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ON THE STABILISATION OF EXPLOSIVE INSTABILITIES BY NONLINEAR FREQUENCY SHIFTS

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The paper presents an analytic solution to the problem of explosive instabilities, including the effects of nonlinear frequency shifts proportional to the square of the amplitude. It is found that such frequency shifts may introduce stabilisation of the explosive instabilities and may lead to soliton-like solutions in time or to solutions which are periodic in time.

Key Words : Stabilisation; Explosive Instabilities; Nonlinear Frequency Shifts; Plasma Physics

1. INTRODUCTION

In plasma physics[1-6] as well as in modern optics[7] nonlinear wave interactions are of great importance and explain new interesting effects. For the optical case[7&8] of coherent three-wave interaction with limited amplitudes the sum of the squares of the amplitudes is conserved, i.e., we have $\sum_j |A_j|^2$ = constant. During the process of interaction, energy is interchanged between stable modes, all having positive wave energy. For interactions with negative energy waves[9], i.e., when $S_k S_{k_1} = S_k S_{k_2} = -1$, where $S_k \equiv sign\, [\partial/\partial\omega\, (\omega\, \varepsilon\, (\omega))]_{\omega=\omega_k}$, the situation is different and we may have nonlinear growth of all amplitudes in time (and/or in space). Neglecting higher order terms in the equations we may obtain unlimited amplitudes for finite times[9-26]. In practical solutions there should exist some effects which limit the amplitudes. Limitation of amplitudes due to high nonlinearity may be caused by frequency shifts proportional to the squares of the amplitudes of the interacting waves[16,17&21]. We solve this problem analytically, assuming that the matrix elements are known. For practical examples such matrix elements may be calculated by applying well-known methods[1-4].

The paper shows how this saturation effect depends on certain assumptions for the initial values, and on the sign of the coupling coefficients of the higher order nonlinear terms which cause the saturation. It is found that the solution describing the saturated explosive instability may be soliton-like in time or may show an oscillatory dependence on time. It is pointed out that dissipative effects may also change the solution from a soliton-like behaviour into a damped oscillatory behaviour, following the first maximum for the amplitude.

2. Basic Equations

We use a coherent wave description to study a closed interacting system of three waves, which are assumed to be resonant ($\omega_0 = \omega_1 - \omega_2; k_0 = k_1 + k_2$). Including terms to third order in the amplitudes of the expansion, the equations of motion for the field amplitudes can be written as follows :

$$\frac{\partial A_k}{\partial t} = - is_k V_{k, k_1, k_2} A_{k_1} A_{k_2} - i \sum_{k_1} W_{k, k_1, -k_1, k} |A_{k_1}|^2 A_k \qquad \text{... (1)}$$

$$\frac{\partial A_{k_1}}{\partial t} = - is_{k_1} V_{k, k_1, -k_2} A_k A^*_{k_2} - i \sum_{k_2} W_{k_1, k_2, -k_2, k_1} |A_{k_2}|^2 A_{k_1} \qquad \text{... (2)}$$

and

$$\frac{\partial A_{k_2}}{\partial t} = - is_{k_2} V_{k_2, k_1, -k_1} A_k A^*_{k_1} - i \sum_{k} W_{k_2, k, -k, k_2} |A_{k_2}|^2 A_{k_2}, \qquad \text{... (3)}$$

where

$$S_k \equiv sign \left[\frac{\partial}{\partial \omega} \omega^2 e \, \varepsilon(\omega) \right]_{\omega = \omega_k}$$

and where the coefficients V_{pqr} and W_{pqrs} are real quantities in the absence of dissipation. Considering the combination of signs which leads to the explosive instability, we assume $S_k S_{k_1} = S_k S_{k_2} = -1$. Eqs. (1) to (3) can be written as

$$\frac{\partial A_k}{\partial t} + iA_k \left[\alpha_0 |A_k|^2 + \alpha_1 |A_{k_1}|^2 + \alpha_2 |A_{k_2}|^2 \right] = - is_k VA_{k_1} A_{k_2}, \qquad \text{... (1a)}$$

