Indian J. pure appl. Math., **31**(9) : 1135-1149, September 2000

STRESS DISTRIBUTION DUE TO GRIFFITH CRACKS AT THE INTERFACE OF TWO DISSIMILAR MEDIA

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(*Received* 28 *December* 1998; *accepted* 2 *September* 1999)

The stress distribution in the neighbourhood of a Griffith Crack and then a pair of coplanar Griffith cracks opened at the interface of two bonded dissimilar media have been considered in this paper and dislocation layers have been utilized to solve the problem. Here the cracks are opened by the shear force applied at infinity, while the crack surfaces are stress free. The solutions of these type of problems have been derived earlier by other methods. The superposition principle is exploited here to adapt the solution of this problem into the framework of our formulations.

Key Words : Dislocation Theory; Two-dimensional Crack Problems; Hilbert Problem

1. INTRODUCTION

Dislocation layers have not been exploited earlier to solve the boundary value problems of classical elasticity. Several problems appeared in literature dealing with cracks located at the interface of two bonded dissimilar half-planes, have been considered by Williams[1], Salganik[2], Erdogan[3], England[4] and Lowengrub.[5 & 6] These authors have based their analysis either on complex variable methods developed by Muskhelishvili[7] or on Fourier transform techniques developed by Sneddon[8].

In this paper the solutions to the title problems by exploiting the modified somigliana integral established by Maiti *et al.*[9-11] have been derived. Attempts were made to solve some problems[12&13] by this approach. Here the stress distribution in the neighbourhood of a single Griffith crack and then a pair of coplanar Griffith cracks opened at the interface of two bonded dissimilar media have been considered. The solutions found in these cases are physically inadmissible, because violent oscillations are found to occur both in displacements and stresses near the crack tips. However, Salganik[2] has differed this view and claimed that the integral transform methods smooth out this characteristic oscillatory phenomena. But he has been proved to be wrong on this account by Lowengrub[5]. England[1] has shown[4] that these effects are confined only to a narrow region near the crack tips and was able to solve this problem based on complex variable formulation given a reasonable estimate of the physical state at the points far away from the crack tips.

In this paper, a single Griffith crack and then a pair of symmetrical Griffith cracks at the interface of two elastic half-planes are opened by the shear force such that $\sigma_{xy} \to P$ as $(x^2 + y^2)^{1/2} \to \infty$. The solutions of these problems reduce to that of a Hilbert problem for a constant shearing force. The solutions thus derived are in complete agreement with those derived by the authors in the homogeneous medium.

2. Displacements and Stresses in a Half-Plane

Somigliana integral[1] was modified by Maiti *et al.*[10] and it has been established there that the displacements u_i in the upper half-plane $y > 0$ may be expressed as

$$u_x(x, y) = -\int_{-\infty}^{\infty} f_x(x')\left[\frac{1}{2\pi}\tan^{-1}\left(\frac{y}{x-x'}\right)+\frac{1}{4\pi(1-\nu)}\frac{y(x-x')}{(x-x')^2+y^2}\right]dx'$$

$$-\int_{-\infty}^{\infty} f_y(x')\left[\frac{1-2\nu}{8\pi(1-\nu)}\log\left\{(x-x')^2+y^2\right\}+\frac{1}{4\pi(1-\nu)}\frac{y^2}{(x-x')^2+y^2}\right]dx' \quad \text{... (2.1)}$$

and

$$u_y(x, y) = \int_{-\infty}^{\infty} f_x(x')\left[\frac{1-s\nu}{8\pi(1-\nu)}\log\left\{(x-x')^2+y^2\right\}+\frac{1}{4\pi(1-\nu)}\frac{(x-x')^2}{(x-x')^2+y^2}\right]dx'$$

$$+\int_{-\infty}^{\infty} f_y(x')\left[\frac{1}{2\pi}\tan^{-1}\left(\frac{x-x'}{y}\right)+\frac{1}{4\pi(1-\nu)}\frac{y(x-x')}{(x-x')^2+y^2}\right]dx', \quad \text{... (2.2)}$$

where ν is the Poisson's ratio. The displacement field, given by (2.1) and (2.2), is the superposition of two displacement fields considered as due to the layers of edge dislocation of densities $-f_x(x)$ and $f_y(x)$ distributed along the line $y = 0$; see, for example, Hirth and Lothe.[15] The corresponding stresses σ_{ij} in the upper half-plane are given by

