MATRIX TRANSFORMATION OF CERTAIN SEQUENCE SPACES THROUGH MATRIX PRODUCT

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(Received 4 January 1999; after revision 14 July 1999; accepted 23 October 1999)

The purpose of the present paper is to find necessary and sufficient conditions for $AB = AoB$ with reference to sequence sets $X_p, Z(p)$ and $V_o$ for $A$ and $B$ in some special classes of infinite matrices.

Key Words : Sequence Spaces; Matrix Transformation; Invariant Mean

1. INTRODUCTION

Let $\omega$ be the family of all real or complex sequences and $\phi$ denote the family of finite sequences. Any linear subspace of $\omega$ is called a sequence space.

The following sequence spaces have been defined and discussed in [1], [2] and [3]. For $1 \leq p < \infty$, $X_p$ is defined to be the space

$$\left\{ x \in \omega ; \sup_{n=1}^{\infty} \left\{ \frac{1}{n} \sum_{i=1}^{n} x_i \right\}^p < \infty \right\}.$$

Let $p = (p_n)$ be a sequence of strictly positive real numbers. $Z(p)$ is defined to be the sequence set

$$\left\{ x \in \omega ; \sup \left\{ \frac{1}{n} \sum_{i=1}^{n} x_i \right\}^{p_n} < \infty \right\}.$$

Note that $Z(p)$ defined above is not a linear space in general. For example, if $p_n = n$ for all $n$ and $x_n = 1$ for all $n$, then $x \in Z(p)$ but $2x \notin Z(p)$. 
Let \( \sigma \) be a one-to-one mapping of the set of positive integers into itself such that \( \sigma^m(n) \neq n \), for all positive integers \( m \) and \( n \). A continuous linear functional \( \phi \) on \( l_\infty \) is said to be an invariant mean if

\begin{enumerate}
  \item \( \phi(x) \geq 0 \) whenever \( x_n \geq 0 \) for all \( n \),
  \item \( \phi(e) = 1 \), where \( e = (1, 1, 1, \ldots) \),
  \item \( \phi(x_{\sigma(n)}) = \phi(x) \) for all \( x \in l_\infty \).
\end{enumerate}

\( V_\sigma \) denotes the space of bounded sequences all of whose invariant means are same.

Let \( X \) and \( Y \) be any two non-empty subsets of \( \omega \). Let \( A = (a_{nk}) \) \( (n, k = 1, 2, 3, \ldots) \) be an infinite matrix of complex numbers.

We write \( Ax = (A_n(x)) \) if \( A_n x = \sum_{k=1}^{\infty} a_{nk} x_k \) converges for each \( n \in N \). If \( x \in X \) implies \( Ax \in Y \), then \( A \) defines a matrix transformation from \( X \) into \( Y \) and we denote it by \( A : X \to Y \). The sequence \( Ax \) is called the \( A \)-transform of \( x \). By \( (X, Y) \) we mean the class of matrices from \( X \) into \( Y \).

Let \( A \) and \( B \) be two infinite matrices. We say that matrix product \( BA \) is defined if for each \( n, k \in N \), \( \sum_{j=1}^{\infty} b_{nj} a_{jk} \) is convergent. We write \( BA = D \) with \( d_{nk} = \sum_{j=1}^{\infty} b_{nj} a_{jk} \) for each \( n \) and \( k \). \( A \) will be called row finite (column finite) if each row (column) of \( A \) lies in \( \phi \). If \( B \) is row finite then \( BA \) is defined for all \( A \)'s. We next note that if \( A \) is column finite then \( BA \) is defined for all \( B \)'s.

The question arises if \( BA \) is defined then does it represent the composition \( BoA \) of maps \( B \) and \( A \) induced by \( B \) and \( A \) respectively.

The following example given by Wilansky\(^4\) shows that the answer is no.

Consider the map \( A : \omega \to \omega \) given by \( (Ax)_n = x_n - x_{n-1} \), where, for notational convenience we take \( x_0 = 0 \). \( A \) is given by the matrix \((a_{nk})\) defined by

\[
a_{nk} = \begin{cases} 
  -1 & n = k - 1, \quad k > 1 \\
  1 & n = k, \quad k \geq 1 \\
  0 & \text{otherwise}.
\end{cases}
\]

Let \( B \) be the matrix with one at each place. Since \( A \) is row finite as well as column finite so \( D = BA \) is defined and for each \( n, k \)

\[
d_{nk} = \sum_{j=1}^{\infty} b_{nj} a_{jk} = \sum_{j=1}^{\infty} a_{jk}.
\]

But for \( x \in c \) (set of convergent sequence), \( (BoA)x \) is defined and is given by

\[
(BoA)x = \sum_{n=1}^{\infty} (x_n - x_{n-1}) = \lim_{n \to \infty} x_n.
\]

In particular, \( BoA = 0 \) on \( \phi \). Thus \( BoA \) is not given by a matrix.
The main purpose of this paper is to find necessary and sufficient conditions for $BA = BoA$ with reference to sequence sets $X_p, Z(p)$ and $V_{\sigma}$.

