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ON CERTAIN FAMILIES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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We introduce the subclass $T_j(n, m, \lambda, \alpha)$ of analytic functions with negative coefficients defined by Salagean operators D^n and D^{n+m}. In this paper we give some properties of functions in the class $T_j(n, m, \lambda, \alpha)$ and obtain numerous sharp results including (for example) coefficient estimates, distorion theorems, closure theorems and modified Hadamard products of several functions belonging to the class $T_j(n, m, \lambda, \alpha)$. We also obtain radii of close to convexity, starlikeness, and convexity for functions belonging to the class $T_j(n, m, \lambda, \alpha)$ and consider integral operators associated with functions belonging to the class $T_j(n, m, \lambda, \alpha)$.

Key Words : Analytic; Salagean Operator; Modified Hadamard Product

1. INTRODUCTION

Let $A(j)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \; (j \in N = \{1, 2, ...\}), \qquad ... (1.1)$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. For a function $f(z)$ in $A(j)$, we define

$$D^0 f(z) = f(z), \qquad ... (1.2)$$

$$D^1 f(z) = Df(z) = zf'(z) \qquad ... (1.3)$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \;\; (n \in N). \qquad ... (1.4)$$

The differential operator D^n was introduced by Salagean[5]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to $A(j)$ is in the class $S_j(n, m, \lambda, \alpha)$ if and only if

$$Re\left\{\frac{(1-\lambda)\,z(D^n f(z))'+\lambda z\,(D^{n+m}f(z))'}{(1-\lambda)D^n f(z)+\lambda D^{n+m}f(z)}\right\}>\alpha(n, m\in N_0=N\cup\{0\}) \quad \text{.... (1.5)}$$

for some $\alpha\,(0\le\alpha<1)$ and $\lambda\,(0\le\lambda\le 1)$, and for all $z\in U$. The operator D^{n+m} was studied by Sekine[7] and Aouf and Salagean[2].

Let $T(j)$ denote the subclass of $A(j)$ consisting of functions of the form

$$f(z)=z-\sum_{k=j+1}^{\infty} a_k z^k\ (a_k\ge 0; j\in N). \quad \text{... (1.6)}$$

Further, we define the class $T_j(n, m, \lambda, \alpha)$ by

$$T_j\,(n, m, \lambda, \alpha)=S_j\,(n, m, \lambda, \alpha)\cap T(j). \quad \text{... (1.7)}$$

We note that by specializing the parameters j, n, m, λ and α, we obtain the following subclasses studied by various authors:

(*i*) $T_j(n, 1, \lambda, \alpha)=P(j, \lambda, \alpha, n)$, $T_j(n, m, 0, \alpha)=P(j, \alpha, n)$ and $T_j(n, 1, 1, \alpha)=P(j, \alpha, n+1)$ (Aouf and Srivastava[3]);

(*ii*) $T_j(0, 1, \lambda, \alpha)=P(j, \lambda, \alpha)$ (Altintas[1]);

(*iii*) $T_j\,(0, 0, 0, \alpha)=T_\alpha(j)$ and $T_j\,(0, 1, 1, \alpha)=T_j(1, 0, 1, \alpha)=C_\alpha(j)$ (Chatterjee[4] and Srivastava *et al.*[9]);

(*iv*) $T_j\,(n, m, 1, \alpha)=T_j(n, m, \alpha)$, where $T_j(n, m, \alpha)$ represents the class of functions $f(z)\in T(j)$ satisfying the condition

$$Re\left\{\frac{z(D^{n+m}f(z))'}{D^{n+m}f(z)}\right\}>\alpha\,(n, m\in N_0; 0\le\alpha<1; z\in U); \text{ and} \quad \text{... (1.8)}$$

(*v*) $T_1(0, 0, 0, \alpha)=T^*(\alpha)$ and $T_1(0, 1, 1, \alpha)=T_1(1, 0, 1, \alpha)=C(\alpha)$ (Silverman[8]).

2. Coefficient Estimates and other Properties of the Class $T_j(n, m, \lambda, \alpha)$

Theorem 1 — *Let the function $f(z)$ be defined by (1.6). Then $f(z)\in T_j\,(n, m, \lambda, \alpha)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n\,(k-\alpha)\,[1+(k^m-1)\lambda]\,a_k\le 1-\alpha. \quad \text{... (2.1)}$$

The result is sharp.

