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BEST APPROXIMATION AND FIXED POINT RESULTS

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This note presents theorems on the existence of best approximations by making use of *-nonexpansive and continuous maps on certain classes of topological vector spaces. Several fixed point theorems are obtained as well.

Key Words : Best Approximation; Fixed Point; Spaces-Topological Vector; Maps-Nonexpensive; Maps-Continuous

We shall need following definitions :

Let M be a nonempty subset of a normed space E. An element $y \in M$ is called a best approximation to $x \in E$ if

$$\| x - y \| = d(x, M) = \inf \{\| x - m \| : m \in M\}.$$

The set of best M-approximations to x is denoted by $P(x)$ and is defined as

$$P(x) = \{m \in M : \| x - m \| = d(x, M)\}.$$

The mapping $P : E \to M$ is called metric projection.

In case $P(x)$ contains only one element for every $x \in E, M$ is called a Chebyshev set. A closed convex subset M of a Hilbert space is Chebyshev and projection map P is nonexpansive.

A Banach space E is said to satisfy Opial's condition if for each $x \in E$ and each sequence $\{x_n\}$ weakly converging to x, $\liminf \| x_n - x \| < \liminf \| x_n - y \|$ holds for all $y \neq x$ in E.

A multivalued map $f : M \to 2^M$ is called (*i*) weakly nonexpansive (cf. [5]) if given $x \in M$ and $u_x \in f(x)$ there is a $u_y \in f(y)$ for each $y \in M$ such that

$$\| u_x - u_y \| \leq \| x - y \|.$$

(*ii*) *-nonexpansive (cf. [5]) if for all x, y in M and $u_x \in f(x)$ with

$$\| x - u_x \| = d(x, f(x))$$

there exists $u_y \in f(y)$ with $\| y - u_y \| = d(y, f(y))$ such that $\| u_x - u_y \| \leq \| x - y \|$.

(*iii*) midpoint concave (cf. [6]) if $\frac{1}{2}f(x)+\frac{1}{2}f(y)\subseteq f\left(\frac{1}{2}x+\frac{1}{2}y\right)$ for all x, y in a convex set M.

(*iv*) demiclosed at 0 if the conditions $x_n \in M, x_n$ converges weakly to $x, y_n \in f(x_n)$ and $y_n \to 0$ imply that $0 \in f(x)$.

(*v*) The map $f: M \to 2^E$ is upper semicontinuous (usc) if for any closed subset B of $E, f^{-1}(B) = \{x \in M : f(x) \cap B \neq \phi\}$ is closed.

Recall that $d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}$.

We follow the technique of Husain and Latif[5] to establish a general result.

Theorem 1 — *Let M be a nonempty weakly compact starshaped subset of a Banach space E. Suppose that $f: M \to 2^E$ is compact valued usc map and $g: M \to 2^M$ defined by $g(x) = \bigcup\{P(y) : y \in f(x), d(f,(x), M) = d(y, M)\}$ is *-nonexpansive. If $g(x)$ is compact Chebyshev for each $x \in M$, then there is a $y_0 \in M$ such that*

$$d(y_0, fy_0) = d(fy_0, M)$$

provided I-g is demiclosed at 0 or E satisfies Opial's condition.

PROOF : Each $g(x)$ is nonempty since each $f(x)$ is compact. As each $g(x)$ is Chebyshev so by the definition of *-nonexpansiveness, each $u_x \in g(x)$ is unique and there is unique $u_y \in g(y)$ for all $y \in M$ such that

$$\| u_x - u_y \| \leq \| x - y \|. \qquad \text{... (1)}$$

We define $f_n : M \to M$ by

$$f_n(x) = \lambda_n u_x + (1 - \lambda_n)w,$$

where w is a starcentre of M, $0 < \lambda_n < 1$ for all $n \geq 1$ and $\lambda_n \to 1$ as $n \to \infty$.

By (1), we get for all $x, y \in M, \| f_n x - f_n y \| < \lambda_n \| x - y \|$.

