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ON A CHARACTERIZATION OF GEODESIC SPHERES

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A previous theorem, due to Hasegawa and Abe, asserts that any Euclidean compact convex connected hypersurface which scalar curvature is bounded from above by a certain function is a geodesic sphere. Two different proofs are given: one using the Weitzenböck-Bochner formula, the other using the Minkowski integral formulae. Actually, this last method provides a natural extension to the above theorem.

Key Words : Hypersurface; *k*th-Mean Curvature; Weitzenböck-Bochner Formula; Minkowski Formulae

1. RESULTS

Any distance sphere $S(p, r)$ of centre p and radius r in $I\!R^{n+1}$ is a totally umbilical hypersurface with constant scalar curvature equal to $n(n-1)/r^2$. On the other hand, Hasegawa and Abe[4] proved the following

Theorem — *If M_n is a closed (compact without boundary) connected orientable Euclidean hypersurface $(n \geq 2)$ with non-negative Ricci curvature and scalar curvature Scal which satisfy*

$$d_p^2 \cdot Scal \leq n\,(n-1)$$

d_p being the Euclidean distance function to some point p of $I\!R^{n+1}$, then M is a geodesic sphere of centre p in $I\!R^{n+1}$.

Let $(\sigma_k)_{1 \leq k \leq n}$ be the kth symmetric elementary functions of the principal curvatures of M and $(H_k = \sigma_k / C_n^k)_{0 \leq k \leq n}$ be the k-th-mean curvatures of M. For example, $H_0 = 1, H_1 = H$ is the mean curvature of M, $n(n-1)\,H_2 = Scal$ and H_n is the Gauss-Kronecker curvature of M. The hypersurface M will be called *convex* if the sectional curvature of M is everywhere non-negative. This is equivalent to say that the Ricci curvature Ric of M is everywhere non-negative: indeed, let $k_1, \ldots, k_n$ be the principal curvatures of M at a point of M and $\upsilon_1, \ldots, \upsilon_n$ an associated orthonormal basis of principal vectors. If $k_i > 0$ and $k_j < 0$, then the assumption on the Ricci curvature and the Gauss formula imply that $k_i(nH - k_i) \geq 0$ and $k_j(nH - k_j) \geq 0$ which provides the contradiction $k_i \leq nH \leq k_j$. So the product of any two principal curvatures is non-negative, that is the sectional curvature of M is non-negative.

The above theorem is a special case of:

The Main Result — Let M_n be a closed connected Euclidean convex hypersurface ($n \geq 2$) which kth-mean curvature H_k satisfies

$$d_p^k \cdot |H_k| \leq 1$$

for some point p of $\mathbb{R}^{n+1}$ and some integer $k \in \{1, ..., n\}$. Then M is a geodesic sphere of center p in $\mathbb{R}^{n+1}$.

Remark : Note that any compact connected Euclidean hypersurface is orientable by the generalized Jordan curve theorem.

As a consequence, we obtain:

Corollary 1 — Let M_n be a closed connected Euclidean convex hypersurface ($n \geq 2$) included in a closed ball $\overline{B}_p(r)$ of $\mathbb{R}^{n+1} (p \in \mathbb{R}^{n+1}, r > 0)$ and which kth-mean curvature H_k satisfies

$$r^k \cdot |H_k| \leq 1$$

for some integer $k \in \{1, ..., n\}$. Then $M = \partial\overline{B}_p(r)$, that is M is the geodesic sphere of centre p and radius r in $\mathbb{R}^{n+1}$.

Corollary 2 — Let M_n be a closed Euclidean convex hypersurface ($n \geq 2$) included in a closed ball $\overline{B}_p(r)$ of $\mathbb{R}^{n+1} (p \in \mathbb{R}^{n+1}, r > 0)$. Then

for $k = 0, 1, ..., n$, $\max\limits_M |H_k| \geq \frac{1}{r^k}$.

Remark : For close results, one can see articles 1, 2 and 5.

2. First Proof

We show in this section how the Weitzenböck-Bochner formula can be used to give a new proof of Hasegawa-Abe result.

Let $\tilde{\nabla}$ (resp. $\langle \cdot, \cdot \rangle$) be the euclidean connection (resp. scalar product) of $\mathbb{R}^{n+1}$, ∇ (resp. $\langle \cdot, \cdot \rangle$) the one induced on M, η a smooth unit vector field normal to M, h the second fundamental form of M valued on the normal bundle of M, A its shape operator and H its mean curvature. We also consider the function $F : \mathbb{R}^{n+1} \to \mathbb{R} : q \mapsto d_p^2(q)/2$, its smooth restriction $f = F_{|M}$ and the *support function* $\alpha = \langle \tilde{\nabla} F, \eta \rangle : M \to \mathbb{R}$. Let finally ξ be a unit vector field on M such that $\nabla f = |\nabla f| \cdot \xi$.