PROOF : Assume that the inequality (2.1) holds true. Then we find that

$$\left|\frac{(1-\lambda)\,z(D^n f(z))'+\lambda z(D^{n+m}f(z))'}{(1-\lambda)\,D^n f(z)+\lambda D^{n+m}f(z)}-1\right|$$

$$\leq \frac{\sum_{k=j+1}^{\infty} k^n (k-1) [1+(k^m-1)\lambda] a_k |z|^{k-1}}{1-\sum_{k=j+1}^{\infty} k^n [1+(k^m-1)\lambda] a_k |Z|^{k-1}}$$

$$\leq \frac{\sum_{k=j+1}^{\infty} k^n(k-1) [1+(k^m-1)\lambda] a_k}{1-\sum_{k=j+1}^{\infty} k^n [1+(k^m-1)\lambda] a_k} \leq 1-\alpha.$$

This shows that the values of the function

$$\Phi(z) = \frac{(1-\lambda) z(D^n f(z))' + \lambda z(D^{n+m} f(z))'}{(1-\lambda) D^n f(z) + \lambda D^{n+m} f(z)} \qquad \dots (2.2)$$

lie in a circle which is centered at $w = 1$ and whose radius is $1 - \alpha$. Hence, $f(z)$ satisfies the condition (1.5).

Conversely, assume that the function $f(z)$ is in the class $T_j(n, m, \lambda, \alpha)$. Then we have

$$Re\left\{\frac{(1-\lambda) z(D^n f(z))' + \lambda z(D^{n+m} f(z))'}{(1-\lambda) D^n f(z) + \lambda D^{n+m} f(z)}\right\}$$

$$= Re\left\{\frac{1-\sum_{k=j+1}^{\infty} k^{n+1} [1+(k^m-1)\lambda] a_k z^{k-1}}{1-\sum_{k=j+1}^{\infty} k^n [1+(k^m-1)\lambda] a_k z^{k-1}}\right\} > \alpha \qquad \dots (2.3)$$

for some $\alpha (0 \leq \alpha < 1)$, $\lambda (0 \leq \lambda \leq 1)$, $n, m \in N_0$ and for all $z \in U$. Choose values of z on the real axis so that $\Phi(z)$ given by (2.2) is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we can see that

$$1-\sum_{k=j+1}^{\infty} k^{n+1} [1+(k^m-1)\lambda] a_k$$

$$\geq \alpha\left\{1-\sum_{k=j+1}^{\infty} k^n [1+(k^m-1)\lambda] a_k\right\}. \qquad \dots (2.4)$$

Thus we have the inequality (2.1).

Finally, the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{k^n (k-\alpha)[1+(k^m-1)\lambda]} z^k \; (k \geq j+1; j \in N) \qquad \text{... (2.5)}$$

is an extremal function for the assertion of Theorem 1.

Corollary 1 — Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then

$$a_k \leq \frac{1-\alpha}{k^n (k-\alpha)[1+(k^m-1)\lambda]} \; (k \geq j+1). \qquad \text{... (2.6)}$$

The equality in (2.6) is attained for the function $f(z)$ given by (2.5).

***Theorem* 2** — *Let* $0 \leq \alpha_1 \leq \alpha_2 < 1, 0 \leq \lambda \leq 1, j \in N$ *and* $n, m \in N_0$. *Then*

$$T_j(n, m, \lambda, \alpha_1) \supseteq T_j(n, m, \lambda, \alpha_2). \qquad \text{... (2.7)}$$

PROOF : Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha_2)$ and let $\alpha_1 = \alpha_2 - \delta$. Then, by Theorem 1, we have

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha_2)[1+(k^m-1)\lambda] a_k \leq 1-\alpha_2 \qquad \text{... (2.8)}$$

and

$$\sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda] a_k \leq \frac{1-\alpha_2}{j+1-\alpha_2} < 1. \qquad \text{... (2.9)}$$

Consequently,

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha_1)[1+(k^m-1)\lambda] a_k = \sum_{k=j+1}^{\infty} k^n(k-\alpha_2)[1+(k^m-1)\lambda] a_k$$

$$+ \delta \sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda] a_k \leq 1-\alpha_1. \qquad \text{... (2.10)}$$

This completes the proof of Theorem 2 with the aid of Theorem 1.

