Indian J. pure appl. Math., **31**(9) : 1113-1122, September 2000

GENERALIZED SET-VALUED STRONGLY NONLINEAR QUASIVARIATIONAL INCLUSIONS

S. H. SHIM*, S. M. KANG**, N. J. HUNG*** AND Y. J. CHO†

*Department of Mathematics, Gyeongsang National University, Chinju 660 701, Korea
**Department of Mathematics, Gyeongsang National University, Chinju 660 701, Korea
***Department of Mathematics, Sichuan University, Chengdu, Sichuan 610 064, P. R. China
†Department of Mathematics, Gyeongsang National University, Chinju 660 701, Korea

(*Received* 28 *January* 1999; *accepted* 2 *September* 1999)

In this paper we introduce and study a new class of generalized set-valued strongly nonlinear quasivariational inclusions. Using the resolvent operator technique for maximal monotone mapping, we construct some new iterative algorithms for solving this class of generalized set-valued strongly nonlinear quasivariational inclusion. We prove the existence of solution for this kind of generalized set-valued strongly nonlinear quasivariational inclusion without compactness and the convergence of iterative sequences generated by the algorithms.

Key Words : Quasivariational Inclusion; Variational Inclusion; Variational Inequality; Set-Valued Mapping; Iterative Algorithm

1. INTRODUCTION

Variational inequality theory and complementarity problem theory are very powerful tool of the current mathematical technology. In recent years, classical variational inequality and complementarity problem have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional, structural, transportation, elasticity, and applied sciences, etc., see [1]-[20], [31], [33]-[42], [44]-[50] and the references therein. A useful and an important generation of variational inequalities is a mixed variational inequality containing nonlinear term. Due to the presence of the nonlinear term, the projection method cannot be used to study the existence of a solution for the mixed variational inequalities.

In 1994, Hassouni and Moudafi[19] used the resolvent operator technique for maximal monotone mapping to study a new class of mixed variational inequalities for single-valued mappings. In 1996, Huang[22] extended this technique for a new class of general mixed variational inequalities with non-compact set-valued mappings and Adly[1] modified this technique for another new general mixed variational inequalities for single-valued mappings, which includes the mixed variational inequality considered by Hassouni and Moudafi[19] as special cases. For related works, we refer to[13, 21, 24, 29, 37, 45].

Inspired and motivated by recent research works[1, 20-22, 24,36,39,46 & 48], in this paper, we introduce and study a new class of quasivariational inclusion, which is called the generalized set-valued strongly nonlinear quasivariational inclusion. We establish the equivalence between generalized set-valued strongly nonlinear quasivariational inclusion without compactness and the convergence of iterative sequence generated by the algorithms. The results shown in this paper improve and extend the previously many known results in this area.

2. PRELIMINARIES

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$, and inner product $\langle \cdot, \cdot \rangle$. Given set-valued mappings $G, S, T : H \to 2^H$ (where 2^H denotes the family of all nonempty subsets of H) and single-valued mappings $p, m : H \to H$ and $N : H \times H \to H$. Suppose that $M : H \to 2^H$ is a maximal monotone mapping. We consider the following problem :

Find $u \in H, x \in Su, y \in Tu, z \in Gu$ such that

$$\left.\begin{array}{l} p(u) - m(z) \in dom\,(M), \\ 0 \in N(x, y) + M(p(u) - m(z)). \end{array}\right\} \qquad \dots (2.1)$$

The problem (2.1) is called the generalized set-valued strongly nonlinear quasivariational inclusion, which has potential applications in mechanics, physics, differential equations, pure and applied sciences (see [48]).

Now, we give some special cases of the problem (2.1) as follows:

(I) A well-known example[30 & 43] of a maximal monotone mapping is the subdifferential of a proper lower semicontinuous convex function. Letting $M = \partial\varphi$, where $\varphi : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $\partial\varphi$ denotes the subdifferential of function φ, then the problem (2.1) is equivalent to finding $u \in H, x \in Su, y \in Tu, z \in Gu$ such that

$$\left.\begin{array}{l} p(u) - m(z) \in dom\,(\partial\varphi), \\ \langle N(x, y), v - p(u) \rangle \geq \varphi(p(u) - m(z)) - \varphi(v) \end{array}\right\} \qquad \dots (2.2)$$

for all $v \in H$. The problem (2.2) is called the generalized set-valued nonlinear quasivariational inclusion and it appears to be a new one.

