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ATOMIC DECOMPOSITION IN $L^p(\mathcal{R}^n)$ $(1 < p < \infty)^*$

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The object of this paper is to give an atomic decomposition in L^p $(1 < p < 2)$ with tent space.

Key Words : $L^{p,s}(\mathcal{R}^n)$; **Tent Space** $T_p^{s,q}$; $\left(p, T_q^{s,q}\right)$ **atom;** (p, L^2) **Atom and Atomic Decomposition**

1. INTRODUCTION

In 1974, Coifman found and proved the first atomic decomposition for $H^p(\mathcal{R})$[1], then *R*. Latter generalized it to $H^p(\mathcal{R}^n)$ in 1978[4]; and Frazier and Jawerth gave an atomic decomposition for Besov space in 1985[3]. But we do not have an atomic decomposition for $L^p(\mathcal{R}^n)$ $(1 < p < 2)$ such that the characteristic function does not appear in the estimate of the structure of this atomic decomposition. Here we give such a result.

The objects of this paper are :

(I) Isomorphism between $L^{p,s}$ and the tent space $T_p^{s,2}$;

(II) $\left(p, T_q^{s,q}\right)$ atom and the related atomic decomposition in $T_p^{s,q}$; and

(III) (p, L^2) atom and the related decomposition.

2. ISOMORPHISM BETWEEN $L^{p,s}$ AND TENT SPACE $T_p^{s,2}$

For arbitrary $j \in Z$ and $k \in Z^n$, set $Q(j,k) = \{x : 2^j x - k \in (0,1)^n\}$ which is a dyadic cube. Denote $\mathcal{D} = \{Q_{j,k}, j \in Z, k \in Z^n\}$, $\Xi = \{\varepsilon \in \{0, 1\}^n, |\varepsilon| = \varepsilon_1 + \ldots\ldots + \varepsilon_n \neq 0\}$ and $A = \{(\varepsilon, Q), \varepsilon \in XI, Q \in \mathcal{D}\}$. $\psi(x)$ is derived from an multiresolution analyse $\{V_j\}_{j \in Z}$ in $L^2(\mathcal{R})$, generated by $\varphi(x) \in C^{2+|s|}$ where $\{\varphi(x+k)\}_{k \in Z}$ is a orthogonal basis and there exists a positive integer M, such that supp $\varphi(x) \subset [-2^M, 2^M]$, sup p $\psi(x) \subset [-2^M, 2^M]$. For $x \in \mathcal{R}$, denote $\Phi^{(0)}(x) = \varphi(x)$,

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$\Phi^{(1)}(x) = \psi(x)$. For $j \in \mathcal{Z}, k \in (k_1 \ldots\ldots k_n) \in \mathcal{Z}^n, \varepsilon \in \Xi, Q_{j,k} \in \mathcal{D}, x = (x_1, \ldots\ldots x_n) \in \mathcal{R}^n, \psi_{\varepsilon, Q_{j,k}}(x) = \Phi^{(\varepsilon)}_{Q_{j,k}}(x) = \Phi^{(\varepsilon)}_{j,k}(x) = 2^{\frac{1}{2}nj} \prod_{i=1}^{n} \Phi^{(\varepsilon)_i}(2^j x_i - k_i)$. Then $\{\psi_{\varepsilon, Q}\}_{(\varepsilon, Q) \in \Lambda}$ is a basis of $L^2(\mathcal{R}^n)$.

$\alpha = \{\alpha_{\varepsilon, Q}\}_{(\varepsilon, Q) \in \Lambda}$ is a sequence with index set Λ and it corresponds to the function $S_{s,q}(\alpha)(x) = \left(\sum_{(\varepsilon, Q) \in \Lambda} |Q|^{-q\left(\frac{s}{n}+\frac{1}{2}\right)} |\alpha_{\varepsilon, Q}|^q X_Q(x) \right)^{\frac{1}{q}}$ where $X_Q(x)$ is the characteristic function at Q.

Definition 1 — We say that α belongs to a tent space $T_p^{s,q}$, if $\|\alpha\|_{T_p^{s,q}} = \| S_{s,q}(\alpha) \|_{L^p} < +\infty$.

