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OSCILLATIONS OF IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

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In this paper, the impulsive delay differential equation

$$\left.\begin{aligned}&(r(t)x'(t))' + P(t)f(x(t-\tau)) = 0,\ t \ge t_0,\ t \ne t_k,\ k = 1, 2, \ldots,\\ &x(t_k^+) = g_k(x(t_k)),\quad x'(t_k^+) = h_k(x'(t_k)),\ k = 1, 2, \ldots\end{aligned}\right\} \quad \ldots \text{(E)}$$

is considered, where $0 \le t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim_{k \to +\infty} t_k = +\infty$, τ is a positive constant, $r \in C(R, (0, +\infty))$, $P \in C(R, [0, +\infty))$, $f \in C(R, R)$, $xf(x) > 0$ for all $x \ne 0$, and $f'(x) \ge 0$. Some sufficient conditions are obtained ensuring that all solutions of eq. (E) oscillate.

Key Words : Impulsive Delay Differential Equation; Oscillation

1. INTRODUCTION AND PRELIMINARIES

Differential equations with impulses provide an adequate mathematical model of many evolutionary processes that suddenly change their state at certain moments. The first investigators of oscillation on the first order delay differential equations with impulses were Gopalsamy and Zhang[4]. Later on, there have been extensive studies in this area[5]. However, there are not much concerning the oscillatory properties of the second order impulsive delay differential equations and the second order impulsive ordinary differential equations[9-12]. In this paper, we will consider the following second order impulsive delay differential equation

$$\left.\begin{aligned}&(r(t)x'(t))' + P(t)f(x(t-\tau)) = 0,\ t \ge t_0,\ t \ne t_k,\ k = 1, 2, \ldots,\\ &x(t_k^+) = g_k(x(t_k)),\qquad x'(t_k^+) = h_k(x'(t_k)),\ k = 1, 2, \ldots\end{aligned}\right\} \quad \ldots (1)$$

where $0 \le t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim_{k \to +\infty} t_k = +\infty$, and τ is a positive constant.

Throughout this paper, we always assume that

(*i*) $r \in C(R, (0, +\infty))$, $P \in C(R, [0, +\infty))$, $f \in C(R, R)$, $xf(x) > 0$ for all $x \neq 0$, and $f'(x) \geq 0$.

(*ii*) $g_k, h_k \in C(R, R)$ and there exist positive numbers $a_k, \bar{a}_k, b_k, \bar{b}_k$ such that

$$\bar{a}_k \leq \frac{g_k(x)}{x} \leq a_k, \bar{b}_k \leq \frac{h_k(x)}{x} \leq b_k \text{ for all } x \neq 0, k = 1, 2, \ldots .$$

Let $J \subset R$ be an interval, we define $PC(J, R) = \{x : J \to R; x(t)$ is continuous everywhere except some t_k's at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-) = x(t_k)\}$; $PC'(J, R) = \{x \in PC(J, R) : x(t)$ is continuously differentiable everywhere except some $t'_k s$ at which $x'(t_k^-)$ and $x'(t_k^+)$ exist and $x'(t_k^-) = x'(t_k)\}$.

Let $t_0 \geq 0$, $\phi \in PC([t_0 - \tau, t_0], R)$. By a solution of eq. (1) we mean that a real valued function $x \in PC([t_0 - \tau, +\infty), R) \cap PC'([t_0, +\infty), R)$ which satisfies

(*iii*) for any $t \in [t_0 - \tau, t_0]$, $x(t) = \phi(t)$

(*iv*) for any $t \in [t_0, +\infty)$, $t \neq t_k$, $k = 1, 2, \ldots, x(t)$ satisfies

$$(r(t)x'(t))' + P(t)f(x(t - \tau)) = 0.$$

(*v*) for any $k = 1, 2, \ldots$, $x(t_k^+) = g_k(x(t_k))$, $x'(t_k^+) = h_k(x'(t_k))$.

Let t_0 be a given initial point and let $\phi \in PC([t_0 - \tau, t_0], R)$ be a given initial function, then one can show by using the method of steps that eq. (1) has a unique solution on $[t_0, +\infty)$ satisfying the initial condition $x(t) = \phi(t)$ for $t \in [t_0 - \tau, t_0]$.