***Theorem* 3** — *Let* $0 \leq \alpha < 1, 0 \leq \lambda_1 \leq \lambda_2 \leq 1, j \in N$ *and* $n, m \in N_0$.

Then

$$T_j(n, m, \lambda_1, \alpha) \supseteq T_j(n, m, \lambda_2, \alpha). \qquad \text{... (2.11)}$$

PROOF : It follows from Theorem 1 that

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda_1] a_k$$

$$\leq \sum_{k=j+1}^{\infty} k^n (k-\alpha) [1 + (k^m - 1) \lambda_2] a_k \leq 1 - \alpha$$

for $f(z) \in T_j(n, m, \lambda_2, \alpha)$.

Theorem 4 — *For* $0 \leq \alpha < 1, 0 \leq \lambda \leq 1, j \in N$ *and* $n, m \in N_0$,

$$T_j(n+1, m, \lambda, \alpha) \subseteq T_j(n, m, \lambda, \alpha). \quad \text{... (2.12)}$$

The proof of Theorem 4 follows also from Theorem 1.

3. Growth and Distortion Theorems

Theorem 5 — *Let the function* $f(z)$ *defined by (1.6) be in the class* $T_j(n, m, \lambda, \alpha)$. *Then for* $|z| = r < 1$,

$$|D^i f(z)| \geq r - \frac{1-\alpha}{(j+1)^{n-i} (j+1-\alpha) [1 + [(j+1)^m - 1]\lambda]} r^{j+1} \quad \text{... (3.1)}$$

and

$$|D^i f(z)| \leq r + \frac{1-\alpha}{(j+1)^{n-i} (j+1-\alpha) [1 + [(j+1)^m - 1]\lambda]} r^{j+1} \quad \text{... (3.2)}$$

for $z \in U$ *and* $0 \leq i \leq n$. *The equalities in (3.1) and (3.2) are attained for the function* $f(z)$ *given by*

$$f(z) = z - \frac{1-\alpha}{(j+1)^n (j+1-\alpha) [1 + [(j+1)^m - 1]\lambda]} z^{j+1} \; (z = \pm r). \quad \text{... (3.3)}$$

Proof : Note that $f(z) \in T_j(n, m, \lambda, \alpha)$ if and only if

$$D^i f(z) \in T_j(n-i, m, \lambda, \alpha)$$

and that

$$D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i a_k z^k. \quad \text{... (3.4)}$$

By Theorem 1, we know that

$$(j+1)^{n-i} (j+1-\alpha) [1 + [(j+1)^m - 1] \lambda] \sum_{k=j+1}^{\infty} k^i a_k$$

$$\leq \sum_{k=j+1}^{\infty} k^n (k-\alpha) [1 + (k^m - 1)\lambda] a_k \leq 1 - \alpha, \quad \text{... (3.5)}$$

that is, that

$$\sum_{k=j+1}^{\infty} k^i a_k \leq \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]}. \qquad \text{... (3.6)}$$

The assertions (3.1) and (3.2) of Theorem 5 would now follow readily from (3.4) and (3.6).

Finally, we note that the equalities in (3.1) and (3.2) are attained for the function $f(z)$ defined by

$$D^i f(z) = z - \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} z^{j+1}. \qquad \text{... (3.7)}$$

This completes the proof of Theorem 5.

Corollary 2 — Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then, for $|z| = r < 1$,

$$|f(z)| \geq r - \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^{j+1} \qquad \text{... (3.8)}$$

and

$$|f(z)| \leq r - \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^{j+1} \; (z \in U). \qquad \text{... (3.9)}$$

The equalities in (3.8) and (3.9) are attained for the function $f(z)$ given by (3.3).

PROOF : Taking $i = 0$ in Theorem 5, we immediately obtain (3.8) and (3.9).

Corollary 3 — Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then for $|z| = r < 1$,

$$|f'(z)| \geq 1 - \frac{1-\alpha}{(j+1)^{n-1}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^j \qquad \text{... (3.10)}$$

and

$$|f'(z)| \geq 1 - \frac{1-\alpha}{(j+1)^{n-1}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^j \; (z \in U). \qquad \text{... (3.11)}$$

The equalities in (3.10) and (3.11) are attained for the function $f(z)$ given by (3.3).