$$\sigma_{xx}(x, y) = \frac{\mu}{2\pi(1-\nu)}\left[\int_{-\infty}^{\infty} f_x(x')\frac{y\left\{3(x-x')^2+y^2\right\}}{\left\{(x-x')^2+y^2\right\}^2}dx'\right.$$

$$\left.+\int_{-\infty}^{\infty} f_y(x')\frac{(x-x')\left\{y^2-(x-x')^2\right\}}{\left\{(x-x')^2+y^2\right\}^2}dx'\right], \quad \text{... (2.3)}$$

$$\sigma_{yy}(x, y) = \frac{\mu}{2\pi(1-\nu)}\left[\int_{-\infty}^{\infty} f_x(x')\frac{y\left\{y^2-(x-x')^2\right\}}{\left\{(x-x')^2+y^2\right\}^2}dx'\right.$$

$$\left.-\int_{-\infty}^{\infty} f_y(x')\frac{(x-x')\left\{(x-x')^2+3y^2\right\}}{\left\{(x-x')^2+y^2\right\}^2}dx'\right], \quad \text{... (2.4)}$$

and

$$\sigma_{xy}(x, y) = \frac{\mu}{2\pi(1-\nu)}\left[\int_{-\infty}^{\infty} f_x(x') \frac{(x-x')\{y^2-(x-x')^2\}}{\{(x-x')^2+y^2\}^2} dx' + \int_{-\infty}^{\infty} f_y(x') \frac{y\{y^2-(x-x')^2\}}{\{(x-x')^2+y^2\}^2} dx'\right], \quad \ldots (2.5)$$

where μ is the shear modulus.

The boundary displacements and stresses can now be derived, in the limit as $y \to 0$, as

$$u_x(x, 0) = -\frac{1}{2}\int_{x}^{\infty} f_x(x')dx' - \frac{1-2\nu}{4\pi(1-\nu))}\int_{-\infty}^{\infty} f_y(x') \log|x-x'|\, dx', \quad \ldots (2.6)$$

$$u_y(x, 0) = \frac{1-2\nu}{4\pi(1-\nu)}\int_{-\infty}^{\infty} f_x(x' \log|x-x'|\, dx' + \frac{1}{4\pi(1-\nu)}$$

$$\int_{-\infty}^{\infty} f_x(x')\, dx' - \frac{1}{2}\int_{-\infty}^{\infty} f_y(x')dx' \quad \ldots (2.7)$$

$$\sigma_{yy}(x, 0) = \frac{\mu}{2\pi(1-\nu)}\int_{-\infty}^{\infty} \frac{f_y(x')\, dx'}{x'-x} \quad \ldots (2.8)$$

and

$$\sigma_{xy}(x, 0) = \frac{\mu}{2\pi(1-\nu)}\int_{-\infty}^{\infty} \frac{f_x(x')\, dx'}{x'-x}. \quad \ldots (2.9)$$

From the above relations one can easily derive

$$\frac{du_x(x, 0)}{dx} = \frac{1}{2} f_x(x) + \frac{1-2\nu}{2\mu}\sigma_{yy}(x, 0) \quad \ldots (2.10)$$

and

$$\frac{du_y(x, 0)}{dx} = \frac{1}{2} f_y(x) - \frac{1-2\nu}{2\mu}\sigma_{xy}(x, 0), \quad \ldots (2.11)$$

which give dislocation densities in terms of boundary displacements and stresses.

The application of infinite Hilbert transforms to (2.8) and (2.9) yields

$$f_x(x) = -\frac{2(1-\nu)}{\pi\mu}\int_{-\infty}^{\infty} \frac{\sigma_{xy}(x', 0)\, dx'}{x'-x}, \quad \ldots (2.12)$$

and

$$f_y(x) = -\frac{2(1-\nu)}{\pi\mu}\int_{-\infty}^{\infty}\frac{\sigma_{yy}(x',0)\,dx'}{x'-x}, \qquad \text{... (2.13)}$$

which, when substituted into (2.10) and (2.11), give

$$U'(x,0) = \frac{1-2\nu}{2\mu}\sigma_{yy}(x,0) - \frac{1-\nu}{\pi\nu}\int_{-\infty}^{\infty}\frac{\sigma_{xy}(x',0)\,dx'}{x'-x} \qquad \text{... (21.4)}$$

and

$$V'(x,0) = -\frac{1-2\nu}{2\mu}\sigma_{xy}(x,0) - \frac{1-\nu}{\pi\mu}\int_{-\infty}^{\infty}\frac{\sigma_{yy}(x',0)\,dx'}{x'-x}, \qquad \text{... (2.15)}$$