A solution of eq. (1) is said to be nonoscillatory if this solution is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

Lemma 1 — Assume that[1]

(a1) the sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ with $\lim_{k \to +\infty} t_k = +\infty$;

(a2) $m \in PC'(R_+, R)$ is left continuous at t_k for $k = 1, 2, \ldots$;

(a3) for $k = 1, 2, \ldots, t \geq t_0$,

$$m'(t) \leq p(t)m(t) + q(t), t \neq t_k, \quad \ldots (2)$$

$$m(t_k^+) \leq d_k m(t_k) + b_k, \quad \ldots (3)$$

where $p, q \in C(R_+, R)$, $d_k \geq 0$, and b_k are real constants.

Then

$$m(t) \leq m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^{t} p(s)ds\right)$$

$$+ \int_{t_0}^{t} \prod_{s < t_k < t} d_k \exp\left(\int_s^t p(\sigma) d\sigma \right) q(s) ds \quad \ldots (4)$$

$$+ \sum_{t_0 < t_k < t} \prod_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^{t} p(s) ds \right) b_k.$$

Remark 1 : If the inequalities (2) and (3) are reversed then in the conclusion the inequality (4) is also reversed.

2. Main Results

Lemma 2 — Let $x(t)$ be a solution of eq. (1). Assume that there exists some $T \geq t_0$ such that $x(t - \tau) > 0$ for $t \geq T$ and the following conditions hold

(h1) conditions (i) and (ii) are valid; and

(h2) $\lim_{t \to +\infty} \int_{t_0}^{t} \frac{1}{r(s)} \prod_{t_0 < t_k < s} \frac{\bar{b}_k}{a_k} ds = +\infty.$

Then $x'(t) \geq 0$ for $t \in [T, t_l] \cup \left(\bigcup_{k=l}^{+\infty} (t_k, t_{k+1}] \right)$, where $l = \min \{k : t_k \geq T\}$.

Proof : At first, we shall prove that $x'(t_k) \geq 0$ for any $k \geq l$. If it is not true, then there exists some j such that $j \geq l$ and $x'(t_j) < 0$. From eq. (1) and (ii), we have

$$x'(t_j^+) = h_j(x'(t_j)) \leq \bar{b}_j x'(t_j) < 0.$$

Let $x'(t_j^+) = -\alpha \, (\alpha > 0)$. From eq. (1) and (i), for $t \in \bigcup_{i=1}^{+\infty} (t_{j+i-1}, t_{j+i}]$ we have

$$(r(t)\, x'(t))' = -P(t) f(x(t - \tau)) \leq 0.$$

Hence, $r(t)\, x'(t)$ is monotonically nonincreasing in $(t_{j+i-1}, t_{j+i}]$, $i = 1, 2, \ldots$.

So,

$$x'(t_{j+1}) \leq \frac{r(t_j^+)}{r(t_{j+1})} x'(t_j^+) = -\frac{r(t_j)}{r(t_{j+1})} \alpha < 0,$$

$$x'(t_{j+2}) \leq \frac{r(t_{j+1}^+)}{r(t_{j+2})} x'(t_{j+1}^+) = \frac{r(t_{j+1})}{r(t_{j+2})} h_{j+1}(x'(t_{j+1}))$$

$$\leq \frac{r(t_{j+1})}{r(t_{j+2})} \bar{b}_{j+1} x'(t_{j+1}) \leq -\frac{r(t_j)}{r(t_{j+2})} \bar{b}_{j+1} \alpha < 0,$$

$$x'(t_{j+3}) \le \frac{r(t_{j+2}^+)}{r(t_{j+3})} x'(t_{j+2}^+) \le \frac{r(t_{j+2})}{r(t_{j+3})} \bar{b}_{j+2} x'(t_{j+2})$$

$$\le -\frac{r(t_j)}{r(t_{j+3})} \bar{b}_{j+2} \bar{b}_{j+1} \alpha < 0.$$

We can easily show that, for any positive integer $n \ge 2$,

$$x'(t_{j+n}) \le -\frac{r(t_j)}{r(t_{j+n})} \left(\prod_{i=1}^{n-1} \bar{b}_{j+i} \right) \alpha < 0.$$