PROOF : Setting $i = 1$ in Theorem 5, and making use of the definition (1.3), we arrive at Corollary 3.

4. Convex Linear Combinations

In this section, we shall prove that the class $T_j(n, m, \lambda, \alpha)$ is closed under convex linear combinations.

***Theorem* 6** — *$T_j(n, m, \lambda, \alpha)$ is a convex set.*

PROOF : Let the functions

$$f_v(z) = z - \sum_{k=j+1}^{\infty} a_{k,v} Z^k \ (a_{k,v} \geq 0;\ v = 1, 2) \quad \text{... (4.1)}$$

be in the class $T_j(n, m, \lambda, \alpha)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \mu f_1(z) + (1 - \mu) f_2(z) \ (0 \leq \mu \leq 1) \quad \text{... (4.2)}$$

is also in the class $T_j(n, m, \lambda, \alpha)$. Since, for $0 \leq \mu \leq 1$,

$$h(z) = z - \sum_{k=j+1}^{\infty} [\mu a_{k,1} + (1 - \mu)\, a_{k,2}]\, z^k, \quad \text{... (4.3)}$$

with the aid of Theorem 1, we have

$$\sum_{k=j+1}^{\infty} k^n (k - \alpha)\, [1 + (k^m - 1)\lambda]\, [\mu a_{k,1} + (1 - \mu) a_{k,2}] \leq 1 - \alpha, \quad \text{... (4.4)}$$

which implies that $f(z) \in T_j\,(n, m, \lambda, \alpha)$. Hence, $T_j\,(n, m, \lambda, \alpha)$ is a convex set.

***Theorem* 7** — *Let*

$$f_j(z) = z \quad \text{... (4.5)}$$

and

$$f_k(z) = z - \frac{1 - \alpha}{k^n (k - \alpha)\, [1 + (k^m - 1)\lambda]}\, z^k \ (k \geq j + 1;\ n, m, \in N_0) \quad \text{... (4.6)}$$

for $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$. Then $f(z)$ is in the class $T_j\,(n, m, \lambda, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z), \quad \text{... (4.7)}$$

where

$$\mu_k \geq 0\ (k \geq j) \text{ and } \sum_{k=j}^{\infty} \mu_k = 1. \quad \text{... (4.8)}$$

PROOF : Assume that

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z) = Z - \sum_{k=j+1}^{\infty} \frac{1-\alpha}{k^n (k-\alpha)[1+(k^m-1)\lambda]} \mu_k z^k. \quad \text{... (4.9)}$$

Then it follows that

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} \cdot \frac{1-\alpha}{k^n (k-\alpha)[1+(k^m-1)\lambda]} \mu_k$$

$$= \sum_{k=j+1}^{\infty} \mu_k = 1 - \mu_j \leq 1. \quad \text{... (4.10)}$$

So, by Theorem 1, $f(z) \in T_j(n, m, \lambda, \alpha)$.

Conversely, assume that the function $f(z)$ defined by (1.6) belongs to the class $T_j(n, m, \lambda, \alpha)$. Then

$$a_k \leq \frac{1-\alpha}{k^n (k-\alpha)[1+(k^m-1)\lambda]} \; (k \geq j+1; n, m \in N_0). \quad \text{... (4.11)}$$

Setting

$$\mu_k = \frac{k^n (k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_k \; (k \geq j+1; n, m \in N_0) \quad \text{... (4.12)}$$

and

$$\mu_j = 1 - \sum_{k=j+1}^{\infty} \mu_k, \quad \text{... (4.13)}$$

we can see that $f(z)$ can be expressed in the form (4.7). This completes the proof of Theorem 7.

5. Radii of Close-to-Convexity, Starlikeness, and Convexity

***Theorem* 8** — *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then $f(z)$ is close-to-convex of order $\rho (0 \leq \rho < 1)$ in $|z| < r_1$, where*

$$r_1 = r_1(n, m, \lambda, \alpha, \rho) = \inf_k \left[\frac{(1-\rho)k^{n-1}(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} \right]^{\frac{1}{k-1}} (k \geq j+1). \quad \text{... (5.1)}$$

The result is sharp, the extremal function $f(z)$ being given by (2.5).