$$\frac{\partial A_{k_1}}{\partial t} + iA_{k_1} \left[\alpha'_0 |A_k|^2 + \alpha'_1 |A_{k_1}|^2 + \alpha'_2 |A_{k_2}|^2 \right] = - is_{k_1} VA_k A^*_{k_2} \qquad \text{... (2a)}$$

and

$$\frac{\partial A_{k_2}}{\partial t} + iA_{k_2} \left[\alpha''_0 |A_k|^2 + \alpha''_1 |A_{k_1}|^2 + \alpha''_2 |A_{k_2}|^2 \right] = - is_{k_2} VA_k A^*_{k_1}, \qquad \text{... (3a)}$$

where

$$\begin{pmatrix} \alpha_0 \, \alpha_1 \, \alpha_2 \\ \alpha'_0 \, \alpha'_1 \, \alpha'_2 \\ \alpha''_0 \, \alpha''_1 \, \alpha''_2 \end{pmatrix} = \begin{pmatrix} W_{k, k, -k, k} & W_{k, k_1, -k_1, k} & W_{k, k_2, -k_2, k} \\ W_{k_1, k, -k, k_1} & W_{k_1, k_1, -k_1, k_2} & W_{k_1, k_2, -k_2, k_1} \\ W_{k_2, k, -k, k_2} & W_{k_2, k, -k, k_2} & W_{k_2, k_2, -k_2, k_2} \end{pmatrix} \qquad \text{... (4)}$$

Introducing $A_j = u_j \exp(i\varphi_j)$ we have from eqs. (1*a*) to (3*a*),

$$\frac{\partial u_0}{\partial t} = V u_1 u_2 \cos \Phi, \qquad \dots (5)$$

$$\frac{\partial u_1}{\partial t} = V u_0 u_2 \cos \Phi, \qquad \dots (6)$$

$$\frac{\partial u_2}{\partial t} = V u_0 u_1 \cos \Phi \qquad \dots (7)$$

and

$$\frac{\partial \Phi}{\partial t} + \delta\omega = -V\left[\frac{u_1 u_2}{u_0} + \frac{u_0 u_2}{u_1} + \frac{u_0 u_1}{u_2}\right]\sin \Phi, \qquad \dots (8)$$

where

$$\Phi = \varphi_0 - \varphi_1 - \varphi_2 + \frac{\pi}{2}, \qquad \dots (9)$$

$$\delta\omega = \Sigma\ \beta_j u_j^2 \qquad \dots (10)$$

and

$$\beta_j = \alpha_j - \alpha'_j - \alpha''_j\ . \qquad \dots (10a)$$

3. Constants of Motion and Wave Solutions

From eqs. (5-7) we have the independent constants of motion equations

$$u_k^2 - u_{k_1}^2 = M_1, \qquad \dots (11)$$

$$u_k^2 - u_{k_2}^2 = M_2 \qquad \dots (12)$$

and from eq. (8)

$$\left[\frac{\partial \Phi}{\partial t} + (\beta_0 + \beta_1 + \beta_2)\, u_k^2 - (\beta_1 M_1 + \beta_2 M_2\right] \cot an\ \Phi = -\frac{\partial}{\partial t} \ln (u_k u_{k_1} u_{k_2}) \qquad \dots (13)$$

The β-terms entering eq. (13) introduce an effective change in the eigen frequencies of the waves. The constant of motion for $\beta_i = 0$, (i = 0, 1, 2)[15], $u_k u_{k_1} u_{k_2} \sin \Phi = \Gamma_0$ does not exist for $\beta_i \neq 0$ (cf. [20]). The presence of the β-terms will in fact influence the evolution of phases that would otherwise lead to unbounded solutions for the amplitudes in a finite time[15]. Let us introduce the notations

$$u_{k_j}^2 = n_{k_j} \quad \beta_0 + \beta_1 + \beta_2 = \gamma \;, \beta_1 M_1 + \beta_2 M_2 = \delta. \qquad \text{... (14)}$$