Consider the following impulsive differential inequalities

$$(r(t)x'(t))' \le 0,\ t > t_j,\ t \ne t_k,\ k = j+1, j+2, \ ...,$$

$$x'(t_k^+) \le \bar{b}_k x'(t_k),\ k = j+1,\ j+2,\ ... \ .$$

Let $m(t) = rt(t)\,x'(t)$, then

$$m'(t) \le 0,\ t > t_j,\ t \ne t_k,\ k = j+1, j+2,\ ... \ ,$$

$$m(t_k^+) \le \bar{b}_k m(t_k),\ k = j+1,\ j+2,\ ... \ .$$

From Lemma 1, we have

$$m(t) \le m(t_j^+) \prod_{t_j < t_k < t} \bar{b}_k, \qquad ... (5)$$

i.e.,

$$x'(t) \le \frac{r(t_j)\,x'(t_j^+)}{r(t)} \prod_{t_j < t_k < t} \bar{b}_k. \qquad ... (6)$$

Then, using the facts that $x(t_k^+) \le a_k x(t_k)$ holds for $k = j+1,\ j+2,\ ...,$ from Lemma 1, we have

$$x(t) \le x(t_j^+) \prod_{t_j < t_k < t} a_k + \int_{t_j^+}^{t} \prod_{s < t_k < t} a_k \left(\frac{r(t_j)\,x'(t_j^+)}{r(s)} \prod_{t_j < t_k < s} \bar{b}_k \right) ds \qquad ... (7)$$

$$= \prod_{t_j < t_k < t} a_k \left[x(t_j^+) - \alpha r(t_j) \int_{t_j^+}^{t} \frac{1}{r(s)} \prod_{t_j < t_k < s} \frac{\bar{b}_k}{a_k} ds \right].$$

Since $x(t) > 0$ for $t \ge T$, the last inequality contradicts (h2) of Lemma 2. Therefore, $x'(t_k) \ge 0$ for $k \ge l$. The condition (ii) implies $x'(t_k^+) \ge \bar{b}_k x'(t_k) \ge 0$ for any $k \ge l$. Because $r(t)x'(t)$ is nonincreasing in $(t_k, t_{k+1}]$, it is clear that $x'(t) \geq \frac{r(t_{k+1})}{r(t)} x'(t_{k+1}) \geq 0$ for $t \in (t_k, t_{k+1}], k \geq l$; and $x'(t) \geq \frac{r(t_l)}{r(t)} x'(t_l) \geq 0$ for $t \in [T, t_l]$. Thus the proof of Lemma 2 is complete.

Remark 2 : In the case that $x(t)$ is eventually negative, under the conditions (h1) and (h2), it can be proved similarly that $x'(t) \leq 0$ for $t \in [T, t_l] \cup \left(\bigcup_{k=l}^{+\infty} (t_k, t_{k+1}] \right)$, where l = min $\{k : t_k \geq T\}$.

In the following discussion, $\int_{\pm\varepsilon}^{\pm\infty} \frac{du}{f(u)} < +\infty$ denotes

$$\int_{\varepsilon}^{+\infty} \frac{du}{f(u)} < +\infty \text{ and } \int_{-\varepsilon}^{-\infty} \frac{du}{f(u)} < +\infty. \qquad \text{... (8)}$$

Theorem 1 — *Assume that the conditions (h1) and (h2) of Lemma 2 hold and there exists a positive integer* k_0 *such that* $\bar{a}_k \geq 1, b_k \leq 1$ *for* $k \geq k_0$. *If*

$$\int_{\pm\varepsilon}^{\pm\infty} \frac{du}{f(u)} < +\infty \qquad \text{... (9)}$$

holds for some $\varepsilon > 0$, *and*

$$\sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \left(\lim_{t \to +\infty} \int_s^t \prod_{s < t_k < u} \frac{1}{b_k} P(u) du \right) ds = +\infty, \qquad \text{... (10)}$$

then all solutions of eq. (1) oscillate.