So each f_n is a contraction and hence has a fixed point $x_n \in M$. The sequence $\{x_n\}$ in M has a subsequence denoted by $\{x_n\}$ converging weakly to $y_0 \in M$. By the definition of f_n, there is unique $u_n \in g(x_n)$ such that

$$x_n = f_n(x_n) = \lambda_n u_n + (1 - \lambda_n)w.$$

Obviously

$$y_n = u_n - x_n = (1 - \lambda_n)(u_n - w) \to 0 \text{ as } n \to \infty. \qquad \text{... (2)}$$

I-g is Demiclosed at 0 — The sequence $\{x_n\}$ converges weakly to y_0 and from (2), $y_n \to 0$ where $y_n = u_n - x_n \in g(x_n) - x_n$. Thus $0 \in (I - g)(y_0)$ implies that $y_0 \in g(y_0)$. It follows that for some

$y \in f(y_0)$ with $d(f(y_0) = d(y, M), M), y_0 \in P(y)$. Since $d(y_0, f(y_0)) \leq d(y_0, y) = d(y, M) = d(fy_0 M) \leq d(y_0, fy_0)$ so we have $d(y_0, fy_0) = d(fy_0, M)$.

E Satisfies Opial's Condition — The map g being *-nonexpansive is weakly nonexpansive so for each $u_n \in g(x_n)$ there is $v_n \in g(y_0)$ such that

$$\| u_n - v_n \| \leq \| x_n - y_0 \| \quad \text{... (3)}$$

As $g(y_0)$ is compact so $\{v_n\}$ converges to some $a \in g(y_0)$.

Combination of (3) with the facts $y_n \to 0$ and $v_n \to a$ implies

$$\liminf \| y_n + x_n - v_n \| = \liminf \| x_n - a \| \leq \liminf \| x_n - y_0 \|.$$

By Opial's condition, we have

$$\liminf \| x_n - y_0 \| < \liminf \| x_n - a \|.$$

Thus we have $y_0 = a \in g(y_0)$. Now result follows from the argument used above.

For single valued map the concept of *-nonexpansive coincides with that of a nonexpansive map. Thus we have the following Ky Fan type best approximation result which generalizes Theorem 3 due to Carbone[1].

Corollary 2 — Let M be a weakly compact starshaped subset of a Banach space E which satisfies Opial's condition. If $f: M \to E$ is continuous map and $P \circ f: M \to M$ is nonexpansive, then there is a $y_0 \in M$ such that

$$\| y_0 - fy_0 \| = d(fy_0, M).$$

A closed bounded subset of a Hilbert space H is weakly compact, H satisfies Opial's condition and I-T is demiclosed for any nonexpansive map T defined on a closed convex set in H.

Corollary 3 — Let M be a closed bounded and convex subset of a Hilbert space H. If $f: M \to H$ is nonexpansive map, then there is a $y_0 \in M$ such that

$$\| y_0 - fy_0 \| = d(fy_0, M).$$

Corollary 4[7] — Let B_r be a closed ball of radius r and centre 0 in a Hilbert space H. Let $f: B_r \to H$ be a nonexpansive map with the property that if $fx = \alpha x$ for some $x \in \partial B_r$, then $\alpha \leq 1$. Then f has a fixed point.

An other result in this direction is presented below:

Theorem 5 — *Let M be a closed subspace of a normed space E and $f: M \to E$ be a continuous map. If $P \circ f: M \to M$ is linear nonexpansive map such that for some $y_0 \in M$, we have $(P \circ f)^2 (y_0) - 2(P \circ f)(y_0) + y_0 = 0$, then*

$$\| y_0 - fy_0 \| = d(fy_0, M).$$

In case $fy_0 \in M$, then f has a fixed point.

PROOF : Let $g = P \circ f$. Then g is a linear nonexpansive self map on the closed subspace M of E such that

$$g^2 y_0 - 2g y_0 + y_0 = 0 \quad \text{... (4)}$$

we show that y_0 is a fixed point of g. From (4) and linearity of g, we obtain

$$(g - I)\,(g - I)\, y_0 = 0.$$

Put $$(g - I)\, y_0 = y \quad \text{... (5)}$$

Then $(g - I)y = 0$ implies that $gy = y$.

From (5), we get $gy_0 = y_0 + y$ and hence $g^n y_0 = y_0 + ny$ for all $n \geq 1$.