(II) If G is the identity mapping, $m = 0$ and $M = \partial\varphi$, where $\varphi : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $\partial\varphi$ denotes the subdifferential of function φ, then the problem (2.1) is equivalent to finding $u \in H, x \in Su, y \in Tu$ such that

$$\left.\begin{array}{l} p(u) \in dom\,(\partial\varphi), \\ \langle N(x, y), v - p(u) \rangle \geq \varphi(p(u) - \varphi(v) - \varphi(v) \end{array}\right\} \qquad \dots (2.3)$$

for all $v \in H$. The problem (2.3) is called the generalized set-valued mixed variational inequality studied by Noor, Noor and Rassias[39].

(III) If G is the identity mapping, $N(x, y) = x - Ay$, where $A : H \to H$ and $M = \partial\varphi$, where $\varphi : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $\partial\varphi$ denotes the subdifferential of function φ, then the problem (2.1) is equivalent to finding $u \in H, x \in Su, y \in Tu$ such that

$$\left.\begin{array}{l} p(u) \in dom\,(\partial\varphi) + m(u), \\ \langle x - Ay, v + m(u) - p(u) \rangle \geq \varphi(p(u) - m(u)) - \varphi(v) \end{array}\right\} \qquad \dots (2.4)$$

for all $v \in H$. The problem (2.4) is called the generalized set-valued quasivariational inclusion studied by Yuan[48].

(IV) If G is the identity mapping and $m = 0$, then the problem (2.1) is equivalent to finding $u \in H, x \in Su, y \in Tu$ such that

$$\left.\begin{array}{l} p(u) \in dom\,(M), \\ 0 \in N(x, y) + M(p(u)), \end{array}\right\} \quad \text{... (2.5)}$$

which is called the generalized set-valued variational inclusion, which was studied by Noor[36], by using the compactness condition.

(V) If S and T are single-valued mappings and G is the identity mapping, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\left.\begin{array}{l} p(u) - m(u) \in dom\,(M), \\ 0 \in N\,(Su, Tu) + M(p(u) - m(u)), \end{array}\right\} \quad \text{... (2.6)}$$

which is called the generalized strongly nonlinear quasivariational inclusion.

(VI) If S and T are single-valued mappings, G is an identity mapping, $N(x, y) = x - y$, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\left.\begin{array}{l} p(u) - m(u) \in dom(M), \\ 0 \in Su - Tu + M(p(u) - m(u)), \end{array}\right\} \quad \text{... (2.7)}$$

which is called the generalized mixed variational inequality studied by Uko[46].

(VII) If G is the identity mapping, $m = 0$, S and T are both single-valued mappings, $N(x, y) = Sx - Ty$ and $M = \partial\varphi$, where $\varphi : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $\partial\varphi$ denotes the subdifferential of function φ, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\left.\begin{array}{l} p(u) \in dom(\partial\varphi), \\ \langle Sx - Ty, v - p(u) \rangle \geq \varphi(p(u)) - \varphi(v) \end{array}\right\} \quad \text{... (2.8)}$$

for all $v \in H$, which is known as the variational inclusion problem introduced and studied by Hassouni and Moudafi[19].

(VIII) If $M = \partial\varphi$, where $\varphi = \delta_K$, the indicator functions of closed convex set K in H defined by

$$\varphi(x) = \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K. \end{cases}$$

Then the problem (2.1) is equivalent to finding $u \in H, x \in Su, y \in Tu, z \in Gu$ such that

$$\left.\begin{array}{l} p(u) \in K(z) = m(z) + K, \\ \langle N(x, y), v - p(u) \rangle \geq 0 \end{array}\right\} \quad \text{... (2.9)}$$

for all $v \in K(z)$. The problem (2.9) is called the generalized set-valued strongly nonlinear quasivariational inequality.

(IX) If $N(x, y) = Ax + By$, where A and B are both single-valued mappings from H to H and $M = \partial\varphi$, then the problem (2.9) is equivalent to finding $u \in H$, $x \in Su, y \in Tu, z \in Gu$ such that

$$\left.\begin{array}{l} p(u) \in K(z) = m(z) + K, \\ \langle Ax + By, v - p(u) \rangle \geq 0 \end{array}\right\} \quad \text{... (2.10)}$$

for all $v \in K(z)$. The problem (2.10) is called the completely generalized strongly nonlinear implicit quasivariational inequality studied by Huang[20].