PROOF : Let $x(t)$ be a nonoscillatory solution of eq. (1). Without loss of generality, we can assume $k_0 = 1$ and $x(t) > 0$ for $t \geq t_0$. From Lemma 2, we can find $x'(t) \geq 0$ for $t \geq t_0$. It is clear that $x'(t-\tau) \geq 0$ for $t \geq t_0 + \tau$. Since $\bar{a}_k \geq 1$ for k = 1, 2, ..., we have

$$x(t_0^+) \leq x(t_1) \leq x(t_1^+) \leq x(t_2) \leq x(t_2^+) \leq \ldots \qquad \text{... (11)}$$

Obviously, $x(t)$ is monotonically nondecreasing in $[t_0, +\infty)$. From eq. (1) we have

$$(r(t)x'(t))' = -P(t) f(x(t-\tau)), t \geq t_0, t \neq t_k,$$

$$x'(t_k^+) \leq b_k x'(t_k), k = 1, 2, \ldots. \qquad \text{... (12)}$$

Let $m(t) = r(t)\,x'(t)$ for $t \ge t_0$, then we have

$$m'(t) = -P(t)\,f(x(t-\tau)),\ t \ge t_0,\ t \ne t_k,$$

$$m(t_k^+) \le b_k\, m(t_k),\ k = 1, 2, \ldots. \qquad \ldots (13)$$

From Lemma 1, it follows that

$$m(t) \le m(s) \prod_{s<t_k<t} b_k - \int_s^t \prod_{u<t_k<t} b_k\, P(u)\,f(x(u-\tau))\,du,\quad t_0 \le s \le t, \qquad \ldots (14)$$

i.e.,

$$r(t)x'(t) \le r(s)x'(s) \prod_{s<t_k<t} b_k - \int_s^t \prod_{u<t_k<t} b_k\, P(u)f(x(u-\tau))du,\quad t_0 \le s \le t, \qquad \ldots (15)$$

especially,

$$r(s)x'(s) \le r(s-\tau)\,x'(s-\tau) \prod_{s-\tau<t_k<s} b_k,\qquad s \ge t_0 + \tau. \qquad \ldots (16)$$

From (15), we have

$$x'(s) \ge \frac{r(t)\,x'(t)}{r(s) \prod\limits_{s<t_k<t} b_k} + \frac{\int_s^t \prod\limits_{u<t_k<t} b_k P(u)f(x(u-\tau))du}{r(s) \prod\limits_{s<t_k<t} b_k}$$

$$\ge \frac{1}{r(s)} \int_s^t \prod_{s<t_k<u} \frac{1}{b_k}\ P(u)\,f(x(u-\tau))\,du. \qquad \ldots (17)$$

From the facts that $xf(x) > 0$ holds for $x \ne 0$ and $f(x)$ is nondecreasing, we have

$$\frac{x'(s)}{f(x(s-\tau))} \ge \frac{1}{r(s)} \int_s^t \prod_{s<t_k<u} \frac{1}{b_k} P(u) \frac{f(x(u-\tau))}{f(x(s-\tau))}\,du \ge \frac{1}{r(s)} \int_s^t \prod_{s<t_k<u} \frac{1}{b_k} P(u)du. \qquad \ldots (18)$$

For $s \in (t_k, t_{k+1}]$, $b_k \le 1$, $k = 1, 2$, from (16), we obtain

$$\int_{t_k^+}^{t_{k+1}} \frac{r(s)\,x'(s)}{r(s-\tau)\,f(x(s-\tau))}\,ds \le \int_{t_k^+}^{t_{k+1}} \frac{x'(s-\tau)}{f(x(s-\tau))}\,ds = \int_{x(t_k^+-\tau)}^{x(t_{k+1}-\tau)} \frac{du}{f(u)}. \qquad \ldots (19)$$

From (18) (19), we have

$$\sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \left(\lim_{t \to +\infty} \int_s^t \prod_{s < t_k < u} \frac{1}{b_k} P(u)du \right) ds$$

$$\leq \sum_{k=0}^{+\infty} \int_{x(t_k^+ - \tau)}^{x(t_{k+1} - \tau)} \frac{du}{f(u)} \leq \int_{x(t_0^+ - \tau)}^{+\infty} \frac{du}{f(u)}, \qquad \text{... (20)}$$

which is in contradiction with conditions (9) and (10) of Theorem 1. That is, all solutions of eq. (1) oscillate. The proof of Theorem 1 is complete.