where $u_x(x, 0) = U(x, 0)$, $u_y(x, 0) = V(x, 0)$. Using the subscripts 1 and 2 in the material constants for the upper half-plane and lower half-plane respectively, one can derive

$$U'(x,0+) = \frac{1-2\nu_1}{2\mu_1}\sigma_{yy}(x,0+) - \frac{1-\nu_1}{\pi\mu_1}\int_{-\infty}^{\infty}\frac{\sigma_{xy}(x',0+)\,dx'}{x'-x}, \qquad \text{... (2.16)}$$

and

$$V'(x,0+) = \frac{1-2\nu_2}{2\mu_1}\sigma_{xy}(x,0+) - \frac{1-\nu_1}{\pi\mu_1}\int_{-\infty}^{\infty}\frac{\sigma_{yy}(x',0+)\,dx'}{x'-x}, \qquad \text{... (2.17)}$$

$$U'(x,0-) = \frac{1-2\nu_2}{2\mu_2}\sigma_{yy}(x,0-) + \frac{1-\nu_2}{\pi\mu_2}\int_{-\infty}^{\infty}\frac{\sigma_{xy}(x',0-)\,dx'}{x'-x} \qquad \text{... (2.18)}$$

and

$$V'(x,0-) = \frac{1-2\nu_1}{2\mu_2}\sigma_{xy}(x,0-) + \frac{1-\nu_2}{\pi\mu_2}\int_{-\infty}^{\infty}\frac{\sigma_{yy}(x',0-)\,dx'}{x'-x}. \qquad \text{... (2.19)}$$

The solutions to these crack problems essentially hinges on the boundary relations (2.16) - (2.19).

3. Stress Distribution Due to a Single Crack

Consider a two-dimensional infinite heterogeneous medium containing a single thin crack, which occupies the region $|x| \le a, y = 0$ and is opened by a constant shearing force P at the interface of two dissimilar elastic half-planes. The relevant boundary conditions on the line $y = 0$ are given by

$$\sigma_{yy}(x, 0+) = 0, \sigma_{xy}(x, 0+) = -P, |x| \leq a, \quad \text{... (3.1)}$$

$$\sigma_{yy}(x, 0-) = 0, \sigma_{xy}(x, 0-) = -P, |x| \leq a, \quad \text{... (3.2)}$$

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-), \sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-), \text{ for } |x| > a \quad \text{... (3.3)}$$

and

$$u_x(x, 0+) = u_x(x, 0-), u_y(x, 0+) = u_y(x, 0-), \text{ for } |x| \geq a, \quad \text{... (3.4)}$$

with all the stress components vanishing as $(x^2 + y^2)^{1/2} \to \infty$.

The boundary conditions (3.1) - (3.3), when applied to (2.17) and (2.19), yield for $|x| \leq a$

$$\frac{1 - \nu_2}{\mu_2} v^1(x, 0+) + \frac{1 - \nu_1}{\mu_1} v^1(x, 0-) = LP, \quad \text{... (3.5)}$$

where

$$L = \frac{(1 - 2\nu_1)(1 - \nu_2) + (1 - 2\nu_2)(1 - \nu_1)}{2\mu_1 \mu_2}. \quad \text{... (3.6)}$$

Further, from the boundary conditions and (2.16) and (2.18) one can derive for

$|x| \leq a$

$$\frac{1 - \nu_2}{\mu_2} U^1(x, 0+) + \frac{1 - \mu_2}{\mu_1} U^1(x, 0-) = 0. \quad \text{... (3.7)}$$

From (3.5) and (3.7) it is easy to derive

$$\phi^+(x) + g\phi^-(x) = iTP, |x| \leq a, \quad \text{... (3.8)}$$

where

$$\phi^+(x) = \phi(x + i0) = U^1(x, 0+) + iV^1(x, 0+) \quad \text{... (3.9)}$$

$$\phi^-(x) = \phi(x - i0) = U^1(x, 0-) + iV^1(x, 0-), \quad \text{... (3.10)}$$

$$g = \frac{(1 - \nu_1)\mu_2}{(1 - \nu_2)\mu_1} \quad \text{... (3.11)}$$

and

$$T = \frac{L\mu_2}{1 - \mu\nu_2}. \quad \text{... (3.12)}$$

To solve the Hilbert problem (3.8) a sectionally holomorphic function $\phi(z)$ is to be found in the entire plane cut along the segment $[-a, a]$ which is given by