Meyer has proved the following result in Theorem 3 of Section 1, Chapter VI of [5].

Theorem 1 — *If* $1 < p < \infty$, *and* $f(x) = \sum_{(\varepsilon, Q) \in \Lambda} a_{\varepsilon, Q} \psi_{\varepsilon, Q}(x)$ *then* $f(x) \in L^{p,s}$ *if and only if* $\left\{ a_{\varepsilon, Q} \right\}_{(\varepsilon, Q) \in \Lambda} \in T_p^{s,2}$.

We know that if $s = 0$, then $L^{p,s} = L^p$.

Hence $\| \alpha_{\varepsilon, Q} \| T_p^{s,2}$ gives another norm of $f(x) \in L^{p,s}$. And we adopt this norm in the following sections.

3. $(p, T_q^{s,q})$ Atom and the Related Atomic Decompositions in $T_p^{s,q}$ for $1 \le p < q$

Definition 2 — We say that $\alpha = \left\{ \alpha_{\varepsilon, Q} \right\}_{(\varepsilon, Q) \in \Lambda}$ is a $(p, T_q^{s,q}, R)$ atom if there is a dyadic cube $R \in \mathcal{D}$ such that (*i*) if $Q \not\subset R$, then $\alpha_{\varepsilon, Q} = 0$; (*ii*) $\| S_{s,q}(\alpha) \|_{L^p} \le |R|^{\frac{1}{q} - \frac{1}{p}}$.

We can neglect R and say also that α is a $(p, T_q^{s,p})$ atom. It is evident that, if $q \ge p$ and α is a $(p, T_q^{s,p})$ atom, then $\alpha \in T_p^{s,q}$ and $\|\alpha\|_{T_p^{s,q}} \le 1$.

Supp α = $\{Q \in \mathcal{D}$ there exists an ε such that $\varepsilon \in \Xi$ and $a_{\varepsilon, Q} \neq 0\}$.

Theorem 2 — *For* $1 \le p < q < +\infty$, *there exist two positive constants* c_1 *and* c_2 *such that for an arbitrary sequence* $\alpha = \left\{ \alpha_{\varepsilon, Q} \right\}_{(\varepsilon, Q) \in \Lambda} \in T_p^{s,q}$, *there exist a sequence* $\{\lambda_m\}$ *and a sequence of* $(p, T_q^{s,q})$ *atoms* α_m *such that* $\alpha = \sum_m \lambda_m \alpha_m$ *and* λ_m *satisfy the condition*

$$C_1\left(\sum_m |\lambda_m|^p\right)^{\frac{1}{p}} \leq \| \alpha \|_{T_p^{s,q}} \leq C_2\left(\sum_m |\lambda_m|^p\right)^{\frac{1}{p}}.$$

PROOF : For $k \in \mathcal{Z}$, we denote E_k to be the set of x which satisfy the condition that $S_{s,q}(\alpha)(x) > 2^k$. Hence, $E_k \supset E_{k+1}$ and $\sum_{-\infty}^{+\infty} 2^{pk} |E_k| \leq 2 \int S^p_{s,q}(\alpha)(x)dx.$

We denote $\mathcal{D}_k$ the set of the dyadic cubes which satisfy that $|Q \cap E_k| \geq \beta |Q|$ where $0 < \beta < 1$, and $E_k^* = \bigcup_{Q \in \mathcal{D}_k} Q$. We have $|E_k^*| \leq \frac{1}{\beta} |E_k|$ and $\sum_{-\infty}^{+\infty} 2^{pk} |E_k^*| \leq \frac{2}{\beta} \| \alpha \| T_p^{p_{s,q}}$.

Denote $Q(k, l) \in \mathcal{D}_k$ to be the largest dyadic cubes such that $Q(k, l_1) \cap Q(k, l_2) = \phi$ for $l_1 \neq l_2$, and $E_k^* = \underset{l}{U} Q(k, l)$. Denote by $\Delta(k) = \{Q \in \mathcal{D}, Q \in \mathcal{D}_k$ and $Q \notin \mathcal{D}_{k+1}\}$ and $\Delta(k, l) = \{Q \in \mathcal{D}, Q \in \Delta(k)$ and $Q \subset Q(k, l)\}$.

