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MATRIX TRANSFORMATION OF CERTAIN SEQUENCE SPACES THROUGH MATRIX PRODUCT

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The purpose of the present paper is to find necessary and sufficient conditions for $AB = AoB$ with reference to sequence sets X_p, $Z(p)$ and V_σ for A and B in some special classes of infinite matrices.

Key Words : Sequence Spaces; Matrix Transformation; Invariant Mean

1. INTRODUCTION

Let ω be the family of all real or complex sequences and ϕ denote the family of finite sequences. Any linear subspace of ω is called a sequence space.

The following sequence spaces have been defined and discussed in [1], [2] and [3]. For $1 \leq p < \infty$, X_p is defined to be the space

$$\left\{ x \in \omega;\ \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{i=1}^{n} x_i \right|^p < \infty \right\}.$$

Let $\mathbf{p} = (p_n)$ be a sequence of strictly positive real numbers. $Z(p)$ is defined to be the sequence set

$$\left\{ x \in \omega;\ \sup \left| \frac{1}{n} \sum_{i=1}^{n} x_i \right|^{p_n} < \infty \right\}.$$

Note that $Z(p)$ defined above is not a linear space in general. For example, if $p_n = n$ for all n and $x_n = 1$ for all n, then $x \in Z(p)$ but $2x \notin Z(p)$.

Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) \neq n$, for all positive integers m and n. A continuous linear functional ϕ on l_∞ is said to be an invariant mean if

(i) $\phi(x) \geq 0$ whenever $x_n \geq 0$ for all n,

(ii) $\phi(e) = 1$, where $e = (1, 1, 1, ...)$,

and (iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

V_σ denotes the space of bounded sequences all of whose invariant means are same.

Let X and Y be any two non-empty subsets of ω. Let $A = (a_{nk})$ $(n, k = 1, 2, 3, ...)$ be an infinite matrix of complex numbers.

We write $Ax = (A_n(x))$ if $A_n x = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for each $n \in N$. If $x \in X$ implies $Ax \in Y$, then A defines a matrix transformation from X into Y and we denote it by $A : X \to Y$. The sequence Ax is called the A-transform of x. By (X, Y) we mean the class of matrices from X into Y.

Let A and B be two infinite matrices. We say that matrix product BA is defined if for each $n, k \in N$, $\sum_{j=1}^{\infty} b_{nj} a_{jk}$ is convergent. We write $BA = D$ with $d_{nk} = \sum_{j=1}^{\infty} b_{nj} a_{jk}$ for each n and k. A will be called row finite (column finite) if each row (column) of A lies in ϕ. If B is row finite then BA is defined for all A's. We next note that if A is column finite then BA is defined for all B's.

The question arises if BA is defined then does it represent the composition BoA of maps B and A induced by B and A respectively.

The following example given by Wilansky[4] shows that the answer is no.

Consider the map $A : \omega \to \omega$ given by $(Ax)_n = x_n - x_{n-1}$, where, for notational convenience we take $x_0 = 0$. A is given by the matrix (a_{nk}) defined by

$$a_{nk} = \begin{cases} -1 & n = k-1 \quad k > 1 \\ 1 & n = k \quad k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let B be the matrix with one at each place. Since A is row finite as well as column finite so $D = BA$ is defined and for each n, k

$$d_{nk} = \sum_{j=1}^{\infty} b_{nj} a_{jk} = \sum_{j=1}^{\infty} a_{jk} .$$

But for $x \in c$ (set of convergent sequence), $(BoA)x$ is defined and is given by

$$(BoA)x = \sum_{n=1}^{\infty} (x_n - x_{n-1}) = \lim x_n,$$

In particular, $BoA = 0$ on ϕ. Thus BoA is not given by a matrix.

The main purpose of this paper is to find necessary and sufficient conditions for $BA = BoA$ with reference to sequence sets X_p, $Z(p)$ and V_σ .