PROOF : We must show that

$|f'(z) - 1| \leq 1 - \rho$ for $|z| < r_1(n, m, \lambda, \alpha, \rho)$, where $r_1(n, m, \lambda, \alpha, \rho)$ is given by (5.1). Indeed we find from the definition (1.6) that

$$|f'(z)-1| \leq \sum_{k=j+1}^{\infty} k a_k |z|^{k-1}.$$

Thus

$$|f'(z)-1| \leq 1-\rho,$$

if

$$\sum_{k=j+1}^{\infty} \left(\frac{k}{1-\rho}\right) a_k |z|^{k-1} \leq 1. \qquad \text{... (5.2)}$$

But, by Theorem 1, (5.2) will be true if

$$\left(\frac{k}{1-\rho}\right) |z|^{k-1} \leq \frac{k^n (k-\alpha)\,[1+(k^m-1)\,\lambda]}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)\,k^{n-1}(k-\alpha)\,[1+(k^m-1)\,\lambda]}{1-\alpha}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \qquad \text{... (5.3)}$$

Theorem 8 follows easily from (5.3).

Theorem 9 — *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then $f(z)$ is starlike of order $\rho (0 \leq \rho < 1)$ in $|z| < r_2$, where $r_2 = r_2(n, m, \lambda, \alpha, \rho)$ =*

$$\inf_k \left[\frac{(1-\rho)\,k^n (k-\alpha)\,[1+(k^m-1)\,\lambda]}{(k-\rho)\,(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \qquad \text{... (5.4)}$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

PROOF : It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho$$

for $|z| < r_2(n, m, \lambda, \alpha, \rho)$, where $r_2(n, m, \lambda, \alpha, \rho)$ is given by (5.4). Indeed we find, again from the definition (1.6), that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{k=j+1}^{\infty} (k-1)\, a_k |z|^{k-1}}{1 - \sum_{k=j+1}^{\infty} a_k |z|^{k-1}}.$$

Thus

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho$$

if

$$\sum_{k=j+1}^{\infty} \left(\frac{k-\rho}{1-\rho} \right) a_k |z|^{k-1} \leq 1. \qquad \dots (5.5)$$

But, by Theorem 1, (5.5) will be true if

$$\left(\frac{k-\rho}{1-\rho} \right) |Z|^{k-1} \leq \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho) k^n (k-\alpha) [1+(k^m-1)\lambda]}{(k-\rho)(1-\alpha)} \right]^{\frac{1}{k-1}} \quad (k \geq j+1). \qquad \dots (5.6)$$

Theorem 9 follows easily from (5.6).

Corollary 4 — Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then $f(z)$ is convex of order $\rho (0 \leq \rho < 1)$ in $|z| < r_3$, where

$$r_3 = r_3(n, m, \lambda, \alpha, \rho) = \inf_k \left[\frac{(1-\rho) k^{n-1} (k-\alpha) [1+(k^m-1)\lambda]}{(k-\rho)(1-\alpha)} \right]^{\frac{1}{k-1}} \quad (k \geq j+1). \qquad \dots (5.7)$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

6. Modified Hadamard Products

Let the functions $f_v(z) = (v = 1, 2)$ be defined by (4.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{k=j+1}^{\infty} a_{k,1} a_{k,2} z^k. \qquad \dots (6.1)$$

Theorem 10 — *Let each of the functions $f_v(z)$ (v = 1, 2) defined by (4.1) be in the class $T_j(n, m, \lambda, \alpha)$. Then*

$$f_1 * f_2(z) \in T_j(n, m, \lambda, \beta(j, n, m, \lambda, \alpha)),$$

where

$$\beta(j, n, m, \lambda, \alpha) = 1 - \frac{j(1-\alpha)^2}{(j+1)^n (j+1-\alpha)^2 [1+\lambda[(j+1)^m - 1)]] - (1-\alpha)^2}. \qquad \dots (6.2)$$

The result is sharp.