From eqs. (11-14) we then have

$$\left[n_k (n_k - M_1)(n_k - M_2) \right]^{-1/2} \frac{\partial}{\partial t} \left\{ \left[n_k (n_k - M_1)(n_k - M_2) \right]^{1/2} \sin \Phi \right\}$$

$$= -(\gamma n_k - \delta) \cos \Phi. \qquad \text{... (15)}$$

By introducing furthermore the notation

$$\left[n_k (n_k - M_1)(n_k - M_2) \right]^{1/2} \sin \Phi = y \qquad \text{... (16)}$$

we obtain from eq. (15),

$$\frac{\partial y}{\partial t} [n_k (n_k - M_1)(n_k - M_2)]^{1/2} (\gamma n_k - \delta) \cos \Phi. \qquad \text{... (17)}$$

We also have from eq. (5),

$$\frac{\partial n_k}{\partial t} = 2V \sqrt{n_k (n_k - M_1)(n_k - M_2)} \cos \Phi. \qquad \text{... (18)}$$

From eqs. (17) and (18) we then find

$$\frac{dy}{dn_k} = -\frac{1}{2V} (\gamma n_k - \delta) \qquad \text{... (19)}$$

which can be integrated to give

$$y = -\frac{\gamma n_k^2}{4V} + \frac{\delta}{2V} n_k + \Gamma, \qquad \text{... (20)}$$

where the constant Γ is determined by the initial conditions, and where for $\delta = \gamma = 0$ we recognize the constant of motion

$$\Gamma_0 = u_k u_{k_1} u_{k_2} \sin \Phi. \qquad \text{... (20a)}$$

We now consider eq. (18).. From eq. (16) we have

$$\cos \Phi = \pm \frac{\left[n_k (n_k - M_1)(n_k - M_2) - y^2 \right]^{1/2}}{\left[n_k (n_k - M_1)(n_k - M_2) \right]^{1/2}} \qquad \text{... (21)}$$

and we obtain from eq. (18)

$$\frac{\partial n_k}{\partial t} = \pm 2V\left\{ -n_k^4\frac{\gamma^2}{16V^2} + n_k^3\left(1+\frac{\gamma\delta}{4V^2}\right) - n_k^2\left[M_1+M_2+\left(\frac{\delta}{2V}\right)^2 - \frac{\gamma\Gamma}{2V}\right] + n_k\left(M_1M_2-\frac{\delta\Gamma}{V}\right) - \Gamma^2\right\}^{1/2} \quad \text{... (22)}$$

Eq. (22) can be rewritten in the form

$$\frac{1}{2}\left(\frac{\partial n_k}{\partial t}\right)^2 + \Pi\,(n_k) = 0 \quad \text{... (23)}$$

where

$$\Pi(n) = 2V^2\left\{ \frac{\gamma^2}{16V^2}n_k^4 - \left(1-\frac{\gamma\delta}{4V^2}\right)n_k^3 + \left[M_1+M_2+\left(\frac{\delta}{2V}\right)^2 - \frac{\gamma\Gamma}{2V}\right]n_k^2 - \left(M_1M_2-\frac{\delta\Gamma}{V}\right)n_k + \Gamma^2\right\} \quad \text{...(24)}$$

and where we may regard eq. (23) as an analogue of the law of conservation of energy. In the following we will consider different cases for which $\Gamma = 0$ or $\Gamma \neq 0$.

3.1. THE CONSTANTS OF MOTION $M_1 = M_2 = 0$ AND $\Gamma = 0$

Let us investigate the solutions to. eqs. (23) and (24) under the assumption that the constants of motion are (11) and (12) $M_1 = M_2 = 0$ and $\Gamma = 0$ in (20), (23) and (24), where then also in (14) the quantity $\delta = 0$.

According to expression (24) we have for this case

$$\Pi(n) = 2V\left(\frac{\gamma^2}{16V^2}n^4 - n^3\right) \quad \text{... (25)}$$

in which we have introduced the notation n for n_k. We notice that $\Pi(n)$ in (25) has a minimum value

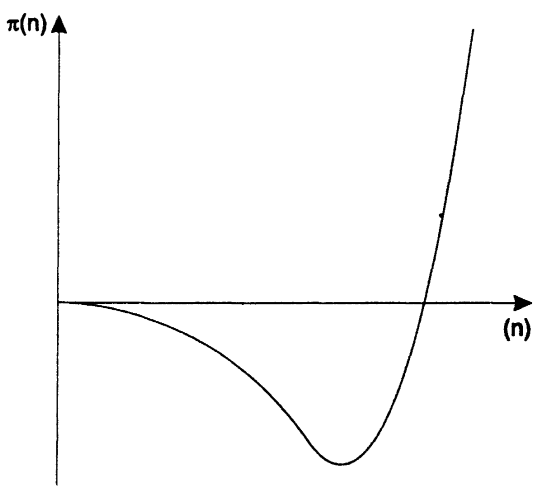

FIG. 1(*a*) The function $\Pi(n)$, (25).

