{m} [M(|x_\sigma^k(n) - L|/r)]^{p_k} = 0, \right.$$

uniformly in n, for some $r > 0\}$

$$[V_\sigma, M]^0_{(p_k)} = \left\{ x \in w : \lim_m \frac{1}{m} \sum_{k=1}^{m} [M(|x_{\sigma^k}(n)|/r)]^{p_k} = 0, \right.$$

uniformly in n, for some $r > 0\}$,

$$[V_\sigma, M]^\infty_{(p_k)} = \left\{ x \in w : \sup_{m,n} \frac{1}{m} \sum_{k=1}^{m} [M(|x_{\sigma^k}(n)|/r)]^{p_k} < \infty, \right.$$

for some $r > 0\}$.

The sequence spaces $W(M, p)$, $W_0(M,p)$ and $W_\infty(M,p)$ were defined by Parashar and Choudhary[8] and generalized by Esi[1] defined as below :

$$W(A, M, p) = \left\{ x \in w : \lim_m \sum_k a_{mk} [M(|x_k - L|/r)]^{p_k} = 0, \text{ for some } r > 0 \text{ and } L > 0 \right\},$$

$$W_0(A, M, p) = \left\{ x \in w : \lim_m \sum_k a_{mk} [M(|x_k|/r]^{p_k} = 0, \text{ for some } r > 0 \right\},$$

$$W_\infty(A, M, p) = \left\{ x \in w : \sup_m \sum_k a_{mk} [M(|x_k|/r)]^{p_k} < \infty, \text{ for some } r > 0 \right\},$$

where $A = (a_{mk})$ is a nonnegative regular matrix and $p = (p_k)$ is any sequence of positive real numbers.

Theorem 1 — *Let $p = (p_k)$ be bounded. Then $W_0(A_\sigma, M, p, s)$, $W(A_\sigma, M, p, s)$ and $W_\infty(A_\sigma, M, p, s)$ are linear spaces over the set of complex numbers C.*

PROOF : Using the same technique of Theorem 1 of Nuray and Gülcü[7], it is easy to prove the theorem.

Theorem 2 — *Let $H = max\ (1, sup_k\ p_k)$. Then $W_0(A_\sigma, M, p, s)$ is a linear topological space paranormed by*

$$G(x) = \text{Inf} \left\{ r^{pm/H} : \left(\sum_k a_{mk} k^{-s} [M_k(|x_{\sigma^k}(n)|/r]^{p_k} \right)^{1/H} \leq 1, m = 1, 2, 3, \ldots n = 1, 2, 3, \ldots \right\}.$$

PROOF : Clearly $G(x) = G(-x)$, The subadditivity of G follows from Theorem 1. Since $M_k(0) = 0$ for all k, we get Inf $\{r^{Pm/H}\} = 0$ for $x = 0$. Conversely, suppose that $G(x) = 0$, then, it is easy to see that $x = 0$. Finally using the same technique of Theorem 2 of Nuray and Gülcü[7], it can be easily seen that scalar multiplication is continuous. This completes the proof.

Remark : It can be easily verified that when $s = 0$ and $M_k(x) = x$ for all k, then the paranorm defined in $W_0(A_\sigma, M, p, s)$ and paranorm defined in $[A_\sigma, p]_0$ are the same.

In order to discuss further result we need the following definition.

Definition — An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists, constant $K > 0$, such that $M(2u) < KM(u)$ $(u \geq 0)$. The Δ_2-condition is equivalent to the satisfaction of inequality $M(Lu) \leq KLM(u)$ for all values of u and for $L > 1$.

***Theorem* 3** — *Let A be a nonnegative regular matrix, and $M = (M_k)$ be a sequence of Orlicz functions which satisfies Δ_2-condition for all k. Then*

i) $[A_\sigma, p]_0 \subset W_0(A_\sigma, M, p, s)$

ii) $[A_\sigma, p] \subset W(A_\sigma, M, p, s)$

iii) $[A_\sigma, p]_\infty \subset W_\infty(A_\sigma, M, p, s)$

PROOF : Let $x \in [A_\sigma, p]$, then

$$S_{mn} = \sum_{k=1}^{m} a_{mk} \, | x_\sigma^k(n) - L |^{p_k} \to 0, \text{ as } m \to \infty, \text{ uniformly in } n. \qquad \dots (2.1)$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \varepsilon$ for $0 \leq t \leq \delta$ and for all k. Write $Z_{\sigma^k}(n) = | x_{\sigma^k}(n) - L.|$ and consider

$$\sum_{k=1}^{m} a_{mk} \, k^{-s} \, [M_k(| z_{\sigma^k}(n) |)]^{p_k} = \Sigma_1 + \Sigma_2$$

where the first summation is over $z_{\sigma^k}(n) \leq \delta$ and the second summation is over $z_{\sigma^k}(n) > \delta$. Since M_k is continuous for all k

$$\Sigma_1 < \varepsilon^H \sum_k a_{mk} \, k^{-s}$$

and for $z_{\sigma^k}(n) > \delta$ we use the fact that

$$z_{\sigma^k}(n) < z_{\sigma^k}(n) < 1 + z_{\sigma^k}(n)/\delta.$$

Since M_k is nondecreasing and convex for all k, it follows that

$$M_k(z_{\sigma^k(n)}) < M_k(1 + {}_{\sigma^k(n)}/\delta) < \frac{1}{2} M_k(2) + \frac{1}{2} M_k(2_{\sigma^k(n)}/\delta).$$