We have

Proposition 1 — *(i) If* $|k - k'| + |l - l'| \neq 0$, *then* $\Delta(k, l) \cap \Delta(k', l') = \phi$;

(ii) $\bigcup_{k,l} \Delta(k, l) = \bigcup_k \Delta(k) = \bigcup_k \mathcal{D}_k$; *and*

(iii) The support of α *is contained in* $\bigcup_k \mathcal{D}k$.

PROOF : (*i*) If $k = k', l \neq l'$, since $Q(k, l') \cap Q(k, l) = \phi$, we have $\Delta(k, l) \cap \Delta(k, l') = \phi$.

If $k \neq k'$, since $\Delta(k) \cap \Delta(k') = \phi$, we have $\Delta(k, l) \cap \Delta(k, l') = \phi$.

(*ii*) By definition, we have $\Delta(k) = \bigcup_l \Delta(k, l)$, so we have $\bigcup_{(k,l)} \Delta(k, l) = \bigcup_k \Delta(k)$, and we have $\Delta(k) = \mathcal{D}_k / \mathcal{D}_{k+1}$.

So $\bigcup_k \Delta(k) = \bigcup_k \mathcal{D}_k$.

(*iii*) If $\alpha_{\varepsilon, Q} \neq 0$, then there exists an integer k such that $|\alpha_{\varepsilon, Q}| |Q|^{-\left(\frac{s}{n} + \frac{1}{2}\right)} > 2^k$ so we have

$$S_{s,q}(\alpha)(x) > 2^k \text{ and } |Q \cap E_k| = |Q| \geq \beta |Q|.$$

Thus $Q \in \mathcal{D}_k$, that is to say, the support of α is contained in $\bigcup_k \mathcal{D}_k$.

Then we define

$$\lambda_{k,l} = |Q(k,l)|^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{\substack{Q \in \Lambda(k,l) \\ \varepsilon \in \Xi}} |Q|^{1-q\left(\frac{s}{n}+\frac{1}{2}\right)} |\alpha_{\varepsilon,Q}|^q \right)^{\frac{1}{q}}$$

and

$$\alpha_{k,l}(\varepsilon, Q) = \begin{cases} \dfrac{1}{\lambda_{k,l}} \alpha_{\varepsilon,Q} & Q \in \Delta(k,l) \\ 0 & Q \notin \Delta(k,l) \end{cases}$$

According to the definition of $\alpha_{k,l}(\varepsilon, Q)$, we have that $\{\left\{ a_{k,l}(\varepsilon,Q) \right\}_{(\varepsilon,Q)\in\Lambda}$ is a $\left(p, T_q^{s,q}, Q(k,l) \right)$ atom. According to Proposition 1, we have

$$\alpha_{\varepsilon,Q} = \sum_k \sum_l \lambda_{k,l}\, \alpha_{k,l}(\varepsilon, Q).$$

As to the estimate of $\lambda_{k,l}$, we prove first

Lemma 1 — $\displaystyle\sum_{Q \in \Lambda(k,l)} \sum_{\varepsilon \in \Xi} |a_{\varepsilon,Q}|^q |Q|^{1-q\left(\frac{s}{n}+\frac{1}{2}\right)} \le \frac{1}{1-\beta} \int_{Q(k,l)/E_{k+1}} (S_{s,q}(\alpha)(x))^q\, dx.$

PROOF : According to the definition of $\Delta(k, 1)$ if $Q \in \Delta(k, 1)$. We have $Q \subset Q(k, 1)$ and $Q \notin \mathcal{D}_{k+1}$, so we have that $|Q \cap E_{k+1}| < \beta |Q|$ and $|Q/E_{k+1}| \ge (1-\beta)|Q|$.