***Theorem* 2** — *Assume that the conditions (h1) and (h2) of Lemma 2 hold, and there exists a positive integer k_0 such that $b_k \leq 1$ for $k \geq k_0$. Suppose that $f(ab) \geq f(a) f(b)$ for any $ab > 0, t_{k+1} - t_k = \tau$ for all $k = 1, 2, \dots$. If*

$$\int_{\pm\varepsilon}^{\pm\infty} \frac{du}{f(u)} < +\infty \qquad \text{... (21)}$$

holds for some $\varepsilon > 0$, and

$$\sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \left(\lim_{t \to +\infty} \int_s^t \prod_{s < t_k < u} \frac{1}{c_k} P(u)du \right) ds = +\infty, \qquad \text{... (22)}$$

where

$$c_k = \begin{cases} b_1, & \text{if } k = 1, \\ \dfrac{b_k}{f(a_{k-1})}, & \text{if } k = 2, 3, \dots. \end{cases}$$

Then all solutions of eq. (1) oscillate.

PROOF : Let $x(t)$ be a nonoscillatory solution of eq. (1). Suppose that $x(t) > 0$ for $t \geq t_0$ and $k_0 = 1$. From Lemma 2, $x'(t) \geq 0$ for $t \geq t_0$. It is clear that $x'(t-\tau) \geq 0$ for $t \geq t_0 + \tau$.

Let

$$u(t) = \frac{r(t)x'(t)}{f(x(t-\tau))}.$$

Then, $u(t_k^+) \geq 0$ for $k = 1, 2, \dots$, $u(t) \geq 0$ for $t \geq t_0$. Using (i) and eq. (1), for $t \neq t_k$, we have

$$u'(t) = \frac{(r(t)x'(t))'}{f(x(t-\tau))} - \frac{r(t)x'(t)x'(t-\tau)f'(x(t-\tau))}{f^2(x(t-\tau))} \leq -P(t). \qquad \text{... (23)}$$

If $k = 1$,

$$u(t_1^+) = \frac{r(t_1^+)\, x'(t_1^+)}{f(x(t_1^+ - \tau))} \le \frac{b_1 r(t_1) x'(t_1)}{f(x(t_1 - \tau))} = c_1 u(t_1). \quad \text{... (24)}$$

If $k = 2, 3, \ldots,$

$$u(t_k^+) = \frac{r(t_k^+)\, x'(t_k^+)}{f(x(t_k^+ - \tau))} \le \frac{b_k r(t_k)\, x'(t_k)}{f(x(t_{k-1}^+))} \le \frac{b_k r(t_k)\, x'(t_k)}{f(\overline{a}_{k-1}\, x(t_{k-1}))}$$

$$\le \frac{b_k r(t_k) x'(t_k)}{f(\overline{a}_{k-1}) f(x(t_{k-1}))} = \frac{b_k r(t_k) x'(t_k)}{f(\overline{a}_{k-1}) f(x(t_k - \tau))} = c_k u(t_k). \quad \text{... (25)}$$

Consider the following impulsive differential inequalities

$$u'(t) \le -P(t),\ t \ge t_0,\ t \ne t_k,\ k = 1, 2, \ldots,$$

$$u(t_k^+) \le c_k u(t_k),\ k = 1, 2, \ldots . \quad \text{... (26)}$$

From Lemma 1, it follows that

$$u(t) \le u(s) \prod_{s < t_k < t} c_k - \int_s^t \prod_{v < t_k < t} c_k\, P(v) dv,\ t_0 \le s \le t. \quad \text{... (27)}$$

From the above inequality, we have

$$u(s) \ge \frac{u(t)}{\prod_{s < t_k < t} c_k} + \frac{\int_s^t \prod_{v < t_k < t} c_k\, P(v)\, dv}{\prod_{s < t_k < t} c_k} \quad \text{... (28)}$$

$$\ge \int_s^t \prod_{s < t_k < v} \frac{1}{c_k} P(v)\, dv,$$

i.e.,

$$\frac{x'(s)}{f(x(s-\tau))} \ge \frac{1}{r(s)} \int_s^t \prod_{s < t_k < v} \frac{1}{c_k} P(v) dv. \quad \text{... (29)}$$