Consider $n \,\|\, y \,\| = \|\, g^n\, y_0 - y_0 \,\|$

$$\leq \|\, g^n y_0 - g^n\, 0 \,\| + \|\, y_0 \,\|$$

$$\leq 2 \,\|\, y_0 \,\|$$

Thus $\|\, y \,\| \leq (2 \,\|\, y_0 \,\|)/n$ for all $n \geq 1$.

As $n \to \infty$, we get $y = 0$ and so (5) implies that $gy_0 = y_0$.

Now $(P \circ f)y_0 = y_0$ gives

$$\|\, y_0 - fy_0 \,\| = d(fy_0, M) \text{ as desired.}$$

The example to follow illustrates that even if f is not nonexpansive and not linear, the map $P \circ f$ may be nonexpansive and linear on the closed convex set containing zero.

Example 6 — Let $f : [-2, 0] \to \boldsymbol{R}$ be given by $f(x) = 2x + 6$.

Then f is neither nonexpansive nor linear. But $(P \circ f)\, x = P(2x + 6) = 0$ for all $x \in [-2, 0]$ implies that $\|\, 0 - f(0) \,\| = d(f, 0), [-2, 0])$.

We now state Theorem 3.2 [6] in an equivalent form for our needs.

Theorem 7 — *Let M be a closed subset of a topological vector space (tvs) E and $f : M \to 2^M$ be a closed-valued usc multifunction. If*

(i) $S = \{x - y : y \in f(x), x \in M\}$ is convex; and

(ii) $f(M)$ is compact, then f has a fixed point.

Proposition 8[6] — *Let M be a closed convex subset of a tvs E and $f : M \to 2^E$ be a closed-valued usc multifunction such that $f(M)$ is compact. Then the set S is convex if and only if f is midpoint concave.*

Definition 9 — Let M be any subset of a metrizable *tvs* E and g be a self map on M. For any map $f : M \to E$, we define a multivalued function $G_f : M \to M$ by

$$G_f(y) = \{t \in M : d(gt, fy) \leq m(y)\} \text{ where } m(y) = 1/2\, [d(gy, fy) + d(fy, M)].$$

Properties described below are easy to prove (cf. [8]) :

(i) For each $y \in M, gy$ $(P \circ f)y$ if and only if $y \in G_f(y)$

(ii) $G_f(y) \neq \phi$ for each $y \in M$ provided g is onto.

(iii) If f and g are continuous, then $G_f(y)$ is closed for every $y \in M$ and G_f has closed graph.

We are now in a position to establish a theorem similar to main result due to Prolla[8].

Theorem 10 — *Let M be a nonempty compact convex subset of a metrizable tvs E and g be a continuous onto self map on M. Suppose $f: M \to E$ is a continuous map such that the map G_f is midpoint concave. Then there is $y_0 \in M$ such that*

$$d(gy_0, fy_0) = d(fy_0, M)$$

PROOF : Since G_f (x) is nonempty closed so it is compact subset of M for each $x \in M$. As M is a compact Hausdorff space and G_f has closed graph so G_f is USE (see [4], p. 205). It now follows that G_f (M) is compact (cf. (6), p. 135). Now Theorem 7 and Prop. 8 imply that G_f has a fixed point in M. That is there is some $y_0 \in M$ such that $y_0 \in G_f(y_0)$. By Property (i) $gy_0 \in P(fy_0)$. Hence, $d(gy_0, fy_0) = d(fy_0, M)$.

Remarks 11 : (i) If $fx \in M$ for all $x \in M$ in above theorem, then a coincidence result is obtained, that is, there is a $y_0 \in M$ such that $fy_0 = gy_0$.

(ii) In Example 5, f is midpoint affine and P is midpoint concave.

(iii) If we consider $G_f = F = g^{-1} \circ P \circ f$ as in theorem 4[2] or theorem 3[9], then F is midpoint concave provided g^{-1} and P are so and f is midpoint affine.

(iv) It would be interesting to find conditions under which the map G_f is midpoint concave.

(v) Theorem A and Theorem 1 in [1] coincide for Banach spaces with strictly convex dual (e.g. Hilbert spaces) because for such spaces each Chebyshev set with continuous metric projection is convex and metric projection for locally compact Chebyshev set M in a normed space E is continuous.

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