But we have also

$$(S_{s,q}(\alpha)(x))^q \ge \sum_{Q \in \Lambda(k,l)} \sum_{\varepsilon \in \Xi} |a_{\varepsilon,Q}|^q |Q|^{-q\left(\frac{s}{n}+\frac{1}{2}\right)} X_Q(x).$$

Hence, we have

$$\int_{Q(k,l)/E_{k+1}} (S_{s,q}(\alpha)(x))^q\, dx \ge \sum_{Q \in \Lambda(k,l)} \sum_{\varepsilon \in \Xi} |a_{\varepsilon,Q}|^q |Q|^{-q\left(\frac{s}{n}+\frac{1}{2}\right)} |Q/E_{k+1}|$$

$$\ge (1-\beta) \sum_{Q \in \Lambda(k,l)} \sum_{\varepsilon \in \Xi} |\alpha_{\varepsilon,Q}|^1 |Q|^{1-q\left(\frac{s}{n}+\frac{1}{2}\right)}.$$

Return now to the inequalities for $\lambda_{k,l}$.

Apply the hypothesis that $1 \le p < q < +\infty$ and the inequality

$\int \left(\sum_m f_m(x) \right)^t dx \le \sum_m (f_m(x))^t\, dx$ for $0 < t \le 1$ and $f_m(x) \ge 0$, we obtain that

$$\| \alpha \| T_p^{p_{s,q}} = \int \left(\sum_k \sum_l \lambda_{k,l}^q \sum_{\substack{(\varepsilon, Q) \in \Lambda \\ Q \in \Lambda(k,l)}} | Q |^{-q\left(\frac{s}{n}+\frac{1}{2}\right)} | \alpha_{k,l}(\varepsilon Q) \right)^q X_Q(x))^{p/q}\, dx$$

$$\le \sum_k \sum_l \lambda_{k,l}^p \int (S_{s,q}(a_{k,l}(x))^p\, dx.$$

Since $\alpha_{k,1}$ is an atom, we have

$$\| \alpha \|^p T_p^{s,q} \le \sum_k \sum_l \lambda_{k,l}^p \,.$$

Furthermore, we have

$$\sum_k \sum_l \lambda_{k,l}^p \le C_\beta \sum_k \sum_l | Q(k,l) |^{1-\frac{p}{q}} \left(\int_{Q(k,l)/E_{k+1}} (S_{s,q}(\alpha)(x))^q\, dx \right)^{\frac{p}{q}}.$$

Apply then the definition of E_{k+1}, we have

$$\sum_k \sum_l \lambda_{k,l}^p \le C_\beta \sum_k \sum_t 2^{p^k} | Q(k,l) |$$

$$\le C_\beta \sum_k 2^{p^k} | E_k^* |$$

$$\le C_\beta \| \alpha \|^p\, T_p^{s,q}$$

4. (p, L^2) Atom and Atomic Decomposition

Denote $|R|$ the volume of the cube R, $[s]$ the integer part of s, then we define

Definition 3 — $a(x)$ is a $(p, L^{2,s}, R)$ atom if (i) supp $a(x) \subset R$, (ii) $a(x) \in L^{2,s}$ and $\| a(x) \| L^{2,s} \le | R |^{\frac{1}{2}-\frac{1}{p}}$, and (iii) $\int x^\alpha a(x) dx = 0$ for $0 \le | \alpha | \le [s]$.

When $s = 0$, we say $a(x)$ is a (p, L^2, R) atom, and when R is neglected, we say that $a(x)$ is a $(p, L^{2,s})$ atom. When $s < 0$, condition *(iii)* is not required.

***Theorem* 3** — *For $1 < p < 2$, there exist two positive constants c_1 and c_2 such that for an arbitrary function $f(x)$ $L^{p,s}$, there exist a sequence $\{\lambda_m\} \in l^p$ and a sequence of $(p, \dot{L}^{2,s})$ atoms $\{\alpha_m(x)\}$ such that $f(x) = \sum_m \lambda_m \alpha_m(x)$ and*

$$C_1\left(\sum_m |\lambda_m|^p\right)^{\frac{1}{p}} \leq \| f(x) \|_{\dot{L}^{p,s}} \leq C_2\left(\sum^{m} |\lambda_m|^p\right)^{\frac{1}{p}}.$$