(X) If G is the identity mapping, $N(x, y) = Ax + By$, where A and B are both single-valued mappings from H to H and $M = \partial \varphi$, then the problem (2.10) is equivalent to finding $u \in H, x \in Su, y \in Tu$ such that

$$\left.\begin{array}{l} p(u) \in K(u) = m(u) + K, \\ \langle Ax + By, v - p(u) \rangle \geq 0 \end{array}\right\} \quad \ldots (2.11)$$

for all $v \in K(u)$. The problem (2.11) is called the generalized strongly nonlinear implicit quasivariational inequality studied by Huang[20].

For a suitable choice of the mappings S, T, G, N, p, m, M and the space H, a number of known classes mixed variational inequalities, variational inequalities, quasi-variational inequalities, complementarity problems and quasi (implicit) complementarity problems of[1,7,9,13,14,19,27,29,34,40,42,44,46,48&49] can be obtained as special cases of the generalized set-valued strongly nonlinear quasivariational inclusion (2.1). Further, these type of quasivariational inclusion enable us to study many important problems arising in mechaincs, physics, optimization and control, nonlinear programming, economics, finance, regional, structural, transportation, elasticity, and applied sciences in a general and unified framework.

3. ITERATIVE ALGORITHMS

It is well known (see [7], [30]) that if M is a maximal monotone mapping from H to 2^H, then for every $\mu > 0$, the resolvent $(I + \mu M)^{-1}$ is a well defined single-valued nonexpansive operator mapping H into itself. By using the resolvent operator technique it is possible to convert the problem (2.1) into an equivalent equation which is easier to handle. To do this, if we multiply all the terms in (2.1) with some $\rho > 0$ and add $p(u) - m(z)$, then we obtain

$$p(u) - m(z) - \rho N(x, y) \in p(u) - m(z) + \rho M(p(u) - m(z)).$$

Therefore, we have the following :

Lemma 3.1 — (u, x, y, z) is a solution of the problem (2.1) if and only if (u, x, y, z) satisfies the relation

$$p(u) = m(z) + J_\rho^M (p(u) - m(z) - \rho N(x, y)),$$

where $\rho > 0$ is a constant, $J_\rho^M = (I + \rho M)^{-1}$ and I is the identity mapping on H.

Based on Lemma 3.1 and Nadler[32], we now give some new general and unified algorithms for the problem (2.1) as follows:

Let $p, m : H \to H, N : H \times H \to H$ and $S, T, G : H \to CB(H)$, where $CB(H)$ denotes the family of all nonempty bounded closed subsets of H. For given $u_0 \in H$, we take $x_0 \in Su_0, y_0 \in Tu_0, z_0 \in Gu_0$ and let

$$u_1 = u_0 - p(u_0) + m(z_0) + J_\rho^M (p(u_0) - m(z_0) - \rho N(x_0, y_0)).$$

Since $x_0 \in Su_0 \in CB(H), y_0 \in CB(H)$ and $z_0 \in Gu_0 \in CB(H)$, by Nadler[32], there exist $x_1 \in Su_1$, $y_1 \in Tu_1$ and $z_1 \in Gu_1$ such that

$$\| x_0 - x_1 \| \leq (1+1) H(Su_0, Su_1),$$

$$\| y_0 - y_1 \| \leq (1+1) H(Tu_0, Tu_1),$$

$$\| z_0 - z_1 \| \leq (1+1) H(Gu_0, Gu_1),$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$. Let

$$u_2 = u_1 - p(u_1) + m(z_1) + J_\rho^M (p(u_1) - m(z_1) - \rho N(x_1, y_1)).$$

Since $x_1 \in Su_1 \in CB(H), y_1 \in Tu_1 \in CB(H)$ and $z_1 \in Gu_1 \in CB(H)$, there exist $x_2 \in Su_2, y_2 \in Tu_2$ and $z_2 \in Gu_2$ such that

$$\| x_1 - x_2 \| \leq \left(1 + \frac{1}{2}\right) H(Su_1, Su_2),$$

$$\| y_1 - y_2 \| \leq \left(1 + \frac{1}{2}\right) H(Tu_1, Tu_2),$$

and

$$\| z_1 - z_2 \| \leq \left(1 + \frac{1}{2}\right) H(Gu_1, Gu_2).$$

By induction, we can obtain our algorithm for the problem (2.1) as follows :