Since $b_k \le 1$ for $k = 1, 2, \ldots,$ we have for $s \in (t_k, t_{k+1}]$

$$\frac{x'(s-\tau)}{f(x(s-\tau))} \ge \frac{r(s) x'(s)}{r(s-\tau) f(x(s-\tau))} \ge \frac{1}{r(s-\tau)} \int_s^t \prod_{s < t_k < v} \frac{1}{c_k} P(v)\, dv. \quad \text{... (30)}$$

For $s \in (t_k, t_{k+1}], k = 1, 2, \ldots,$

$$\int_{t_k^+}^{t_{k+1}} \frac{r(s)x'(s)}{r(s-\tau)f(x(s-\tau))}\,ds \le \int_{t_k^+}^{t_{k+1}} \frac{x'(s-\tau)}{f(x(s-\tau))}\,ds = \int_{x(t_k^+-\tau)}^{x(t_{k+1}-\tau)} \frac{dv}{f(v)}. \quad \ldots (31)$$

Hence,

$$\sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \left(\lim_{t\to+\infty} \int_s^t \prod_{s<t_k<v} \frac{1}{c_k} P(v)\,dv \right) ds$$

$$\le \sum_{k=0}^{+\infty} \int_{x(t_k^+-\tau)}^{x(t_{k+1}-\tau)} \frac{dv}{f(v)} \le \int_{x(t_0^+-\tau)}^{+\infty} \frac{dv}{f(v)}, \quad \ldots (32)$$

which is in contradiction with conditions (21) and (22) of Theorem 2. That is, all solutions of eq. (1) oscillate. The proof of Theorem 2 is complete.

From Theorem 1 and Theorem 2, we can immediately obtain the following corollaries.

Corollary 1 — Assume that the conditions ($h1$) and ($h2$) of Lemma 2 hold and there exists a positive integer k_0 such that $\bar{a}_k \ge 1, b_k \le 1$ for $k \ge k_0$. If

$$\int_{\pm\varepsilon}^{\pm\infty} \frac{du}{f(u)} < +\infty$$

holds for some $\varepsilon > 0$, and

$$\int_{t_0}^{+\infty} \frac{1}{r(s-\tau)} \int_s^{+\infty} P(t)dtds = +\infty,$$

then all solutions of eq. (1) oscillate

PROOF : Without loss of generality, let $k_0 = 1$. Since $b_k \le 1$, we have

$$\sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \left(\lim_{t\to+\infty} \int_s^t \prod_{s<t_k<u} \frac{1}{b_k} P(u)\,du \right) ds$$

$$= \sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \left(\lim_{n\to+\infty} \int_s^{t_k+n+1} \prod_{s<t_k<u} \frac{1}{b_k} P(u)du \right) ds$$

$$= \sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \lim_{n \to +\infty} \left(\int_s^{t_{k+1}} \prod_{s<t_k<u} \frac{1}{b_k} P(u)du + \int_{t_{k+1}^+}^{t_{k+2}} \prod_{s<t_k<u} \frac{1}{b_k} P(u)du \right.$$

$$\left. + \ldots + \int_{t_{k+n}^+}^{t_{k+n+1}} \prod_{s<t_k<u} \frac{1}{b_k} P(u)du \right) ds$$

$$= \sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \lim_{n \to +\infty} \left(\int_s^{t_{k+1}} P(u)du + \frac{1}{b_{k+1}} \int_{t_{k+1}^+}^{t_{k+2}} P(u)du \right.$$

$$\left. + \ldots + \frac{1}{b_{k+1} b_{k+2} \cdots b_{k+n}} \int_{t_{k+n}^+}^{t_{k+n+1}} P(u)du \right) ds$$

$$\geq \sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \lim_{n \to +\infty} \left(\int_s^{t_{k+1}} P(u)du + \int_{t_{k+1}^+}^{t_{k+2}} P(u)du \right.$$

$$\left. + \ldots + \int_{t_{k+n}^+}^{t_{k+n+1}} P(u)du \right) ds$$

$$= \sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \left(\lim_{n \to +\infty} \int_s^{t_{k+n+1}} P(u)du \right) ds$$

$$= \int_{t_0}^{+\infty} \frac{1}{r(s-\tau)} \int_s^{+\infty} P(u)du\, ds = +\infty.$$

In view of Theorem 1, we find that all solutions of eq. (1) oscillate.