PROOF : If $f(x) = \sum_{(\varepsilon, Q) \in \Lambda} \alpha_{\varepsilon, Q}\, \psi_{\varepsilon, Q}(x) \in \dot{L}^{p,s}$, according to Theorem 1, $\{\alpha_{\varepsilon, Q}\}_{(\varepsilon, Q) \in \Lambda} \in T_p^{s,2}$. According to Theorem 2, there exist a sequence $\{\lambda_{k,l}\}$ and a sequence of $(p, T_2^{s,2}, Q(k,l))$ atoms $\left\{\alpha_{k,l}(\varepsilon, Q)\right\}_{(\varepsilon, Q) \in \Lambda} \in T_p^{s,2}$.

We define

$$f_{k,l}(x) = (2^{M+1}+1)^{n\left(\frac{1}{2} - \frac{1}{p}\right)} \sum_{(\varepsilon, Q) \in \Lambda} \alpha_{k,l}(\varepsilon, Q)\, \psi_{\varepsilon, Q}(x)$$

and

$$\tilde{\lambda}_{k,l} = (2^{M+1}+1)^{n\left(\frac{1}{2} - \frac{1}{p}\right)} \lambda_{k,l}. \text{ Denote } Q(k,l) = \prod_{i=1}^{n} (x_i^{k,l}, x_i^{k,l} + u_{k,l})$$

and

$$\tilde{Q}(k,l) = \prod_{i=1}^{n} \left(x_i^{k,l} - 2^M u_{k,l}, x_i^{k,l} + (2^M+1)\, u_{k,l}\right),$$

then we have

(*i*) $\operatorname{supp} f_{k,l}(x) \subset Q(k,l) + [-2^M u_{k,l}, 2^M u_{k,l}] \subset \tilde{Q}(k,l)$,

(*ii*) $\| f_{k,l} \|_{\dot{L}^{p,s}} = (2^{M+1}+1)^{n\left(\frac{1}{2} - \frac{1}{p}\right)} \| \alpha_{k,l}(\varepsilon, Q) \|_{T_p^{s,2}}$

$$\leq (2^{M+1}+1)^{n\left(\frac{1}{2} - \frac{1}{p}\right)} | Q(k,1) |^{n\left(\frac{1}{2} - \frac{1}{p}\right)}$$

$$\leq | \tilde{Q}(k,1) |^{n\left(\frac{1}{2} - \frac{1}{p}\right)}.$$

(*iii*) Since $\int_{x^a} \psi_{\varepsilon, Q}(x)dx = 0$ for $0 \le |a| \le [s] + 2$, we have $\int x^{\alpha} f_{k,1}(x) = 0$ for $0 \le |a| \le [s]$.

That is to say, $f_{k,l}(x)$ is a $(p, \dot{L}^{2,s}, Q(k,l))$ atom. So we have $f(x) = \sum_k \sum_l \lambda_{k,l} f_{k,l}(x)$ is an atomic decomposition and $\lambda_{k,l}$ satisfy the condition,

$$C_1 \left(\sum_k \sum_l |\lambda_{k,l}|^p \right)^{\frac{1}{p}} \le \| f(x) \|_{\dot{L}^{p,s}} \le C_2 \left(\sum_k \sum_l |\lambda_{k,l}|^p \right)^{\frac{1}{p}}.$$

Remark : $\forall \cdot f(x)$ one defines :

$$Mf(x) = \sup_{Q \ni n} \frac{1}{|Q|} \int_Q |f(y)| \, dy$$

Then one has that

(i) The conclusion in Theorem 2 is true also for $p > q$, but one has to change the $(p, T_q^{s,q}, R)$ atom to $(p, T_p^{s,q}, R)$ atom $(\dot{p}', p)$.

(ii) The conclusion in Theorem 3 is also true for $p > 2$.

(iii) The only change for the proofs of these two Theorems is to change the definition of E_k in the proof of Theorem 2 as following :

We denote E_k to be the set of X which satisfy the condition that $MS,_q(\alpha)(x) > 2^k$.

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