Algorithm 3.1 — Suppose that $p, m : H \to H, N : H \times H \to H$ and $S, T, G : H \to CB(H)$. For given $u_0 \in H, x_0 \in Su_0, y_0 \in Tu_0$ and $z_0 \in Gu_0$, compute $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ from the iterative schemes

$$\left.\begin{aligned} & u_{n+1} = u_n - p(u_n) + m(z_n) + J_\rho^M (p(u_n) - m(z_n) - \rho N(x_n, y_n)), \\ & x_n \in Su_n, \| x_n - x_{n+1} \| \leq \left(1 + \frac{1}{n+1}\right) H(Su_n, Su_{n+1}), \\ & y_n \in Tu_n, \| y_n - y_{n+1} \| \leq \left(1 + \frac{1}{n+1}\right) H(Tu_n, Tu_{n+1}), \\ & z_n \in Gu_n, \| z_n - z_{n+1} \| \leq \left(1 + \frac{1}{n+1}\right) H(Gu_n, Gu_{n+1}) \end{aligned}\right\} \quad \ldots (3.1)$$

for $n = 0, 1, 2, \ldots$, where $\rho > 0$ is a constant.

From Algorithm 3.1, we can get an algorithm for the problem (2.2) as follows:

Algorithm 3.2 — Suppose that $p, m : H \to H, N : H \times H \to H$ and $S, T, G : H \to CB(H)$. For given $u_0 \in H, x_0 \in Su_0, y_0 \in Tu_0$ and $z_0 \in Gu_0$, compute $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ from the iterative schemes

$$\left.\begin{aligned}
&u_{n+1} = u_n - p(u_n) + m(z_n) + J_\rho^{\partial\varphi}(p(u_n) - m(z_n) - \rho N(x_n, y_n)),\\
&x_n \in Su_n, \| x_n - x_{n+1} \| \leq \left(1 + \frac{1}{n+1}\right) H(Su_n, Su_{n+1}),\\
&y_n \in Tu_n, \| y_n - y_{n+1} \| \leq \left(1 + \frac{1}{n+1}\right) H(Tu_n, Tu_{n+1}),\\
&z_n \in Gu_n, \| z_n - z_{n+1} \|, \leq \left(1 + \frac{1}{n+1}\right) H(Gu_n, Gu_{n+1})
\end{aligned}\right\} \quad \ldots (3.2)$$

for $n = 0, 1, 2, \ldots$, where $\rho > 0$ is a constant and $J_\rho^{\partial\varphi} = (I = \rho\partial\varphi)^{-1}$.

Remark 3.1 : For a suitable choice of the mappings S, T, G, N, p, m, M and the space H, many known iterative algorithms for solving various classes of variational inequalities and complementarity problems in [1], [7]-[9], [13], [14], [19]-[27], [29], [34]-[40], [42], [44]-[46], [48] and [49] can be obtained as special cases of Algorithms 3.1 and 3.2.

4. Existence and Convergence Theorems

In this section, we prove the existence of a solution of the problem (2.1) and the convergence of iterative sequence generated by Algorithm 3.1.

Lipschitz continuous with constants δ and σ, respectively, and $m : H \to H$ be Lipschitz continuous with constants λ. If there exists a constant $\rho > 0$ such that

$$\left.\begin{aligned}
&\left| \rho - \frac{\alpha + \xi\gamma k - 1)}{\eta^2\beta^2 - \xi^2\gamma^2} \right| < \frac{\sqrt{(\alpha + \xi\gamma(k-1))^2 - (\eta^2\beta^2 - \xi^2\gamma^2)k(2-k)}}{\eta^2\beta^2 - \xi^2\gamma^2},\\
&\alpha > (1-k)\,\xi\gamma + \sqrt{(\eta^2\beta^2 - \xi^2\gamma^2)\,k(2-k)},\ \eta\beta > \xi\gamma,\\
&\rho\xi\gamma < 1 - k,\ k = 2\lambda s + 2\sqrt{1 - 2\delta + \sigma^2},\ k < 1,
\end{aligned}\right. \quad \ldots (4.1)$$

then there exists $u \in H, x \in Su, y \in Tu$ and $z \in Gu$ satisfying the problem (2.1). Moreover,

$$u_n \to u,\ x_n \to x,\ y_n \to y,\ z_n \to z\ (n \to \infty),$$

where the sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined in Algorithm 3.1.