$$\phi(z) = \frac{iTP\,X(z)}{2\pi i} \int_{-a}^{a} \frac{dx'}{x^{+}(x')\,(x'-z)} + CX(z), \qquad \text{... (3.13)}$$

where C is the arbitrary complex constant and

$$X(z) = (z+a)^{-\nu}(z-a)^{\nu-1}, \qquad \text{... (3.14)}$$

is a Plemelj function satisfying the relation

$$X^{+}(x) = -gX^{-}(x),\; x \in [-a, a], \qquad \text{... (3.15)}$$

with

$$e^{2\pi i\nu} = -g. \qquad \text{... (3.16)}$$

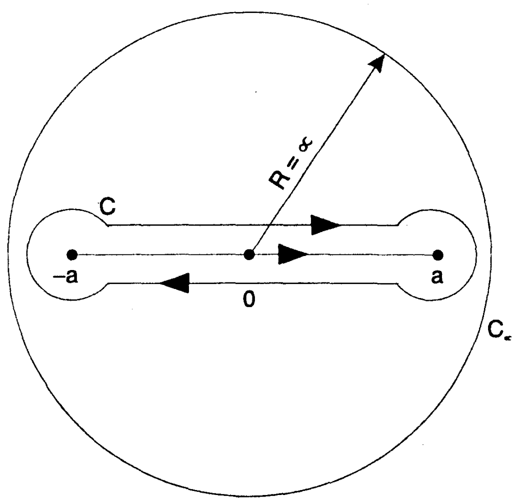
From (3.8), it is obvious that $\phi(z) \sim o(z^{-2})$ for large $|z|$ then $C = 0$.

Following England[16] one can derive from (3.13)

$$\phi(z) = \frac{iTP}{1+g}\,[1 - X(z)\,[z - 2ai\beta], \qquad \text{... (3.17)}$$

where

$$\beta = \frac{\log g}{2\pi} \qquad \text{... (3.18)}$$



The circle C_∞ around lacet C

whence it is easy to see that

$$\phi(x+i0)-\phi(x-i0)=-\frac{TP}{g^{3/2}}\left[\frac{x\cos\beta\theta+2\beta a\sin\beta\theta}{(a^2-x^2)^{1/2}}\right]$$

$$-\frac{iTP}{g^{3/2}}\left[\frac{x\sin\beta\theta-2\beta a\cos\beta\theta}{(a^2-x^2)^{1/2}}\right], \quad \dots (3.19)$$

where

$$\theta=\log\left(\frac{a+x}{a-x}\right), \quad \dots (3.20)$$

for $0\le x<a$. Separating real and imaginary parts in (3.19) one obtains

$$U'(x,0+)-U'(x,0)=-\frac{TP}{g^{3/2}}\left[\frac{x\cos\beta\theta+2\beta a\sin\beta\theta}{(a^2-x^2)^{1/2}}\right] \quad \dots (3.21)$$

and

$$V'(x,0+)-V'(x,0-)=-\frac{TP}{g^{3/2}}\left[\frac{x\sin\beta\theta-2\beta a\cos\beta\theta}{(a^2-x^2)^{1/2}}\right]. \quad \dots (3.22)$$

Also using (3.7) and (3.21) and after integration one gets

$$U(x,0+)=\frac{TP}{(1+g)\,g^{1/2}}[(a^2-x^2)^{1/2}\cos\beta\theta],\ 0\le x<a \quad \dots (3.23)$$

and

$$U(x,0-)=\frac{TP}{(1+g)\,g^{3/2}}[(a^2-x^2)^{1/2}\cos\beta\theta],\ 0\le x<a. \quad \dots (3.24)$$

For $x>a$.