2. MAIN RESULTS

Throughout this section $C$ denote the same matrix. Let $C : \omega \to \omega$ be the linear map given by

$$Cx = \left| \frac{1}{n} \sum_{i=1}^{n} x_i \right|.$$

Then $C$ is one-one and onto and

$C^{-1}$ is given by

$$C^{-1}y = (ny_n - (n - 1)y_{n-1}),$$

where, for notational convenience, we put $y_0 = 0$. Then the sequence sets $X_p$ and $Z(p)$ can also be written as

$$X_p = C^{-1}(l_p)$$

and

$$Z(p) = C^{-1}(l_{\infty}(p)),$$

where

$$l_p = \left\{ x \in \omega; \sum_{n=1}^{\infty} |x_n|^p < \infty \right\} (1, p < \infty)$$

and

$$l_{\infty}(p) = \left\{ x \in \omega; \sup_{n} |x_n|^p < \infty \right\}.$$

In matrix notation $C$ and $R = C^{-1}$ correspond to the matrices

$$C = (c_{nk}) \quad \text{and} \quad R = (r_{nk}) \quad \text{given by}$$

$$c_{nk} = \begin{cases} \frac{1}{n} & k \leq n \\ n & k > n \end{cases}.$$
and
\[ r_{nk} = \begin{cases} 
  -(n-1) & k = n-1, n > 1, \\
  n & k = n, n \geq 1, \\
  0, & \text{otherwise}. 
\end{cases} \]

Let \( A = (a_{nk}) \) be any infinite matrix. In this case the matrix product \( AC^{-1} \) exists and is given by
\[ AC^{-1} = (k (a_{nk} - a_{n,k+1})). \]

**Theorem 1** — Let \( A \) be an infinite matrix. Consider an \( x \in \omega \).

(i) Suppose \((AoC^{-1})x\) and \( AC^{-1}(x) \) both exist in \( \omega \). Then \((AoC^{-1})x = AC^{-1}(x) \) if and only if \( \lim_{k \to \infty} ka_{nk}x_k = 0 \) for each \( n \).

(ii) If \( \lim_{k \to \infty} ka_{nk}x_k = 0 \) for each \( n \), then \((AoC^{-1})x \) exists in \( \omega \) if and only if \( AC^{-1}(x) \) exists in \( \omega \).

**Proof (i)**: We first note that, since \((AoC^{-1})x\) exists, we have for each \( n \),
\[ \sum_{k=1}^{\infty} a_{nk}(kx_k - (k-1)x_{k-1}) \text{ converges and equals } ((AoC^{-1})x)_n. \]
Again since \( AC^{-1}(x) \) exists, we have,
\[ \sum_{k=1}^{\infty} k(a_{nk} - a_{n,k+1})x_k \text{ converges and equals } (AC^{-1}(x))_n. \]

Now suppose \((AoC^{-1})x = AC^{-1}(x)\). Let \( n \in N \).

Then
\[ \sum_{k=1}^{\infty} a_{nk}(kx_k - (k-1)x_{k-1}) = \sum_{k=1}^{\infty} k(a_{nk} - a_{n,k+1})x_k. \]
So
\[ \lim_{s \to \infty} \left[ \sum_{k=1}^{s-1} k(a_{nk} - a_{n,k+1})x_k + sa_{ns}x_s \right] = \lim_{s \to \infty} \sum_{k=1}^{s} k(a_{nk} - a_{n,k+1})x_k. \]

Therefore, \((AoC^{-1})x = AC^{-1}(x)\) if and only if for each \( n \),
\[ \lim_{k \to \infty} ka_{nk}x_k = 0. \]

(ii) This follows on noting, as in the proof of (i), for each \( n \) and \( s \),
\[ \sum_{k=1}^{s} a_{nk}(kx_k - (k-1)x_{k-1}) = \sum_{k=1}^{s-1} k(a_{nk} - a_{n,k+1})x_k + sa_{ns}x_s. \]

**Theorem 2** — Suppose either \( A \in (X_p, \omega) \) or \( AC^{-1} \in (l_p, \omega) \). Then both \( AoC^{-1} \) and \( AC^{-1} \) define map on \( l_p \) to \( \omega \) with \( AoC^{-1} = AC^{-1} \) on \( l_p \) if and only if for each \( n \), \((ka_{nk})_{k=1}^{\infty} \) is bounded.
Proof: Suppose for some \( n, (ka_{nk})_{k=1}^{\infty} \) is unbounded, let \( r > 1/p \), then there exists a strictly increasing sequence \((k_j)\) such that

\[ \|k_j a_{nk_j}\| > j^r \quad \text{for each } j. \]

Put

\[ x_m = \begin{cases} 
\frac{1}{j^r} & \text{for } m = k_j, j = 1, 2, \\
0 & \text{otherwise.}
\end{cases} \]

Then \( x = (x_m) \in l_p \) and, for each \( j \), \( \|k_j a_{nk_j} x_{k_j} \| > 1 \). So, \( \lim_{k \to \infty} ka_{nk} x_k \) can not be zero.