Corollary 2 — Assume that the conditions ($h1$) and ($h2$) of Lemma 2 hold, and there exists a positive integer k_0 and a constant $\alpha > 0$ such that $\bar{a}_k \geq 1, \frac{1}{b_k} \geq t_{k+1}^{\alpha}$ for $k \geq k_0$. If

$$\int_{\pm\varepsilon}^{\pm\infty} \frac{du}{f(u)} < +\infty$$

holds for some $\varepsilon > 0$, and

$$\sum_{k=0}^{+\infty} [R(t_{k+1}-\tau)-R(t_k^+-\tau)] \int_{t_{k+1}^+}^{+\infty} t^\alpha P(t)dt = +\infty,$$

where

$$R(t) = \int_{t_0}^{t} \frac{ds}{r(s)}.$$

Then all solutions of eq. (1) oscillate.

PROOF : Without loss of generality, let $k_0 = 1, t_1 \geq 1$. Since $\frac{1}{b_k} \geq t_{k+1}^\alpha$ for $k \geq k_0$, we get

$$\frac{1}{b_{k+1}} \geq t_{k+2}^\alpha, \frac{1}{b_{k+1}\, b_{k+2}} \geq t_{k+2}^\alpha \cdot t_{k+3}^\alpha \geq t_{k+3}^\alpha, \ldots,$$

$$\frac{1}{b_{k+1}\, b_{k+2} \ldots b_{k+n}} \geq t_{k+2}^\alpha \cdot t_{k+3}^\alpha \ldots t_{k+n+1}^\alpha \geq t_{k+n+1}^\alpha, \ldots,$$

Similar proof to that of Corollary 1, we obtain

$$\sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \left(\lim_{t\to+\infty} \int_s^t \prod_{s<t_k<u} \frac{1}{b_k} P(u)du \right) ds$$

$$= \sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \lim_{n\to+\infty} \left(\int_s^{t_{k+1}} P(u)du + \frac{1}{b_{k+1}} \int_{t_{k+1}^+}^{t_{k+2}} P(u)du \right.$$

$$\left. + \ldots + \frac{1}{b_{k+1}\, b_{k+2} \cdots b_{k+n}} \int_{t_{k+n}^+}^{t_{k+n+1}} P(u)du \right) ds$$

$$\geq \sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s-\tau)} \left(\lim_{n\to+\infty} \int_{t_{k+1}^+}^{t_{k+n+1}} u^\alpha P(u)du \right) ds$$

$$= \sum_{k=0}^{+\infty} [R(t_{k+1}-\tau)-R(t_k^+-\tau)] \int_{t_{k+1}}^{+\infty} u^\alpha P(u)du = +\infty.$$

By Theorem 1, we find that every solution of eq. (1) oscillates.

Corollary 3 — Assume that the conditions ($h1$) and ($h2$) of Lemma 2 hold, and there exists a positive integer k_0 and a constant $\alpha > 0$ such that $b_k \le 1, \frac{1}{c_k} \ge t_{k+1}^{\alpha}$ for $k \ge k_0$, where

$$c_k = \begin{cases} b_1, & \text{if } k = 1, \\ \dfrac{b_k}{f(\bar{a}_{k-1})}, & \text{if } k = 2, 3, \dots \end{cases}$$

Suppose that $f(ab) \ge f(a) \cdot f(b)$ for any $ab > 0$, $t_{k+1} - t_k = \tau$ for all $k = 1, 2, \dots$, and

$$\int_{\pm\varepsilon}^{\pm\infty} \frac{du}{f(u)} < +\infty$$

holds for some $\varepsilon > 0$. If

$$\sum_{k=0}^{+\infty} [R(t_{k+1} - \tau) - R(t_k^+ - \tau)] \int_{t_{k+1}^+}^{+\infty} t^{\alpha} P(t) dt = +\infty,$$

where

$$R(t) = \int_{t_0}^{t} \frac{ds}{r(s)}.$$

Then all solutions of eq. (1) oscillate

Corollary 3 can be deduced from Theorem 2. Its proof is similar to that of Corollary 2.