$$X^{+}(x)=X^{-}(x)=X(x),\ \text{say}, \quad \dots (3.25)$$

then it is easy to derive

$$\phi(x+i0)+\phi(x-i0)=\frac{2iTP}{1+g}\left\{1-\frac{x\cos\beta\theta_1+2\alpha\beta\sin\beta\theta_1}{(x^2-a^2)^{1/2}}\right\}$$

$$+\frac{2TP}{1+g}\left\{\frac{x\sin\beta\theta_1-2a\beta\cos\beta\theta_1}{(x^2-a^2)^{1/2}}\right\}, \quad \dots (3.26)$$

where

$$\theta_1=\log\left(\frac{x+a}{x-a}\right). \quad \dots (3.27)$$

Separating real and imaginary parts in (3.26) and then using the boundary condition (3.4), it is easy to derive

$$U'(x, 0+) = U'(x, 0-) = \frac{TP}{1+g}\left\{\frac{x \sin \beta\theta_1 - 2a\beta \cos \beta\theta_1}{(x^2 - a^2)^{1/2}}\right\} \quad \text{... (3.28)}$$

and

$$V'(x, 0+) = V'(x, 0-) = \frac{TP}{1+g}\left\{1 - \frac{x \cos \beta\theta_1 + 2a\beta \sin \beta\theta_1}{(x^2 - a^2)^{1/2}}\right\} \quad \text{... (3.29)}$$

whence

$$U(x, 0+) = U(x, 0-) = \frac{TP}{1+g}[(x^2 - a^2)^{1/2} \sin \beta\theta_1]. \quad \text{... (3.30)}$$

Further for $x > a$,

$$U'(x, 0+) = U'(x, 0-) = \frac{T}{1+g}\sigma_{yy}(x, 0+) = \frac{T}{1+g}\sigma_{yy}(x, 0-) \quad \text{... (3.31)}$$

and

$$V'(x, 0+) = V'(x, 0-) = -\frac{T}{1+g}\sigma_{xy}(x, 0+) = -\frac{T}{1+g}\sigma_{xy}(x, 0-). \quad \text{... (3.32)}$$

Whence from (3.28) - (3.32), for $x > a$

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-) = P\left\{\frac{x \sin \beta\theta_1 - 2a\beta \cos \beta\theta_1}{(x^2 - a^2)^{1/2}}\right\} \quad \text{... (3.33)}$$

and

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = -P\left\{\frac{1 - x \cos \beta\theta_1 + 2a\beta \sin \beta\theta_1}{(x^2 - a^2)^{1/2}}\right\}, \quad \text{... (3.34)}$$

there will be change in signs for the corresponding values the region $x < -a$.

It is to be mentioned the solution obtained here exhibits the same oscillatory character as has been observed by Lowengrub[5]. The solution with $g = 1$ is identical with that in homogeneous medium obtained by Sneddon and Lowengrub[17] with $-P$ omitted in (3.34).

4. Stress Distribution Due to Two Symmetrical Coplanar Griffith Cracks

Consider a two-dimensional infinite heterogeneous medium containing a pair of coplaner Griffith cracks, which occupy the regions $-b \leq x \leq -a$, $a \leq x \leq b$, $y = 0$ and are opened at the interface of two bonded dissimilar elastic half-planes by the shearing force $P(x)$ where $P(x)$ is an even function of x. Then the relevant boundary conditions on the line $y = 0$ are given by

$$\sigma_{yy}(x, 0+) = 0, \ \sigma_{xy}(x, 0+) = -P(x), \ a \leq |x| \leq b \quad \text{... (4.1)}$$

and

$$\sigma_{yy}(x, 0-) = 0, \sigma_{xy}(x, 0-) = -P(x), a \le |x| \le b, \quad \text{... (4.2)}$$

and outside the crack regions,

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-), |x| < a, |x| > b, \quad \text{... (4.3)}$$

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-), |x| < a, |x| > b, \quad \text{... (4.4)}$$

$$u_x(x, 0+) = u_x(x, 0-), |x| \le a, |x| \ge b \quad \text{... (4.5)}$$

and

$$u_y(x, 0+) = u_y(x, 0-), |x| \le a, |x| \ge b, \quad \text{... (4.6)}$$

with all the stress components vanishing as $(x^2 + y^2)^{1/2} \to \infty$.

The boundary conditions (4.1)-(4.4), when applied to (2.17) and (2.19), yield for $a \le |x| \le b$

$$\frac{1-\nu_2}{\mu_2} v'(x, 0+) + \frac{1-\nu_1}{\mu_1} v'(x, 0-) = LP(x), \quad \text{... (4.7)}$$

where

$$L = \frac{(1-2\nu_1)(1-\nu_2) + (1-2\nu_2)(1-\nu_1)}{2\mu_1\mu_2}. \quad \text{... (4.8)}$$

Further, from the boundary conditions and (2.16) and (2.18) one can derive for $a \le |x| \le b$

$$\frac{1-\nu_2}{\mu_2} U'(x, 0+) + \frac{1-\nu_1}{\mu}, U'(x, 0-) = 0. \quad \text{... (4.9)}$$