PROOF : From Algorithm 3.1, we have

$$\begin{aligned}
\| u_{n+1} - u_n \| = \| & u_n - u_{n-1} - (p(u_n) - p(u_{n-1})) + m(z_n) - m(z_{n-1})\\
&+ J_\rho^M (p(u_n) - m(z_n) - \rho N(x_n, y_n))\\
&- J_\rho^M (p(u_{n-1}) - m(z_{n-1}) - \rho N(x_{n-1}, y_{n-1})) \|
\end{aligned}$$

$$\leq \| u_n - u_{n-1} - (p(u_n) - p(u_{n-1})) \| + \| m(z_n) - m(z_{n-1}) \|$$

$$+ \| p(u_n) - m(z_n) - \rho N(x_n, y_n)$$

$$- ((p(u_{n-1}) - m(z_{n-1}) - \rho N(x_{n-1}, y_{n-1})) \| \quad \text{... (4.2)}$$

$$\leq \| u_n - u_{n-1} - (p(u_n) - p(u_{n-1})) \| + 2 \| m(z_n) - m(z_{n-1}) \|$$

$$+ \| p(u_n) - \rho N(x_n, y_n) - (p(u_{n-1}) - \rho N(x_{n-1}, y_{n-1})) \|$$

$$\leq 2 \| u_n - u_{n-1} - (p(u_n) - p(u_{n-1})) \| + 2\lambda \| z_n - z_{n-1} \|$$

$$+ \| u_n - u_{n-1} - \rho(N(x_n, y_n)) - N(x_{n-1}, y_{n-1})) \|$$

$$\leq 2 \| u_n - u_{n-1} - (p(u_n) - p(u_{n-1})) \| + 2\lambda \| z_n - z_{n-1} \|$$

$$+ \| u_n - u_{n-1} - \rho(N(x_n, y_n)) - N(x_{n-1}, y_n)) \|$$

$$+ \rho \| N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1}) \|.$$

By the Lipschitz continuity and strong monotonicity of p, we obtain

$$\| u_n - u_{n-1} - (p(u_n) - p(u_{n-1})) \|^2 \leq (1 - 2\delta + \sigma^2) \| u_n - u_{n-1} \|^2. \quad \text{... (4.3)}$$

Since S is H-Lipschitz continuous and strongly monotone with respect to the first argument of N, and N is Lipschitz continuous with respect to the first argument, we have

$$\| u_n - u_{n-1} - \rho(N(x_n, y_n)) - N(x_{n-1}, y_n)) \|^2$$

$$= \| u_n - u_{n-1} \|^2 - 2\rho \langle u_n - u_{n-1}, N(x_n, y_n) - N(x_{n-1}, y_n) \rangle$$

$$+ \rho^2 \| N(x_n, y_n) - N(x_{n-1}, y_n) \|^2 \quad \text{... (4.4)}$$

$$\leq \left(1 - 2\rho\alpha + \rho^2\eta^2 \left(1 + \frac{1}{n} \right)^2 \beta^2 \right) \| u_n - u_{n-1} \|^2.$$

Further, since T, G are H-Lipschitz continuous and N is Lipschitz continuous with respect to the second argument, we have

$$\| N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1}) \| \leq \xi \| y_n - y_{n-1} \|$$

$$\leq \xi\gamma \left(1 + \frac{1}{n} \right) \| u_n - u_{n-1} \|, \quad \text{... (4.5)}$$

and

$$\| z_n - z_{n-1} \| \leq s \left(1 = \frac{1}{n} \right) \| u_n - u_{n-1} \|. \quad \text{... (4.6)}$$

From (4.2)~(4.6), it follows that

$$\| u_n - u_{n+1} \| \leq \theta_n \| u_n - u_{n-1} \|, \qquad \text{... (4.7)}$$

where

$$\theta_n = 2\lambda s\,(1 + n^{-1}) + 2\sqrt{1 - 2\delta + \sigma^2} + \sqrt{1 - 2\rho\alpha + \rho^2\,\eta^2\,\beta^2\,(1 + n^{-1})^2}$$

$$+\ \rho\xi\eta\,(1 + n^{-1}).$$

Letting

$$\theta = k + \sqrt{1 - 2\rho\alpha + \rho^2\,\eta^2\,\beta^2} + \rho\xi\gamma,$$

where $k = 2\lambda s + 2\sqrt{1 - 2\delta + \sigma^2}$, we know $\theta_n \to \theta$ as $n \to \infty$. It follows from (4.1) that $\theta < 1$. Hence $\theta_n < 1$ for n sufficiently large. Therefore, (4.7) implies that $\{u_n\}$ is a Cauchy sequence in H and we can suppose that $u_n \to u \in H$ as $n \to \infty$.