2. Main Results

Throughout this section C denote the same matrix. Let $C: \omega \to \omega$ be the linear map given by

$$Cx = \left| \frac{1}{n} \sum_{i=1}^{n} x_i \right|.$$

Then C is one-one and onto and

C^{-1} is given by

$$C^{-1}y = (ny_n - (n-1)\, y_{n-1}),$$

where, for notational convenience, we put $y_0 = 0$. Then the sequence sets X_p and $Z(p)$ can also be written as

$$X_p = C^{-1}(l_p)$$

and

$$Z(p) = C^{-1}(l_\infty(p)),$$

where

$$l_p = \left\{ x \in \omega;\ \sum_{n=1}^{\infty} |x_n|^p < \infty \right\} (1, p < \infty)$$

and

$$l_\infty(p) = \left\{ x \in \omega;\ \sup_n |x_n|^{p_n} < \infty \right\}.$$

In matrix notation C and $R = C^{-1}$ correspond to the matrices

$C = (c_{nk})$ and $R = (r_{nk})$ given by

$$c_{nk} = \begin{cases} \dfrac{1}{n} & k \le n \\ 0 & k > n \end{cases} .$$

and $$r_{nk} = \begin{cases} -(n-1) & k = n-1, n > 1, \\ n & k = n, n \geq 1. \\ 0, & \text{otherwise.} \end{cases}$$

Let $A = (a_{nk})$ be any infinite matrix. In this case the matrix product AC^{-1} exists and is given by

$$AC^{-1} = (k\,(a_{nk} - a_{n,\,k+1})).$$

Theorem 1 — *Let A be an infinite matrix. Consider an* $x \in \omega$.

(i) Suppose $(AoC^{-1})\,x$ *and* $AC^{-1}(x)$ *both exist in* ω. *Then* $(AoC^{-1})\,x = AC^{-1}(x)$ *if and only if* $\lim_{k \to \infty} ka_{nk}\, x_k = 0$ *for each n.*

(ii) If $\lim_{k \to \infty} ka_{nk}\, x_k = 0$ *for each n, then* $(AoC^{-1})\,x$ *exists in* ω *if and only if if* $AC^{-1}(x)$ *exists in* ω.

PROOF (i) : We first note that, since $(AoC^{-1})\,x$ exists, we have for each n, $\sum_{k=1}^{\infty} a_{nk}\,(kx_k - (k-1)\,x_{k-1})$ converges and equals $((AoC^{-1})x_n)$. Again since $AC^{-1}(x)$ exists, we have, for each n, $\sum_{k=1}^{\infty} k\,(a_{nk} - a_{n,\,k+1})\,x_k$ converges and equals $(AC^{-1}(x))_n$.

Now suppose $(AoC^{-1})x = AC^{-1}(x)$. Let $n \in N$.

Then $$\sum_{k=1}^{\infty} a_{nk}\,(kx_k - (k-1)\,x_{k-1}) = \sum_{k=1}^{\infty} k(a_{nk} - a_{n,\,k+1})\,x_k.$$

So

$$\lim_{s \to \infty} \left[\sum_{k=1}^{s-1} k(a_{nk} - a_{n,\,k+1})\,x_k + sa_{ns}\,x_s \right] = \lim_{s \to \infty} \sum_{k=1}^{s} k(a_{nk} - a_{n,\,k+1})\,x_k.$$

Therefore, $(AoC^{-1})\,x = AC^{-1}(x)$ if and only if for each n,

$$\lim_{k \to \infty} ka_{nk}\,x_k = 0.$$

(ii) This follows on noting, as in the proof of (i), for each n and s.

$$\sum_{k=1}^{s} a_{nk}\,(kx_k - (k-1)\,x_{k-1}) = \sum_{k=1}^{s-1} k(a_{nk} - a_{n,\,k+1})\,x_k + sa_{ns}\,x_s.$$

Theorem 2 — *Suppose either* $A \in (X_p, \omega)$ *or* $AC^{-1} \in (l_p, \omega)$. *Then both* AoC^{-1} *and* AC^{-1} *define map on* l_p *to* ω *with* $AoC^{-1} = AC^{-1}$ *on* l_p, *if and only if for each n,* $(ka_{nk})_{k=1}^{\infty}$ *is bounded.*

PROOF : Suppose for some n, $(ka_{nk})_{k=1}^{\infty}$ is unbounded, Let $r > 1/p$, then there exists a strictly increasing sequence (k_j) such that

$$|k_j a_{nk_j}| > j^r \quad \text{for each } j.$$

Put

$$x_m = \begin{cases} \frac{1}{j^r} & \text{for } m = k_j, j = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Then $x = (x_m) \in l_p$ and, for each j, $|k_j a_{nk_j} x_{k_j}| > 1$. So, $\lim_{k \to \infty} ka_{nk} x_k$ can not be zero.