PROOF : Employing the technique used earlier by Schild and Silverman[6], we need to find the largest $\beta = \beta(j, n, m, \lambda, \alpha)$ such that

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\beta)\,[1+(k^m-1)\lambda]}{1-\beta}\, a_{k,1}\, a_{k,2} \le 1. \qquad \text{... (6.3)}$$

Since

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)\,[1+(k^m-1)\lambda]}{1-\alpha}\, a_{k,1} \le 1 \qquad \text{... (6.4)}$$

and

$$\sum_{k=j+1}^{\infty} \frac{k^n\,(k-\alpha)\,[1+(k^m-1)\,\lambda]}{1-\alpha}\, a_{k,2} \le 1, \qquad \text{... (6.5)}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)\,[1+(k^m-1)\lambda]}{1-\alpha}\, \sqrt{a_{k,1}\, a_{k,2}} \le 1; \text{ and} \qquad \text{... (6.6)}$$

Thus it is sufficient to show that

$$\frac{k^n\,(k-\beta)\,[1+(k^m-1)\,\lambda]}{1-\beta}\, a_{k,1}\, a_{k,2} \le \frac{k^n\,(k-\alpha)\,[1+(k^m-1)\,\lambda}{1-\alpha}\, \sqrt{a_{k,1}\, a_{k,2}}\ (k \ge j+1), \qquad \text{... (6.7)}$$

that is, that

$$\sqrt{a_{k,1}\, a_{k,2}} \le \frac{(k-\alpha)\,(1-\beta)}{(k-\beta)\,(1-\alpha)}\ (k \ge j+1). \qquad \text{... (6.8)}$$

Note that

$$\sqrt{a_{k,1}\, a_{k,2}} \le \frac{1-\alpha}{k^n\,(k-\alpha)\,[1+(k^m-1)\,\lambda]}\ (k \ge j+1). \qquad \text{... (6.9)}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{k^n\,(k-\alpha)\,[1+(k^m-1)\,\lambda]} \le \frac{(k-\alpha)\,(1-\beta)}{k-\beta)\,(1-\alpha)}\ (k \ge j+1), \qquad \text{... (6.10)}$$

or, equivalently, that

$$\beta \le 1 - \frac{(k-1)\,(1-\alpha)^2}{k^n\,(k-\alpha)^2\,[1+(k^m-1)\,\lambda] - (1-\alpha)^2}\ (k \ge j+1). \qquad \text{... (6.11)}$$

Since

$$A(k) = 1 - \frac{(k-1)(1-\alpha)^2}{k^n (k-\alpha)^2 [1 + (k^m - 1)\lambda] - (1-\alpha)^2} \quad \text{... (6.12)}$$

is an increasing function of k $(k \geq j+1)$, letting $k = j + 1$ in (6.12) we obtain

$$\beta \leq A(j+1) = 1 - \frac{j(1-\alpha)^2}{(j+1)^n (j+1-\alpha)^2 [1 + [(j+1)^m - 1]\lambda] - (1-\alpha)^2}, \quad \text{... (6.13)}$$

which proves the main assertion of Theorem 10.

Finally, by taking the functions

$$f_\nu(z) = z - \frac{(1-\alpha)}{(j+1)^n (j+1-\alpha)[1 + [(j+1)^m - 1]\lambda]} z^{j+1} \; (\nu + 1, 2), \quad \text{... (6.14)}$$

we can see that the result is sharp.

Theorem 11 — *Let* $f_1(z) \in T_j(n, m, \lambda, \alpha)$ *and* $f_2(z) \in T_j(n, m, \lambda, \gamma)$.

Then

$$f_1 * f_2 (Z) \in T_j(n, m, \lambda, \xi(j, n, m, \lambda, \alpha, \gamma)),$$

where

$$\xi(j, n, m, \lambda, \alpha, \gamma) = 1 - \frac{j(1-\alpha)(1-\gamma)}{(j+1)^n (j+1-\alpha)(j+1-\gamma)[1 + [(j+1)^m - 1]\lambda] - (1-\alpha)(1-\gamma)}. \quad \text{... 6.(15)}$$

The result is best possible for the functions

$$f_1(z) = z - \frac{1-\alpha}{(j+1)^n (j+1-\alpha)[1 + [(J+1)^m - 1]\lambda]} z^{j+1} \quad \text{... (6.16)}$$

and

$$f_2(z) = z - \frac{1-\gamma}{(j+1)^n (j+1-\gamma)[1 + [(j+1)^m - 1]\lambda]} z^{j+1}. \quad \text{... (6.17)}$$

PROOF : Proceeding as in the proof of Theorem 10, we get

$$\xi \leq 1 - \frac{(k-1)(1-\alpha)(1-\gamma)}{k^n (k-\alpha)(k-\gamma)[1 + (k^m - 1)\lambda] - (1-\alpha)(1-\gamma)} \; (k \geq j+1). \quad \text{... (6.18)}$$

Since the right hand side of (6.18) is an increasing function of k, setting $k = j + 1$ in (6.18) we obtain (6.15). This completes the prove of Theorem 11.