Now we prove that $x_n \to x \in Su, y_n \to y \in Tu$ and $z_n \to z \in Gu$. In fact, it follows from Algorithm 3.1 that

$$\| x_n - x_{n-1} \| \leq \left(1 + \frac{1}{n} \right) \eta \| u_n - u_{n-1} \|,$$

$$\| y_n - y_{n-1} \| \leq \left(1 + \frac{1}{n} \right) \gamma \| u_n - u_{n-1} \|,$$

and

$$\| z_n - z_{n-1} \| \leq \left(1 + \frac{1}{n} \right) \sigma \| u_n - u_{n-1} \|.$$

These imply that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are all Cauchy sequences in H. Let $x_n \to x, y_n \to y, z_n \to z$ as $n \to \infty$. Further we have, as $n \to \infty$,

$$\begin{aligned} d(x,\ Su)\ &=\ \inf\ \{\| x - v \| : v \in Su\} \\ &\leq \| x - x_n \| + d(x_n,\ Su) \\ &\leq \| x - x_n \| + H(Su_n,\ Su) \\ &\leq \| x - x_n \| + \eta \| u_n - u \| \to 0. \end{aligned}$$

Hence, $x \in Su$. Similarly, $y \in Tu, z \in Gu$. From (3.1), we have

$$p(u) = m(z) + J_\rho^M\,(p(u) - m(z) - \rho N(x, y)).$$

Therefore, it follows from Lemma 3.1 that (u, x, y, z) is a solution of the problem (2.1). This completed the proof.

From Theorem 4.1, we have the following result :

Theorem 4.2 — *Let N, S, T, G, p and m be the same as in Theorem 4.1. If there exists a constant* $\rho > 0$ *such that the condition (4.1) in Theorem 4.1 holds, then there exist* $u \in H, x \in Su, y \in Tu$ *and* $z \in Gu$ *satisfying the problem (2.2).*

Moreover,

$$u_n \to u,\ x_n \to x,\ y_n \to y,\ z_n \to z\ (n \to \infty),$$

where the sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *are defined in Algorithm 3.2.*

Remark 4.1 : For an appropriate and suitable choice of the mappings *S, T, G, N, p, m, M* and the space *H*, we can obtain several known results in [1], [7]=[9], [13], [14], [19]-[27], [29], [34]-[40], [42], [44]-[46], [48] and [49] as special cases of Theorems 4.1 and 4.2.