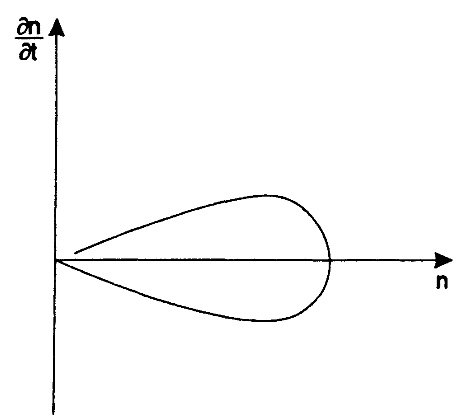

FIG. 1(*b*) The phase plane (29).

$$[\Pi(n)]_{\min} = -\frac{V^2}{2}\left(\frac{12\,V^2}{\gamma^2}\right)^3 \qquad \text{... (26)}$$

for

$$n_{\min} = \frac{12V^2}{\gamma^2} \qquad \text{... (27)}$$

and that

$$\Pi(n) \text{ has a zero for } n = \frac{16V^2}{\gamma^2} \qquad \text{... (28)}$$

beside the triple root of $\Pi(n) = 0$ at the origin (25). The behaviour of $\Pi(n)$ as a function of n is plotted in Fig. 1*a*.

When we consider $M_1 = M_2 = 0, \Gamma = 0$, eq. (23) leads to

$$\frac{\partial n}{\partial t} = \pm 2V\sqrt{-\frac{\gamma}{16V^2}n^4 - n^3} \qquad \text{... (29)}$$

The relation (29) has been plotted in a phase plane description in Fig. 1*b*. The solution for n in this case corresponds to a soliton solution[4] in time, which can be expressed as

$$n(t) = \frac{1}{V^2\left[\frac{D[n(0)]}{V} - t\right]^2 + \frac{\gamma^2}{16V^2}}. \qquad \text{... (30)}$$

where

$$D[N(0)] = \frac{1}{n(0)}\sqrt{n(0) - \frac{\gamma^2}{16V^2}n^2(0)}\,. \qquad \text{... (30a)}$$

The solution for $n(t)$ as given by relation (30) has a maximum value

$$n_{\max} = \frac{16V^2}{\gamma} \qquad \text{... (31)}$$

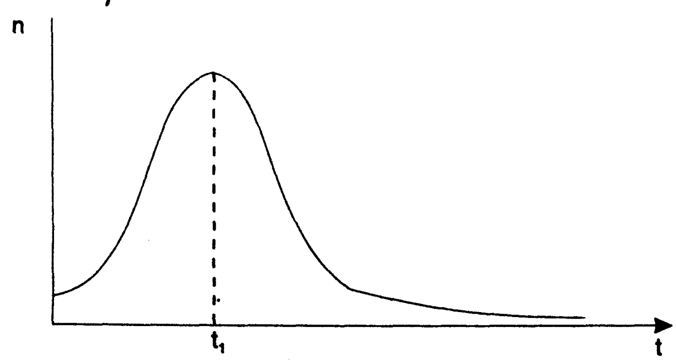

FIG. 2. The solution (30) for $n(t)$.

for

$$t = t_1 = \frac{D[n(0)]}{V} \qquad \dots (32)$$

as depicted in Fig. 2.