Corollary 5 — Let the functions $f_v(z)$ defined by

$$f_v(z) = z - \sum_{k=j+1}^{\infty} a_{k,v} z^k \; (a_{k,v} \geq 0, \; v = 1, 2, 3) \qquad \text{... (6.19)}$$

be in the class $T_j(n, m, \lambda, \alpha)$. Then

$$f_1 * f_2 * f_3(z) \in T_j(n, m, \lambda, \delta(j, n, m, \lambda, \alpha)),$$

where

$$\delta(j, n, m, \lambda, \alpha) = 1 - \frac{j(1-\alpha)^3}{(j+1)^{2n}(j+1-\alpha)^3[1+[(j+1)^m - 1]\lambda]^2 - (1-\alpha)^3}. \qquad \text{... (6.20)}$$

The result is best possible for the functions

$$f_v(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)[1+[(j+1)^m - 1]\lambda]} z^{j+1} \; (v = 1, 2, 3). \qquad \text{... (6.21)}$$

PROOF : From Theorem 10, we have

$$f_1 * f_2(z) \in T_j(n, m, \lambda, \beta(j, n, m, \lambda, \alpha)),$$

where β is given by (6.2). Now, using Theorem 11, we get

$$f_1 * f_2 * f_3(z) \in T_j(n, m, \lambda, \delta(j, n, m, \lambda, \alpha)),$$

where

$$\delta(j, n, m, \lambda, \alpha) = 1 - \frac{j(1-\alpha)(1-\beta)}{(j+1)^n(j+1-\alpha)(j+1-\beta)[1+[(j+1)^m - 1]\lambda] - (1-\alpha)(1-\beta)}$$

$$= 1 - \frac{j(1-\alpha)^3}{(j+1)^{2n}(j+1-\alpha)^3[1+[(j+1)^m - 1]\lambda] - (1-\alpha)^3}.$$

This completes the proof of Corollary 5.

Theorem 12 — *Let the functions* $f_v(z)\,(v = 1, 2)$ *defined by (4.1) be in the class* $T_j(n, m, \lambda, \alpha)$, *then the function*

$$h(z) = z - \sum_{k=j+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \qquad \text{... (6.22)}$$

belongs to the class $T_j(n, m, \lambda, \eta(j, n, m, \lambda, \alpha))$, *where*

$$\eta(j, n, m, \lambda, \alpha) = 1 - \frac{2j(1-\alpha)^2}{(j+1)^n(j+1-\alpha)^2[1+[(j+1)^m - 1]\lambda] - 2(1-\alpha)^2}. \qquad \text{... (6.23)}$$

The result is sharp for the functions $f_v(z) = (v = 1, 2)$ *defined by (6.14).*

PROOF : By virtue of Theorem 1, we obtain

$$\sum_{k=j+1}^{\infty} \left[\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} \right]^2 a_{k,1}^2$$

$$\leq \left[\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_{k,1} \right]^2 \leq 1 \qquad \dots (6.24)$$

and

$$\sum_{k=j+1}^{\infty} \left[\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} \right]^2 a_{k,2}^2$$

$$\leq \left[\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_{k,2} \right]^2 \leq 1. \qquad \dots (6.25)$$

It follows from (6.24) and (6.25) that

$$\sum_{k=j+1}^{\infty} \frac{1}{2} \left[\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} \right]^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1. \qquad \dots (6.26)$$