From (4.7) and (4.9) it is easy to derive

$$\phi^+(x) + g\phi^-(x) = iTP(x), a \le |x| \le b, \quad \text{... (4.10)}$$

where

$$\phi^+(x) = \phi(x + i0) = U'(x, 0+) + iV'(x, 0+), \quad \text{... (4.11)}$$

$$\phi^-(x) = \phi(x + i0) = U'(x, 0+) + iV'(x, 0+), \quad \text{... (4.12)}$$

$$g = \frac{(1-\nu_1)\mu_2}{(1-\nu_2)\mu_1} \quad \text{... (4.13)}$$

and

$$T = \frac{L\mu_2}{1-\nu_2}. \quad \text{... (4.14)}$$

To solve the Hilbert problem (4.10) a sectionally holomorphic function $\phi(z)$ is to be found in the entire plane cut along the segments $[-b, -a]$ and $[a, b]$ for a constant sharing force $P(X) = P$, which is given by

$$\phi(z) = \frac{iTPX(z)}{2\pi i} \int_L \frac{dx'}{X^+(x')(x'-z)} + (c_1 z + c_2)\, X(z), \qquad \text{... (4.15)}$$

where $L = [-b, -a]\, U[a, b]$, c_1 and c_2 are complex constants and

$$X(z) = (z+b)^{-\nu}(z+a)^{\nu-1}(z-a)^{-\nu}(z-b)^{\nu-1}, \qquad \text{... (4.16)}$$

is a Plemelj function satisfying the relation

$$X^+(x) = -g\, X^-(x), x \in L \qquad \text{... (4.17)}$$

with

$$e^{2\pi\nu} = -g. \qquad \text{... (4.18)}$$

Following England[16] one can easily derive from (4.15)

$$\phi(z) = \frac{iTPX(z)}{(1+g)\, 2\pi i}\left[\frac{2\pi i}{X(z)} - L(z)\right] + (c_1 z + c_1)\, X(z), \qquad \text{... (4.19)}$$

where $L(z)$ is an integral over C_∞ (C is the union of the Lacets around the segments $[-b, -a]$ and $[a, b]$) is given by

$$L(z) = 2\pi i\ \{z^2 + (b-a)(2\nu-1)z + \nu(\nu-1)(a^2+b^2) - ab\,(4\nu^2 - 4\nu + 1)\}. \qquad \text{... (4.20)}$$

Hence, from (4.19) and (4.20) it is easy to get

$$\phi(z) = \frac{iTP}{1+g}\ \{1 - (z^2 + Az + B)\, X(z)\}, \qquad \text{... (4.21)}$$

where A and B are constants given by

$$A = (b-a)(2\nu-1) + i\frac{1+g}{TP}c_1 = A_1 + iA_2, \text{ say} \qquad \text{... (4.22)}$$

and

$$B = 2\nu(\nu-1)(a^2+b^2) + ab\,(4\nu^2 - 4\nu + 1) + i\frac{1+g}{TP}c_2 = B_1 + iB_2, \text{ say}, \qquad \text{... (4.23)}$$

whence one can easily derives

$$\phi(x+i0) - \phi(x-i0) = S(x) + iR(x), \qquad \text{... (4.24)}$$

where

$$S(x) = -\frac{TP}{g^{3/2}}\left[\frac{(x^2 + A_1 x + B_1)\cos\beta\theta - (A_2 x + B_2)\sin\beta\theta}{((x^2 - a^2)(b^2 - x^2))^{1/2}}\right] \qquad \text{... (4.25)}$$

and

$$R(x) = -\frac{TP}{g^{3/2}}\left[\frac{(x^2 + A_1 x + B_1)\sin\beta\theta + (A_2 x + B_2)\cos\beta\theta}{((x^2 - a^2)(b^2 - x^2))^{1/2}}\right], \qquad \text{... (4.26)}$$

with

$$\beta = \frac{\log g}{2\pi} \qquad \text{... (4.27)}$$

and

$$\theta = \log\left(\frac{x-a}{x+a}\cdot\frac{x+b}{b-x}\right) \qquad \text{... (4.28)}$$

for $a < x < b_0$.