3.2 CONSTANTS OF MOTION $M_1 = M_2 = 0$ BUT $\Gamma \neq 0$

Let us now treat eqs. (23) and (24) when $M_1 = M_2 = 0$ but $\Gamma \neq 0$. It is then essential to discuss separately the solutions obtained for $\gamma > 0$, and $\gamma < 0$, where γ, as defined by (14), determines the sign of the nonlinear frequency shift in the phase relation (8, 13). We first consider $\gamma > 0$. We then have from eq. (24), (assuming Γ positive),

$$\Pi(n) = 2V^2\left(\frac{\gamma^2}{16V^2}n^4 - n^3 - \frac{\gamma\Gamma}{2V}n^2 + \Gamma^2\right) \qquad \dots (33)$$

The function $\Pi(n)$ given by (33) has three extremum points given by

$$n_{ext}^{(1)} = 0;\ n_{ext}^{(2,3)} = \frac{3 \pm \sqrt{9 + \frac{\gamma^3\Gamma}{V^3}},}{\frac{\gamma^2}{2V^2}} \qquad \dots (34)$$

where the first[(1)] corresponds to a maximum for the function $\Pi(n)$, whereas the two remaining[(2,3)] correspond to minima. The equation $\Pi(n) = 0$, where $\Pi(n)$ is given by (33), can have two positive real roots and two complex roots or four complex roots whereas the possibility of two positive and two negative real roots does not exist for the given form of $\Pi(n)$. The case of four complex roots is of no interest in this connection and we consider in the following only the case of two positive real roots and two complex roots (see Fig. 3).

In accordance with Fig. 3 we wxpect a periodic solution for $a_2 \leq n(t) \leq a_1$. From eqs. (23) and (33) we then have

$$\frac{\partial n}{\partial t} = \pm 2V\left(-\frac{\gamma^2}{16V^2}n^4 + n^3 + \frac{\gamma\Gamma}{2V}n^2 - \Gamma^2\right)^{1/2} \qquad \dots (35)$$

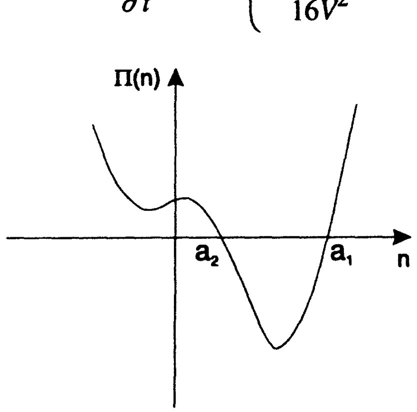

FIG. 3. $\Pi(n)$ according to (33).

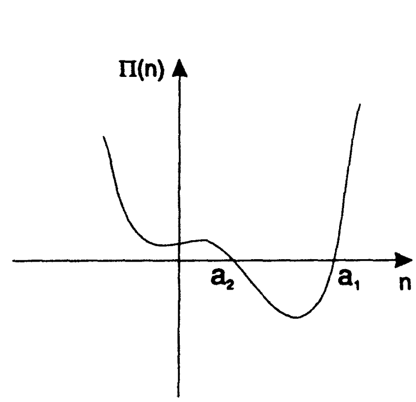

FIG. 4. $\Pi(n)$ according to (43) in the case of two real roots to $\Pi(n) = 0$.

which can be transformed to

$$\int_{n(0)}^{n(t)} \frac{dn}{\sqrt{-\frac{\gamma^2}{16V^2}n^4+n^3+\frac{\gamma\Gamma}{2V}n^2-\Gamma^2}} = \pm 2Vt. \qquad \text{... (36)}$$

We denote the roots of $\Pi(n) = 0$, where $\Pi(n)$ is defined by (33), as follows:

$a_1; a_2$ (two real roots, $a_1 > a_2 > 0$)

$b \pm id$ (two complex roots)

We then have

$$-\frac{\gamma^2}{16V^2}n^4+n^3+\frac{\gamma\Gamma}{2V}n^2-\Gamma^2 \qquad \text{... (37)}$$

$$= -\frac{\gamma^2}{16V^2}(n-a_1)(n-a_2)[n-(b+id)][n-(b-id)].$$

Let us transform the integral in (36) by introducing the following variables:

$$\tan^2\left(\frac{\varphi(t)}{2}\right) = \frac{\cos\theta_1}{\cos\theta_2}\frac{a_1-n}{n-a_2}, \qquad \text{... (38)}$$

where

$$\tan\theta_1 = \frac{a_1-b}{d}; \tan\theta_2 = \frac{a_2-b}{d}. \qquad \text{... (38a)}$$