Therefore, we need to find the largest $\eta = \eta(j, n, m, \lambda, \alpha)$ such that

$$\frac{k^n(k-\eta)[1+(k^m-1)\lambda]}{1-\eta} \leq \frac{1}{2} \left[\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} \right]^2 \quad (k \geq j+1), \qquad \dots (6.27)$$

that is,

$$\eta \leq 1 - \frac{2(k-1)(1-\alpha)^2}{(k-\alpha)^2 k^n [1+(k^m-1)\lambda] - 2(1-\alpha)^2} \quad (k \geq j+1). \qquad \dots (6.28)$$

Since

$$B(k) = 1 - \frac{2(k-1)(1-\alpha)^2}{k^n (k-\alpha)^2 [1+(k^m-1)\lambda] - 2(1-\alpha)^2}. \qquad \dots (6.29)$$

is an increasing function of $k (k \geq j+1)$, we readily have

$$\eta \leq B(j+1) = 1 - \frac{2j(1-\alpha)^2}{(j+1)^n (j+1-\alpha)^2 [1+[(j+1)^m-1]\lambda] - 2(1-\alpha)^2}, \qquad \dots (6.30)$$

and Theorem 12 follows at once.

7. A Family of Integral Operators

Theorem 13 — *Let the function f(z) defined by (1.6) be in the class* $T_j(n, m, \lambda, \alpha)$, *and let c be a real number such that c > – 1. Then the function F(z) defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt \ (c > -1) \quad \dots (7.1)$$

also belongs to the class $T_j(n, m, \lambda, \alpha)$.

PROOF : From the representation (7.1) of $F(z)$, it follows that

$$F(z) = z - \sum_{k=j+1}^{\infty} b_k z^k ,$$

where

$$b_k = \left(\frac{c+1}{c+k} \right) a_k.$$

Therefore, we have

$$\sum_{k=j+1}^{\infty} k^n (k-\alpha)[1+(k^m-1)\lambda] b_k = \sum_{k=j+1}^{\infty} k^n (k-\alpha)[1+(k^m-1)\lambda] \left(\frac{c+1}{c+k} \right) a_k$$

$$\leq \sum_{k=j+1}^{\infty} k^n (k-\alpha)[1+(k^m-1)\lambda] a_k \leq 1-\alpha,$$

since $f(z) \in T_j(n, m, \lambda, \alpha)$. Hence, by Theorem 1, $F(z) \in T_j(n, m, \lambda, \alpha)$.

Theorem 14 — *Let the function*

$$F(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \ (a_k \geq 0, j \in N)$$

be in the class $T_j(n, m, \lambda, \alpha)$, *and let c be a real number such that* $c > -1$. *Then the function f(z) given by (7.1) is univalent in* $|z| < R^*$, *where*

$$R^* = \inf_k \left[\frac{(k-\alpha)\, k^{n-1}[1+(k^m-1)\lambda]\,(c+1)}{(1-\alpha)(c+k)} \right]^{\frac{1}{k-1}} (k \geq j+1). \quad \dots (7.2)$$

The result is sharp.

PROOF : From (7.1), we have

$$f(z) = \frac{z^{1-c}(z^c F(z))'}{c+1} = z - \sum_{k=j+1}^{\infty} \left(\frac{c+k}{c+1}\right) a_k z^k.$$

In order to obtain the required result, it suffices to show that

$$|f'(z) - 1)| < 1 \text{ whenever } |z| < R^*,$$

where R^* is given by (7.2). Now

$$|f'(z) - 1| \leq \sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1} < 1. \qquad \text{... (7.3)}$$

But Theorem 1 confirms that

$$\sum_{k=j+1} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{(1-\alpha)} a_k \leq 1. \qquad \text{.. (7.4}$$

Hence, (7.3) will be satisfied if

$$\frac{k(c+k)}{c+1} |z|^{k-1} < \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{(1-\alpha)},$$

that is, if

$$|z| < \left[\frac{(k-\alpha)k^{n-1}[1+(k^m-1)\lambda](c+1)}{(1-\alpha)(c+k)}\right]^{\frac{1}{k-1}} (k \geq j+1). \qquad \text{... (7.5)}$$

Therefore, the function $f(z)$ given by (7.1) is univalent in $|z| < R^*$. Sharpness of the result follows if we take

$$f(z) = z - \frac{(1-\alpha)(c+k)}{k^n(k-\alpha)[1+(k^m-1)\lambda](c+1)} z^k (k \geq j+1). \qquad \text{... (7.6)}$$

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