Example 1 — Consider

$$\left.\begin{aligned} &x''(t) + \frac{1}{t^2} x^{2n}(t-1) = 0, t \ge 1, t \ne 2^k, k = 1, 2, \dots, \\ &x((2^k)^+) = \frac{k+1}{k} x(2^k), x'((2^k)^+) = x'(2^k), k = 1, 2, \dots, \end{aligned}\right\} \quad \dots (33)$$

where $n > 1$ is a positive integer. Since $a_k = \bar{a}_k = \frac{k+1}{k}, b_k = \bar{b}_k = 1, P(t) = \frac{1}{t^2}, t_0 = 1, t_k = 2^k$ and $f(x) = x^{2n}$, the conditions (h1) of Lemma 2 is satisfied and

$$\lim_{t \to +\infty} \int_{t_0}^{t} \prod_{t_0 < t_k < s} \frac{\bar{b}_k}{a_k} ds = \int_{1}^{+\infty} \prod_{1 < t_k < s} \frac{k}{k+1} ds$$

$$= \int_{1}^{t_1} \prod_{1 < t_k < s} \frac{k}{k+1} ds + \int_{t_1^+}^{t_2} \prod_{1 < t_k < s} \frac{k}{k+1} ds$$

$$+\int_{t_2^+}^{t_3} \prod_{1<t_k<s} \frac{k}{k+1}\,ds + \int_{t_3^+}^{t_4} \prod_{1<t_k<s} \frac{k}{k+1}\,ds + \ldots$$

$$= 1+\frac{1}{2}\times 2+\frac{1}{2}\times\frac{2}{3}\times 2^2+\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times 2^3+\ldots=+\infty.$$

Since

$$\int_{\pm\varepsilon}^{\pm\infty} \frac{du}{f(u)} = \int_{\pm\varepsilon}^{\pm\infty} \frac{du}{u^{2n}} < +\infty,$$

and

$$\int_1^{+\infty}\int_s^{+\infty} P(t)\,dt\,ds = \int_1^{+\infty}\int_s^{+\infty} \frac{1}{t^2}\,dt\,ds = +\infty,$$

from Corollary 1, every solution of eq. (33) is oscillatory.

REFERENCES

1. V. Lakshmikanthan, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
2. I. Gyori and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.
3. L. H. Erbe, Q. K. Kong and B. G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker Inc, 1995.
4. K. Gopalsamy and B. G. Zhang, *J. math. Anal. Appl.* **139** (1989), 110-22.
5. M. P. Chen, J. S. Yu and J. H. Shen, *Comp. math. Appl.* **27** (1994), 1-6.
6. J. H. Shen, *Appl. Math. Comput.* **77** .(1996), 153-165.
7. Yuzhu Zhang, Aimin Zhao and Jurang Yan, *J. math. Anal. Appl.* **205** (1997), 461-70.
8. Alexander Domoshnisky and Michael Drakhin, *J. math. Anal. Appl.* **206** (1997), 254-69.
9. D. Bainov, M. Dimitrova and A. Dishliev, *Appl. Anal.* **63** (1996), 287-97.
10. D. Bainov and M. Dimitrova, *Appl. Anal.* **64** (1997), 57-67.
11. Y. S. Chen and W. Z. Feng, *J. math. Anal. Appl.* **210** (1997), 150-69.
12. C. C. Huang, *J. math. Anal. Appl.* **214** (1997), 378-94.
13. H. E. Gollwitzer, *J. math. Anal. Appl.* **26** (1969), 385-89.
14. Forbes Burkowski, *J. math. Anal. Appl.* **33** (1971), 258-62.
15. K. L. Chiou, *J. math. Anal. Appl.* **45** (1974), 382-403.
16. L. Berezansky and E. Braverman, *J. math. Anal. Appl.* **220** (1998), 719-40.