If $A \in (X_p, \omega)$ then $AoC^{-1}(x)$ exists. On the other hand, if $AC^{-1} \in (l_p, \omega)$ then $AC^{-1}(x)$ exists. The first of the Theorem 1 gives that if AC^{-1} and AoC^{-1} exist they cannot be equal.

Now suppose for each $n (ka_{nk})_{k=1}^{\infty}$ is bounded. Let $x \in l_p$. Then for each n "$\lim_{k \to \infty} ka_{nk} x_k = 0$" and given assumptions on A imply that at least one of AoC^{-1} or AC^{-1} exists. The result now follows from Theorem 1.

Theorem 3 — *Suppose* $(ka_{nk})_{k=1}^{\infty}$ *is bounded for each* n, then the infinite matrix $A \in (X_p, V_\sigma)$ if and only if $AC^{-1} \in (l_p, V_\sigma)$.

PROOF : It follows from Theorem 2 because $A \in (X_p, V_\sigma)$ if and only if AoC^{-1} maps l_p to V_σ.

Lemma — Let $A = (a_{nk})$ be an infinite matrix and Y be any subset of l_∞ then the following hold —

(i) Suppose $\lim_{k \to \infty} ka_{nk} = 0$ for each n. Suppose either $A \in (C^{-1} Y, \omega)$ or $AC^{-1} \in (Y, \omega)$, then both AoC^{-1} and AC^{-1} define maps on Y to ω, with $AoC^{-1} = AC^{-1}$ on Y,

(ii) Suppose there is a sequence $x \in Y$ with $\inf |x_k| > 0$, and $AoC^{-1} = AC^{-1}$ on Y, then $\lim_{k \to \infty} ka_{nk} = 0$ for each n.

(iii) $A \in (C^{-1} Y, \omega)$. Suppose $C^{-1} Y$ contains a sequence z such that $\inf_k \frac{|z_k|}{k} > 0$ then $\lim_{k \to \infty} ka_{nk} = 0$ for each n.

PROOF : (i) Let $x \in Y$. Then $x \in l_\infty$. So, if $\lim_{k \to \infty} ka_{nk} = 0$, for each n, then $\lim_{k \to \infty} ka_{nk} x_k = 0$, for each n. Using part (ii) of Theorem 1, $(AoC^{-1}) x$ exist in ω. Now $(AoC^{-1}) x$ and $(AC^{-1}) x$ both exist in ω and $\lim_{k \to \infty} ka_{nk} x_k = 0$ for each n, Part (i) of Theorem 1 gives $(AoC^{-1}) x = (AC^{-1}) x$. Therefore, $AoC^{-1} = AC^{-1}$ on Y.

(iii) Since $AoC^{-1} = AC^{-1}$ on Y, we have by Theorem 1 part (i) for each $x \in Y$ $\lim_{k \to \infty} ka_{nk} x_k = 0$ for all n. Now there exists a sequence $x \in Y$ such that $\inf |x_k| > 0$. So, we have $\lim_{k \to \infty} ka_{nk} = 0$, for each n.

(iii) $A \in (C^{-1} Y, \omega)$ implies that for each $z \in C^{-1} y$ $\sum_{k=1}^{\infty} a_{nk} z_k$ converges for each n and which gives $\sum_{k=1}^{\infty} ka_{nk} \frac{z_k}{k}$ converges for each n. This gives $\lim_{k \to 0} ka_{nk} = 0$ for each n.

The following theorem now followers from the above Lemma.

Theorem 4 — *Let Y be any subset of l_∞ such that $C^{-1} Y$ contains a sequence z with $\inf_k \frac{|z_k|}{k} > 0$, and $Z \subset \omega$, then an infinite matrix $A \in (C^{-1} Y, Z)$ if and only if (i) and (ii) hold, where*

(i) $\lim_{k \to 0} ka_{nk} = 0$ *for each n,*

(ii) $AC^{-1} \in (Y, Z)$.

Further, in this case, $AoC^{-1} = AC^{-1}$ *on* Y.

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