If \( A \in (X_p, \omega) \) then \( AoC^{-1} (x) \) exists. On the other hand, if \( AC^{-1} \in (l_p, \omega) \) then \( AC^{-1} (x) \) exists. The first of the Theorem 1 gives that if \( AC^{-1} \) and \( AoC^{-1} \) exist they cannot be equal.

Now suppose for each \( n, (ka_{nk})_{k=1}^{\infty} \) is bounded. Let \( x \in l_p \). Then for each \( n \), \( \lim_{k \to \infty} ka_{nk} x_k = 0 \) and given assumptions on \( A \) imply that at least one of \( AoC^{-1} \) or \( AC^{-1} \) exists. The result now follows from Theorem 1.

**Theorem 3** — Suppose \( (ka_{nk})_{k=1}^{\infty} \) is bounded for each \( n \), then the infinite matrix \( A \in (X_p, V_\sigma) \) if and only if \( AC^{-1} \in (l_p, V_\sigma) \).

Proof: It follows from Theorem 2 because \( A \in (X_p, V_\sigma) \) if and only if \( AoC^{-1} \) maps \( l_p \) to \( V_\sigma \).

**Lemma** — Let \( A = (a_{nk}) \) be an infinite matrix and \( Y \) be any subset of \( l_\infty \) then the following hold —

(i) Suppose \( \lim_{k \to \infty} ka_{nk} = 0 \) for each \( n \). Suppose either \( A \in (C^{-1} Y, \omega) \) or \( AC^{-1} \in (Y, \omega) \), then both \( AoC^{-1} \) and \( AC^{-1} \) define maps on \( Y \) to \( \omega \), with \( AoC^{-1} = AC^{-1} \) on \( Y \).

(ii) Suppose there is a sequence \( x \in Y \) with \( \inf_{k} \|x_k\| > 0 \), and \( AoC^{-1} = AC^{-1} \) on \( Y \), then \( \lim_{k \to \infty} ka_{nk} = 0 \) for each \( n \).

(iii) \( A \in (C^{-1} Y, \omega) \). Suppose \( C^{-1} Y \) contains a sequence \( z \) such that \( \inf_{k} \frac{|z_k|}{k} > 0 \) then \( \lim_{k \to \infty} ka_{nk} = 0 \) for each \( n \).

Proof: (i) Let \( x \in Y \). Then \( x \in l_\infty \). So, if \( \lim_{k \to \infty} ka_{nk} = 0 \), for each \( n \), then \( \lim_{k \to \infty} ka_{nk} x_k = 0 \), for each \( n \). Using part (ii) of Theorem 1, \( (AoC^{-1}) x \) exist in \( \omega \). Now \( (AoC^{-1}) x \) and \( (AC^{-1}) x \) both exist in \( \omega \) and \( \lim_{k \to \infty} ka_{nk} x_k = 0 \) for each \( n \). Part (i) of Theorem 1 gives \( (AoC^{-1}) x = (AC^{-1}) x \). Therefore, \( AoC^{-1} = AC^{-1} \) on \( Y \).
(iii) Since $AoC^{-1} = AC^{-1}$ on $Y$, we have by Theorem 1 part (i) for each $x \in Y \lim_{k \to \infty} ka_{nk} x_k = 0$ for all $n$. Now there exists a sequence $x \in Y$ such that $\inf k x_k > 0$. So, we have $\lim_{k \to \infty} ka_{nk} = 0$, for each $n$.

(iii) $A \in (C^{-1} Y, \omega)$ implies that for each $z \in C^{-1} Y \sum_{k=1}^{\infty} a_{nk} z_k$ converges for each $n$ and which gives $\sum_{k=1}^{\infty} ka_{nk} \frac{z_k}{k}$ converges for each $n$. This gives $\lim_{k \to 0} ka_{nk} = 0$ for each $n$.

The following theorem now follows from the above Lemma.

**Theorem 4** — Let $Y$ be any subset of $l_\infty$ such that $C^{-1} Y$ contains a sequence $z$ with $\inf_{k} \frac{1}{k} z_k > 0$, and $Z \subset \omega$, then an infinite matrix $A \in (C^{-1} Y, Z)$ if and only if (i) and (ii) hold, where

(i) $\lim_{k \to 0} ka_{nk} = 0$ for each $n$,

(ii) $AC^{-1} \in (Y, Z)$.

Further, in this case, $AoC^{-1} = AC^{-1}$ on $Y$.

**ACKNOWLEDGEMENT**

Authors are thankful for useful comments of the referee which have improved the presentation of the paper.

**REFERENCES**