REFERENCES

1. S. Adly, *J. math. Anal. Appl.* **201** (1996), 609-30.
2. H. Attouch, Variational Convergence for Functions and Operators, In: *Appl. Math. Ser.* Pitman, London, 1974.
3. C. Baiocchi and A. Capelo, Variational and Quasivariational Inequalities. In: *Application to Free Boundary Problems*, Qisely, New York, 1984.
4. A. Bensoussan, *Stochastic Control by Functional Analysis Method*, North-Holland, Amsterdam, 1982.
5. A. Bensoussan and J. L. Lions, *Impulse Control and Quasivariational Inequalities*, Gauthiers-Villers, Bordas, Paris, 1984.
6. D. Chan and J. S. Pang, *Math. OR* **7** (1982), 211-22.
7. Shih-sen Chang, *Variational Inequality and Complementarity Problem Theory with Applications*, Shanghai Scientific and Tech. Literature Publishing House, Shanghai, 1991.
8. Shih-sen Chang and Nan-jing Huang, *J. math. Anal. Appl.* **158** (1991), 194-202.
9. Shih-sen Chang and Nan-jing Huang, *Math: Japonica* **36** (1991), 1093-1100.
10. R. W. Cottle, J. P. Pang and R. E. Stone, *The Linear Complementarity Problem*, Academic Press, London, 1992.
11. J. Crank, *Free and Moving Boundary Problems*, Clarendon, Oxford, 1984.
12. V. F. Demyanov, G. E. Stavroulakis, L. N. Polyakova and P. D. Panagiotopoulos, *Quasidifferentiability and Nonsmooth Modeling in Mechanics, Engineering and Economics*, Kluwer Academic, Holland, 1996.
13. Xie-ping Ding, *J. math. Anal. Appl.* **210** (1997), 88-101.
14. Xie-ping Ding, *J. math. Anal. Appl.* **173** (1993), 577-87.
15. G. Duvaut and J. L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.
16. F. Giannessi and A. Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum, New York, 1995.
17. R. Glowinski, J. Lions and R. Trmolieres, *Numerical Analysis of Variational Inequalities*, North Holland, Amsterdam, 1982.
18. P. T. Harker and J. S. Pang, *Math. Programming* **48** (1990), 161-220.
19. A. Hassouni and A. Moudafi, *J. math. Anal. Appl.* **185** (1994), 706-712.
20. Nan-jing Huang, *J. math. Anal. Appl.* **216** (1997), 197-210.
21. Nan-jing Huang, *Comput. math. Appl.* **35** (10) (1998), 1-7.
22. Nan-jing Huang, *Appl. Math. Lett.* **9** (3) (1996), 25-29.
23. Nan-jing Huang, *Z. Angw. Math. Mech.* **78** (1998), 427-430.
24. Nan-jing Huang, *Computers Math. Appl.* **35** (10) (1998), 9-14.
25. Nan-jing Huang and S. Y. Cao, *J. Xinjiang Univ.* **10** (4) (1993), 42-47.
26. Nan-jing Huang and X. Q. Hu, *J. Sichuan Univ.* **31** (1994), 306-10.
27. Nan-jing Huang and D. P. Wu, *J. Sichuan Univ.* **33** (1996), 490-93.
28. G. Isac, *Complementarity problems, LNM*, 1528, Springer-Verlag, Berlin, 1992.
29. K. R. Kazmi, *J. math. Anal. Appl.* **209** (1997), 572-84.

30. G. J. Minty, *Pacific J. Math.* **14** (1964), 243-247.
31. U. Mosco, *Implicit Variational Problems and Quasi-variational Inequalities, LNM* **543**, Springer-Verlag, Berlin, 1976.
32. S. B. Nadler, Jr., *Pacific J. Math.* **30** (1969), 475-488.
33. A. Nagurney and S. Siokos, *Mathl. Comput. Modeling.* **25**(1) (1997), 31-49.
34. M. A. Noor, *J. math. Anal. Appl.* **123** (1987), 455-60.
35. M. A. Noor, *J. math. Anal. Appl.* **130** (1988), 344-53.
36. M. A. Noor, *J. math. Anal. Appl.* **228** (1998), 206-20.
37. M. A. Noor, *Appl. Math. Lett.* **11** (4) (1998), 109-13.
38. M. A. Noor and K. I. Noor, *Mathl. Comput. Modeling.* **26**(7) (1997), 109-21.
39. M. A. Noor, K. I. Noor and T. M. Rassias, *J. math. Anal. Appl.* **220** (1998), 741-59.
40. M. A. Noor, K. I. Noor and T. M. Rassias, *J. Comput. appl. Math.* **47** (1993), 285-312.
41. P. D. Panagiotopoulos and G. E. Stavroulakis, *Acta. Mech.* **94** (1992), 171-194.
42. S. M. Robinson, *Math. Programming Stud.* **10** (1979), 128-141.
43. R. Rockaffellar, *Convex Analysis, Princeton Univ. press, Princeton, NJ*, 1970.
44. A. H. Siddiqi and Q. H. Ansari, *J. math. Anal. Appl.* **149** (1990), 444-50.
45. A. H. Siddiqi and Q. H. Ansari, *J. Math. Anal. Appl.* **166** (1992), 386-92.
46. L. U. Uko, *J. math. Anal. Appl.* **220** (1998), 65-76.
47. J. C. Yao, *J. math. Anal. Appl.* **158** (1991), 139-60.
48. George Xian-Zhi Yuan, *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker Inc., 1999.
49. Lu-chuan Zeng, *J. math. Anal. Appl.* **193** (1995), 706-714.
50. D. L. Zhu and P. Marcotte, *SIAM J. Control. Optim.* **6**(3) (1996), 714-26.