Similarly, one can obtain for $-b < x < -a$

$$\phi(x + i0) - \phi(x - i0) = S_1(x) + iR_1(x), \qquad \text{... (4.29)}$$

where

$$S_1(x) = \frac{TP}{g^{3/2}}\left[\frac{(x^2 + A_1 x + B_1)\cos\beta\theta - (A_2 x + B_2)\sin\beta\theta}{\left\{(x^2 - a^2)(b^2 - x^2)\right\}^{1/2}}\right] \qquad \text{... (4.30)}$$

and

$$R_1(x) = \frac{TP}{g^{3/2}}\left[\frac{(x^2 + A_1 x + B_1)\sin\beta\theta + (A_2 x + B_2)\cos\beta\theta}{\left\{(x^2 - a^2)(b^2 - x^2)\right\}^{1/2}}\right]. \qquad \text{... (4.31)}$$

Now in order that the relation (4.24) holds for $-b < x < -a$ one must have $-S(-x) = S_1(x)$ and $R(-x) = R_1(x)$. This is possible only when $A_1 = B_2 = 0$.

Hence from (4.24) and (4.9), it is found that

$$U'(x, 0+) = -\frac{TP}{(1+g)\,g^{1/2}}\left[\frac{(x^2 + B_1)\cos\beta\theta - A_2 x\sin\beta\theta}{\left\{(x^2 - a^2)(b^2 - x^2)\right\}^{1/2}}\right] \qquad \text{... (4.32)}$$

and
$$U'(x,0-)=\frac{TP}{(1+g)\,g^{1/2}}\left[\frac{(x^2+B_1)\cos\beta\theta-A_2x\sin\beta\theta}{\left\{(x^2-a^2)(b^2-x^2)\right\}^{1/2}}\right], \quad \text{... (4.33)}$$

for $a < x < b$. Now for $x > b$ the displacements and stress will be found, where

$$X^+(x)=X^-(x)=X(x),\ \text{say}, \quad \text{... (4.34)}$$

then it is easy to derive

$$\phi(x+i)+\phi(x-io)=\frac{2iTP}{1+g}\left[1-\frac{(x^2+B_1)\cos\beta\,\theta_2-A_2x\sin\beta\theta_2}{\left\{(x^2-a^2)(x^2-b^2)\right\}^{1/2}}\right.$$
$$\left.-\,i\,\frac{(x^2+B_1)\sin\beta\theta_2+A_2\,x\cos\beta\theta_2}{\left\{(x^2-a^2)(x^2-b^2)\right\}^{1/2}}\right], \quad \text{... (4.35)}$$

which, separating real and imaginary parts and then using the boundary conditions (4.5) and (4.6), gives

$$U'(x,0+)=U'(x,0-)=\frac{TP}{1+g}\left[\frac{(x^2+B_1)\sin\beta\theta_2+A_2\,x\cos\beta\theta_2}{\left\{(x^2-a^2)(x^2-b^2)\right\}^{1/2}}\right] \quad \text{... (4.36)}$$

and
$$V'(x,0+)=V'(x,0-)=\frac{TP}{1+g}\left[-\frac{(x^2+B_1)\cos\beta\theta_2-A_2\,x\sin\beta\theta_2}{\left\{(x^2-a^2)(x^2-b^2)\right\}^{1/2}}\right] \quad \text{... (4.37)}$$

where

$$\theta_2=\log\left(\frac{x-a}{x+a}\cdot\frac{x+b}{x-b}\right). \quad \text{... (4.38)}$$

Further for $x > b$

$$U'(x,0+)=U'(x,0-)=\frac{T}{1+g}\,\sigma_{yy}(x,0+)=\frac{T}{1+g}\,\sigma_{yy}(x,0-), \quad \text{... (4.39)}$$

whence one can derive

$$\sigma_{yy}(x,0+)=\sigma_{yy}(x,0-)=P\left[\frac{(x^2+B_1)\sin\beta\theta_2+A_2x\cos\beta\theta_2}{\left\{(x^2-a^2)(x^2-b^2)\right\}^{1/2}}\right]. \quad \text{... (4.40)}$$

Also

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = -\frac{1+g}{T} V'(x, 0+) = -\frac{1+g}{t} V'(x, 0-)$$

$$= P\left[\frac{(x^2+B_1)\cos\beta\,\theta_2 - A_2 x\sin\beta\theta_2}{\left\{(x^2-a^2)(x^2-b^2)\right\}^{1/2}} - 1\right]. \qquad \text{... (4.41)}$$

The stresses and displacements are to be computed for $0 \le x < a$. In this case also the relation (4.34) holds. Proceeding exactly in the same way one can easily obtain