If we define time in such a way that for $t = 0$ we have $n(0) = a_2$, then from eq. (37) $\varphi(0) = \pi$ and we can write for the left hand side of eq. (36)

$$\frac{4V}{\gamma}\mu \int_{\pi}^{\varphi(t)} \frac{d\varphi}{\sqrt{1-\chi^2\sin^2\varphi}}, \qquad \text{... (39)}$$

where

$$\mu = -\frac{(\cos\theta_1\cos\theta_2)^{1/2}}{d}; \chi^2 = \sin^2\left(\frac{\theta_2-\theta_1}{2}\right). \qquad \text{... (39a)}$$

Then

$$\int_{0}^{\varphi(t)} \frac{d\varphi}{\sqrt{1-\chi^2\sin^2\varphi}} = \pm\frac{\gamma}{2\mu}t + \int_{0}^{\pi} \frac{d\varphi}{\sqrt{1-\chi^2\sin^2\varphi}}. \qquad \text{... (40)}$$

Therefore

$$\sin\varphi(t) = sn\left[\pm\frac{\gamma}{2\mu}t + \int_0^{\pi}\frac{d\varphi}{\sqrt{1-\chi^2\sin^2\varphi}}\right] \equiv sn\,\Psi \qquad \text{... (41)}$$

and by using the definition of n from eq. (37) we obtain the periodic solution

$$n(t) = \frac{a_1\cos\theta_1\,(1+cn\,\Psi) + a_2\cos\theta_2\,(1-cn\,\Psi)}{\cos\theta_1\,(1+cn\,\Psi)+\cos\theta_2\,(1-cn\,\Psi)} \qquad \text{... (42)}$$

for $\gamma > 0$.

Let us now consider $\gamma < 0$. From eq. (24) we can then write (assuming Γ positive),

$$\Pi(n) = 2V^2\left(\frac{\gamma^2}{16V^2}n^4 - n^3 + \frac{|\gamma|\Gamma}{V^3}n^2 + \Gamma^2\right). \qquad \text{... (43)}$$

The function $\Pi(n)$ given by (43) has three extremum points, namely,

$$n_{ext}^{(1)} = 0;\ n_{ext}^{(2,3)} = \frac{\dfrac{3\pm\sqrt{9-|\gamma|^3}}{V^3}}{\dfrac{\gamma^2}{2V^2}}, \qquad \text{... (44)}$$

where the first[(1)] corresponds to a minimum for the function $\Pi(n)$, whereas the remaining two correspond to a maximum[(2)] and a minimum[(3)], respectively.

The equation $\Pi(n) = 0$, where $\Pi(n)$ is defined by eq. (43), has either two real roots or no real root. Since the latter case is of no interest here we consider only the first of these possibilities, for which $\Pi(n)$ has been plotted qualitatively in Fig. (4).

Introducing the roots of $\Pi(n) = 0$, where $\Pi(n)$ is defined by (43), we find

$a_1; a_2$ (two real roots, $a_1 > a_2 > 0$)

$b + id$ (two complex roots).

We can write

$$-\frac{\gamma^2}{16V^2}n^4 + n^3 - \frac{|\gamma|\Gamma}{V^3} = n^2 - \Gamma^2$$

$$= -\frac{\gamma^2}{16V^2}(n-a_1)(n-a_2)[n-(b+id)][n-(b-id)]$$

By introducing $\Psi = \int_0^{\pi}\frac{d\varphi}{\sqrt{1-\chi^2\sin^2\varphi}} \pm \frac{\gamma}{2\mu}t$

and making use of relations (38) and (39a) the periodic solution for $a_2 \le n \le a_1$ when $\gamma < 0$ (assuming Γ positive) can again be written in the form (42).

4. Concluding Remarks

We have analysed the influence of nonlinear frequency shifts on explosive instabilities under special initial conditions and found that solutions can be obtained in analytic form for all times. The soliton-type solution in time, found for the case where all waves have the same amplitude, (and $\Gamma = 0$), may be expected to change into an oscillatory type of structure with increasing time if dissipative effects are included. These phenomena as well as the solutions for more general initial conditions remain to be studied by further detailed analytic and computer investigations.

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