A straightforward computation shows that for any vectors fields X, Y on M,

$$\nabla^2 f(X, Y) = \langle X, Y \rangle + \langle \tilde{\nabla} F, h(X, Y) \rangle \quad ... (2.1)$$

which by contraction leads to

$$\Delta f = n + nH\alpha \quad ... (2.2)$$

Considering a local orthonormal basis $\{X_i\}_{1 \leq i \leq n}$ of principal vectors on M, we deduce from equation (2.1), that

$$|\nabla^2 f|^2 = \sum_{i=1}^{n} \{1 + \langle \tilde{\nabla} F, h(X_i, X_i) \rangle\}^2$$

$$= n + \sum_{i=1}^{n} \langle \tilde{\nabla} F, h(X_i, X_i) \rangle^2 + 2 \sum_{i=1}^{n} \langle \tilde{\nabla} F, h(X_i, X_i) \rangle$$

$$= n + \alpha^2 \text{ Trace } (A^2) + 2nH\alpha$$

Since $\tilde{\nabla} F = \nabla f + \alpha\eta$ on M, we have $\alpha^2 = d_p^2 - |\nabla f|^2$ on M. On the other hand, the Gauss formula implies that $Scal = (nH)^2 - \text{Trace } (A^2)$. So, the squared norm of $\nabla^2 f$ can be rearranged into

$$|\nabla^2 f|^2 = \{n(n-1) - d_p^2 \cdot Scal\} + |\nabla f|^2 \cdot \{Scal + Ric\ (\xi, \xi)\}$$

$$+ (\alpha n H)^2 + 2\Delta f - n^2 - Ric\ (\nabla f, \nabla f) \qquad \dots (2.3)$$

At last,

$$\langle \nabla f, \nabla(\Delta f) \rangle = n \langle \nabla f, \nabla(\alpha H) \rangle = n \cdot Div\ (\alpha H \cdot \nabla f) - n\alpha H \Delta f \qquad \dots (2.4)$$

$$= n \cdot Div(\alpha H \cdot \nabla f) - n\Delta f + n^2 - (\alpha n H)^2$$

Integrating the classical Weitzenböck-Bochner formula:

$$\frac{1}{2} \Delta (|\nabla f|^2) = |\nabla^2 f|^2 + \langle \nabla f, \nabla(\Delta f) \rangle + Ric\ (\nabla f, \nabla f)$$

on M and using equations (2.3), (2.4) and the Green theorem, we obtain :

$$0 = \int_M \{\{n(n-1) - d_p^2 \cdot Scal\} + |\nabla f|^2 \cdot \{Scal + Ric\ (\xi, \xi)\}\} \cdot dM$$

The hypothesis yield to :

$$\begin{cases} n\,(n-1) = d_p^2 \cdot Scal \\ |\nabla f|^2 \cdot (Scal + Ric\ (\xi, \xi)) = 0 \end{cases}$$

So $Scal$ is positive on M and $\nabla f = 0$ on M, that is M is included in a geodesic sphere $F^{-1}(r)$ for some nonzero r. As M is closed and open in the connected set $F^{-1}(r)$, M coincides with $F^{-1}(r)$ and this achieves the proof.

3. Proof of the Main Result

This section gives a proof of the main result and by the same way a second new proof of Hasegawa-Abe result.

The classical Minkowski integral formulae say that

$$\text{for } k = 0, 1, \ldots, n-1, \int_M (H_k + \alpha H_{k+1}) \cdot dM = 0. \qquad \ldots (3.1)$$

Since the Ricci curvature of M is non-negative, the principal curvatures of M are of the same sign by the Gauss formula. By reversing the orientation if necessary, we can assume that the principal curvatures are all non-negative. On the other hand, we have[3]

$$H_1 \geq H_2^{1/2} \geq H_3^{1/3} \geq \ldots \geq H_n^{1/n}. \qquad \ldots (3.2)$$

Moreover, if there exists $k \in \{2, \ldots, n\}$ and $q \in M$ such that $H_{k-1}^{1/(k-1)}(q) = H_k^{1/k}(q)$, then q is an umbilical point.

As $|\alpha| \leq d_p$, we deduce from (3.1), (3.2) and the assumption on H_k that

$$H_{k-1} + \alpha H_k \geq H_{k-1} - d_p H_k$$

$$\geq H_k^{(k-1)/k} - d_p H_k = H_k^{(k-1)/k}(1 - d_p H_k^{1/k}) \geq 0$$

By Minkowski formula, all these inequalities are in fact equalities. In particular, $H_{k-1}^{1/(k-1)} = H_k^{1/k}$ on M, that is each point of M is an umbilical point. As M is compact and connected, M is a geodesic sphere in $I\!R^{n+1}$.

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