Since M_k satisfies Δ_2-condition for all k, therefore

$$M_k(z_{\sigma^k(n)}) < \frac{1}{2} K z_{\sigma^k(n)}/\delta . M_k(2) + \frac{1}{2} K z_{\sigma^k(n)}/\delta . M_k(2) = K z_{\sigma^k(n)}/\delta . M_k(2).$$

Since (k^{-s}) is bounded, we write

$$\sum_2 < \max(1, KK_1 \, \delta^{-1} M_k(2)^H S_{m,n}.$$

This and from (2.1) and regularity of A we obtain $[A_\sigma, p] \subset W(A_\sigma, M, p, s)$. Following similar arguments we can prove that $[A_\sigma, p]_0 \subset W_0(A_\sigma, M, p, s)$ and $[A_\sigma, p]_\infty \subset W_\infty(A_\sigma, M, p, s)$.

***Theorem* 4** — *i) Let $0 < \inf_k p_k \leq p_k \leq 1$. Then $W(A_\sigma, M, p, s) \subset W(A_\sigma, M, s)$*

ii) Let $1 \leq p_k \leq \sup_k p_k < \infty$. Then $W(A_\sigma, M, p, s) \subset W(A_\sigma, M, s)$

iii) Let $0 < p_k \leq q_k$ and (q_k/p_k) be bounded. Then $W(A_\sigma, M, p, s) \subset W(A_\sigma, M, p, s)$.

iv) $s_1 \leq s_2$ implies $W(A_\sigma, M, p, s_1) \subset W(A_\sigma, M, p, s_2)$

PROOF : i) Let $x \in W(A_\sigma, M, p, s)$, since $0 < \inf_k p_k \leq 1$, we get

$$\sum_k a_{mk} k^{-1} [M_k(| x_{\sigma^k(n)} - L |/r)] \leq \sum_k a_{mk} k^{-s} [M_k(| x_{\sigma^k(n)} - L |/r)]^{p_k}$$

for each n and hence get $x \in W(A_\sigma, M, p, s)$.

ii) Let $p_k \geq 1$ for each k, $\sup_k p_k < \infty$ and $x \in W(A_\sigma, M, s)$. Then for each $0 < \varepsilon < 1$ there exists a positive integer N such that

$$\sum_k a_{mk} k^{-s} [M_k(| x_{\sigma^k(n)} - L |/r)] \leq \varepsilon < 1 \text{ for each } n \text{ and for all } m \geq N.$$

This implies that

$$\sum_k a_{mk} k^{-s} [M_k(| x_{\sigma^k(n)} - L |/r)]^{p_k} \leq \sum_k a_{mk} k^{-s} [M_k(| x_{\sigma^k(n)} - L |/r)]$$

Thus we get $x \in W(A_\sigma, M, p, s)$.

iii) If we take $w_k = [M_k(| x_{\sigma^k(n)} - L |/r)]^{q_k}$ for all k, then using the same technique of Theorem 2 of Nanda[14], it is easy to prove (iii).

iv) Let $s_1 \leq s_2$. Then $k^{-s_2} < k^{-s_1}$ for all $k \in N$. Since

$$k^{-s_2} [M_k(| x_{\sigma^k(n)} - L |/r)]^{p_k} \leq k^{-s_1} [M_k(| x_{\sigma^k(n)} - |/r)]^{p_k}$$

for all k and n, we have

$$\sum_k a_{mk} k^{-s_2} [M_k(| x_{\sigma^k(n)} - L |/r)]^{p_k} \leq \sum_k a_{mk} k^{-s_1} [M_k(| x_{\sigma^k(n)} - L |/r)]^{p_k}$$

Since $x \in W(A_\sigma, M, p, s_1)$, we get $x \in W(A_\sigma, M, p, s_2)$.

Corollary 5 — Let $A = (C, 1)$ Cesaro matrix and $M = (M_k)$ a sequence of Orlicz functions. Then

i) If $M = (M_k)$ satisfies Δ_2-condition for all k and $s = 0$, then

$$[V_\sigma]^0_{(p_k)} \subset W_0\,[M, p],\ [V_\sigma]_{(p_k)} \subset W[M, p] \text{ and } [V_\sigma]^\infty_{(p_k)} \subset W_\infty\,[M, p],$$

where for instance

$$W_0\,[M, p] = \left\{ x \in w :\ \lim_m \frac{1}{m} \sum_{k=1}^{m} [M_k(|\,x^k_\sigma(n)\,|/r]^{p_k} = 0 \text{ uniformly in } n, \text{ for some } r > 0 \right\},$$

ii) Let $0 < \inf_k p_k \le p_k \le 1$. Then $W(M, p, s) \subset W(M, s)$.

iii) Let $1 \le p_k \le \sup_k p_k < \infty$. Then $W(M, s) \subset W(M, p, s)$.

iv) Let $0 < p_k \le q_k$ and (q_k/p_k) be bounded. Then $W(M, q, s) \subset W(M, p, s)$.

PROOF : It is trivial.

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