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-) = -P\left\{\frac{(x^2+B_1)\sin\beta\theta + A_2 x\cos\beta\theta_1}{\left\{(a^2-x^2)(b^2-x^2)\right\}^{1/2}}\right] \qquad \text{... (4.42)}$$

and

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = -P\left[1 + \frac{(x^2+B_1)\cos\beta\theta - A_2 x\sin\beta\theta}{\left\{(a^2-x^2)(b^2-x^2)\right\}^{1/2}}\right], \qquad \text{... (4.43)}$$

where

$$\theta_1 = \log\left(\frac{a-x}{a+x}\cdot\frac{b+x}{b-x}\right). \qquad \text{... (4.44)}$$

But in our formulations $\sigma_{ij} \sim 0(r^{-2})$ at a large distance and it is seen that

$$\sigma_{yy} \sim -P\left\{\frac{2\beta(b-a)}{x} + \frac{A_2}{x} + 0(x^{-2})\right\}, \qquad \text{... (4.45)}$$

as $x \to \infty$. This shows that $2\beta(b-a) + A_2 = 0$ implying,

$$A_2 = -2\beta(b-a). \qquad \text{... (4.46)}$$

With this value of A_2 eqs. (4.32) and (4.33) take the forms :

$$U'(x, 0+) = -\frac{TP}{(1+g)\,g^{1/2}}\left[\frac{(x^2+B_1)\cos\beta\theta + 2\beta(b-a)\,x\sin\beta\theta}{\left\{(x^2-a^2)(b^2-x^2)\right\}^{1/2}}\right], \qquad \text{... (4.47)}$$

and

$$U'(x, 0-) = \frac{TP}{(1+g)\,g^{3/2}}\left[\frac{(x^2+B_1)\cos\beta\theta + 2\beta(b-a)\,x\sin\beta\theta}{\left\{(x^2-a^2)(b^2-x^2)\right\}^{1/2}}\right], \qquad \text{... (4.48)}$$

whence for $a < x < b$

$$U(x, 0+) = -\frac{TP}{(1+g)g^{1/2}} \int_a^x \frac{(u^2 + B_1)\cos\beta\theta + 2\beta(b-a)\,u\sin\beta\theta}{\left\{(u^2-a^2)(b^2-u^2)\right\}^{1/2}} du \qquad \text{... (4.49)}$$

and

$$U(x, 0+-) = \frac{TP}{(1+g)g^{3/2}} \int_a^x \frac{(u^2 + B_1)\cos\beta\theta + 2\beta(b-a)\,u\sin\beta\theta}{\left\{(u^2-a^2)(b^2-u^2)\right\}^{1/2}} du. \qquad \text{... (4.50)}$$

To determine the constant B_1 the condition that at the crack tips the limiting values of the displacements must be zero will be consider and hence it will be assumed that $U(b-, 0+) = 0$.

This implies that

$$\int_a^b \frac{(u^2 + B_1)\cos\beta\theta + z\beta(b-a)u\sin\beta\theta}{\left\{(u^2-a^2)(b^2-u^2)\right\}^{1/2}} du = 0, \qquad \text{... (4.51)}$$

implying

$$B_1 = -a^2 - \frac{2}{c(\beta; a)} \cdot \int_a^b xc(\beta x)\,dx - \frac{2\beta(b-a)}{c(\beta; a)}\left[aS(\beta; a) + \int_a^b S(\beta; x)\,dx\right], \qquad \text{... (4.52)}$$

where

$$C(\beta; x) = \int_x^b \frac{\cos\beta\theta du}{\left\{(u^2-a^2)(b^2-u^2)\right\}^{1/2}}, \qquad \text{... (4.53)}$$

and

$$S(\beta; x) = \int_x^b \frac{\sin\beta\theta du}{\left\{(u^2-a^2)(b^2-u^2)\right\}^{1/2}} \qquad \text{... (4.54)}$$

with

$$\theta = \log\left(\frac{u-a}{u+a} \cdot \frac{u+b}{b-u}\right). \qquad \text{... (4.55)}$$

Thus all constants has been determined.

For the homogeneous material medium $\beta = 0$ and using the properties of the elliptic integrals one can obtain

$$B_1 = -b^2 \frac{E(k)}{K(k)},$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind. In the limit as $a \to 0$, $B_1 = 0$ and the equations (4.43) and (4.41) are in complete agreement with the results observed by the author (1999) in the case of a single crack of width $2b$ in the homogeneous medium.

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