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ON UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

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In this work, we investigate the univalence of certain integral operators, considering the class of univalent functions defined by the condition $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1, |z| < 1$, where $f(z) = z + a_2 z^2 + \ldots$ is analytic in the unit disc $U = \{z : |z| < 1\}$.

Key Words : Univalence; Integral Operators; Conformal Mapping

1. INTRODUCTION

We denote by A the class of functions which are analytic in the unit disc U and have the form $f(z) = z + a_2 z^2 + \ldots$ Let S denote the class of the functions $f \in A$ which are univalent in U.

In their paper[4] Ozaki and Nunokawa[2] proved the following :

Theorem A — *Let $f \in A$ satisfy the condition*

$$\left|\frac{z^2 f'(Z)}{f^2(z)} - 1\right| < 1, \; z \in U \qquad \ldots (1)$$

then f is univalent in U.

2. PRELIMINARY RESULTS

We will need the following theorems and lemma :

Theorem B[1, 2] — *Let c be a complex number, $|c| \le 1, c \ne -1$. If $f(z) = z + a_2 z^2 + \ldots$ is a regular function in U and*

$$\left| c|z|^2 + \left(1 - |z|^2\right) \frac{zf'(z)}{f(z)} \right| \le 1 \qquad \ldots (2)$$

for all $z \in U$, then the function f is regular and univalent in U.

Theorem C^5 — *Let α be a complex number, $Re\ \alpha > 0$, and c a complex number, $|c| \le 1$, $c \ne -1$. If $f(z) = z + a_2 z^2 + \ldots$ is a regular function in U and*

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \le 1 \qquad \ldots (3)$$

for all $z \in U$, then the function

$$F_\alpha(z) = \left[\int_0^z u^{\alpha-1} f'(u) du \right]^{\frac{1}{\alpha}} = z + \ldots \qquad \ldots (4)$$

is regular and univalent in U.

The Schwarz Lemma[3] — Let the analytic function $f(z)$ be regular in the unit circle $|z| < 1$ and let $f(0) = 0$. If, in $|z| < 1, |f(z)| \le 1$, then

$$|f(z)| \le |z|, |z| < 1 \qquad \ldots (5)$$

where equality can hold only if $f(z) \equiv Kz$ and $|K| = 1$.

3. Main Results

Theorem 1 — *Let $g \in A$ satisfies (1), α be a complex number, $Re\ \alpha > 0$ and c be a complex number, $|c| \le 1, c \ne -1$. If*

$$|g(z)| \le 1,\ z \in U \qquad \ldots (6)$$

and

$$|c| + 3|\alpha - 1| \le 1 \qquad \ldots (7)$$

then the function

$$F_\alpha(z) = \int_0^z \left(\frac{g(u)}{u} \right)^{\alpha - 1} du \qquad \ldots (8)$$

is in the class S.

PROOF : The function $F_\alpha(z)$ is regular in U, From (8) we have

$$F'_\alpha(z) = \left(\frac{g(z)}{z} \right)^{\alpha-1},\ F''_\alpha(z) = (\alpha - 1) \left(\frac{g(z)}{z} \right)^{\alpha-2} \frac{zg'(z) - g(z)}{z^2}$$

and

$$\left| c|z|^2 + (1 - |z|^2) \frac{zF''_\alpha(z)}{F'_\alpha(z)} \right| = \left| c|z|^2 + (1 - |z|^2)(\alpha - 1) \left(\frac{zg'(z)}{g(z)} - 1 \right) \right|. \qquad \ldots (9)$$

From (9) we obtain

$$\left| c|z|^2+(1-|z|^2)\frac{zF''_\alpha(z)}{F'_\alpha(z)} \right| \le |c|+|\alpha-1|\left(\left| \frac{z^2 g'(z)}{g^2(z)} \right| \left| \frac{g(z)}{z} \right| +1 \right). \quad \ldots (10)$$

Using (10) and Schwarz-Lemma we have

$$\left| c|z|^2+(1-|z|^2)\frac{zF''_\alpha(z)}{F'_\alpha(z)} \right| \le |c|+|\alpha-1|\left(\left| \frac{z^2 g'(z)}{g^2(z)} -1 \right| +2 \right) \quad \ldots (11)$$

The function g satisfies (1) and from (11) we get

$$\left| c|z|^2+(1-|z|^2)\frac{zF''_\alpha(z)}{F'_\alpha(z)} \right| \le |c|+3|\alpha-1| \quad \ldots (12)$$

From (12) and (7) we have

$$\left| c|z|^2+(1-|z|^2)\frac{zF''_\alpha(z)}{F'_\alpha(z)} \right| \le 1,\ z\in U.$$

By Theorem B it follows that the function F_α is in the class S.

***Theorem* 2** — *Let $g\in A$ satisfies (1), α be a complex number, Re $\alpha>0$, and c be a complex number, $|c|\le 1, c\ne -1$.*

If

$$|g(z)|\le 1 \quad \ldots (13)$$

for all $z\in U$, and

$$|c|+3\left|\frac{\alpha-1}{\alpha}\right|\le 1 \quad \ldots (14)$$

then the function

$$G_\alpha(z)=\left[\alpha\int_0^z [g(u)]^{\alpha-1}\,du\right]^{\frac{1}{\alpha}} \quad \ldots (15)$$

is in the class S.

PROOF : From (15) we have

$$G_\alpha(z)=\left[\alpha\int_0^z u^{\alpha-1}\left(\frac{g(u)}{u}\right)^{\alpha-1}du\right]^{\frac{1}{\alpha}}. \quad \ldots (16)$$

Let us consider the function

$$p(z) = \int_0^z \left(\frac{g(u)}{u} \right)^{\alpha - 1} du. \quad \text{... (17)}$$

The function p is regular in U.

From (17) we get $p'(z) = \left(\frac{g(z)}{z} \right)^{\alpha - 1}$,

$$p''(z) = (\alpha - 1) \left(\frac{g(z)}{z} \right)^{\alpha - 2} \frac{zg'(z) - g(z)}{z^2}$$

and

$$\left| c\,|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zp''(z)}{\alpha p'(z)} \right| = \left| c\,|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{\alpha - 1}{\alpha} \left(\frac{zg'(z)}{g(z)} - 1 \right) \right|$$

$$\leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| \left(\left| \frac{z^2 g'(z)}{g^2(z)} \right| \frac{|g(z)|}{|z|} + 1 \right), \; z \in U. \quad \text{... (18)}$$

By the Schwarz-Lemma and using (18) we obtain

$$\left| c\,|z|^2 + (1 - |z|^{2\alpha}) \frac{zp''(z)}{\alpha p'(z)} \right| \leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| \left(\left| \frac{z^2 g'(z)}{g^2(z)} - 1 \right| \right) + 2. \quad \text{... (19)}$$

From (19) and since g satisfies the condition (1) we have

$$\left| c\,|z|^2 + (1 - |z|^{2\alpha}) \frac{zp''(z)}{\alpha p'(z)} \right| \leq |c + 3 \left| \frac{\alpha - 1}{\alpha} \right| \quad \text{... (20)}$$

From the condition (14) and (20) we get

$$\left| c\,|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zp''(z)}{\alpha p'(z)} \right| \leq 1, \quad \text{... (21)}$$

for all $z \in U$.

From (17) we have $p'(z) = \left(\frac{g(z)}{z} \right)^{\alpha - 1}$ and by Theorem C it results that the function G_α is in the class S.

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