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CONSTRUCTION OF BIORDERED SETS FROM VECTOR BUNDLES

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Let H be an infinite dimensional Hilbert space and X a compact Hausdorff space. Then $(X, F(H))$, the set of all continuous maps from X to $F(H)$ where $F(H)$ denotes the set of all Fredholm operators on H; is a semigroup (cf: [2] and [3]). In this paper we construct biordered sets from two collections C and C' of vector bundles, where C is a collection of vector bundles of finite dimensional subspaces of H and C' is a collection of vector-bundles of finite co-dimensional subspaces of H over X.

Key Words : Biordered Sets; Vector Bundles; Hilbert Space; Hausdorff Space

INTRODUCTION

The relation between vector bundles and semigroup of Fredholm operators with a parameter has been discussed in [2]. Attempts at locating regular subsemigroups of the above semigroup involves recognition of biordered sets of such semigroups. These biordered sets can be described in a more general way in terms of vector bundles. The present construction of biordered sets from vector bundles is useful in the above situation.

1. PRELIMINARIES

First we give the definition of biordered set. By a partial algebra E, we mean a set E together with a partial binary operation on E where a partial binary operation on E is a mapping from $D_E \subseteq E \times E$ to E. Then D_E is the domain of the partial binary operation and $(e,f) \varepsilon D_E$ if and only if the product ef exists in the partial algebra E. On E we define :

$$\omega^r = \{(e,f) : fe = e\}$$

and

$$\omega^l = \{(e,f) : ef = e\}.$$

$\mathcal{R} = \omega^r \cap (\omega^r)^{-1}$; $\mathcal{L} = \omega^l \cap (\omega^l)^{-1}$ and $\omega = \omega^r \cap \omega^l$. If T is a statement about E then its left right dual is denoted by T^*. It can be seen that when D_E is symmetric, T^* is meaningful whenever T is. We denote that ω^r, ω^l, $\mathcal{R}$, $\mathcal{L}$ and ω are relations on E.

Definition 1.1 — Let E be a partial algebra. Then E is a biordered set if the following axioms and their duals hold. In the following, e, f etc. denote arbitrary elements of E.

(B_1) ω^r and ω^l are quasi orders on E and

$$D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$$

(B_{21}) $f \in \omega^r(e) \Rightarrow f \mathcal{R} fe \omega e$,

(B_{22}) $g \omega^l f, f, g \in \omega^r(e) \Rightarrow ge \omega^l fe.$

(B_{31}) $g \omega^r f \omega^r e \Rightarrow gf = (ge)f.$

(B_{32}) $g \omega^l f, f, g \in \omega^r(e) \Rightarrow (fg)e = (fe)(ge)$

Let $M(e, f)$ denote the quasiordered set $(\omega^l(e) \cap \omega^r(f), <)$ where $<$ is defined by

$$g < h \Leftrightarrow eg \omega^r eh, gf \omega^l hf.$$

Then the set

$$S(e,f) = \{h \in M(e,f) : g < h \ \forall \ g \in M(e,f)\}$$

is called the sandwich set of e and f (in that order).

(B_4) $f, g \in \omega^r(e) \Rightarrow S(f, g)\, e = S(fe, ge).$

The following result proved in ([5], prop 2.4, p 14) is used in verification of axioms of biordered set.

Proposition 1.1 — *Assume that the partial algebra E satisfies all axioms for biordered sets except (B_4) and (B_4)*. Then E satisfies axiom (B_4) of biordered sets if and only if it satisfies the following :*

(B_4') *If* $g, h \in \omega^r(e)$ *and* $ge \omega^l he$ *then there exists* $g_1 \in \omega^r(e)$ *such that* $g_1 \omega^l h$ *and* $g_1 e = ge$. *Further when* g_1 *exists it is unique.*

Now we give the definition of vector bundle.

Definition 1.2 — Let X be a topological space. Let K denote either R or C. A vector bundle over X is a topological space E together with a continuous map $\pi : E \to X$ satisfying the following conditions.

(i) For each $x \in X$, $E_x = \pi^{-1}(x)$ is a vector space over K

(ii) For each $x \in X$, there exists a neighbourhood U of x in X, a natural number n and for each $y \in U$ there exists a linear map f_y of the vector space E_y onto K^n such that the maps. $F : E/U = \pi^{-1}(u) \to U \times K^n$ defined by $F(v)\left(\pi(v), \ f_{\pi(v)}(v) \right)$ is a homeomorphism.

Remark 1.1 : For any collection $\{E_x : x \in X\}$ of subspaces of H we write $E = \bigcup_{x \in X} E_x$ to denote the set $\bigcup_{x \in X} (\{x\} \times E_x)$ which may be considered as the disjoint union of these subspaces. Thus E is a subset of $X \times H$. Also any element of E is a pair (x, v) with $v \in E_x$. We identity $\{x\} \times E_x$ with E_x and when we say $E_x \subseteq E$ it is meant that $\{x\} \times E_x \subseteq E$. Note that $\bigcup_{x \in X} E_x$ is the symbol we use for what is explained above and it is not just the usual union.

Now we give a characterization of the vector bundles of the type given in [1].

Proposition 1.2 — *Let $\{E_x : x \in X\}$ be a family of finite-dimensional subspaces of a Hilbert space H. Then $E = \bigcup_{x \in X} E_x$ is a vector bundle over X if and only if for each $x \in X$, there exists a neighbourhood U of x, a natural number 'n' and n continuous maps $u_i : U \to E$ such that $\{u_i(x) : i = 1, 2, ..., n\}$ forms a basis of E_x for every $x \in U$. Here E is topologised as a subspace of $X \times H$.*

From the above definition we can consider vector bundles as collection of finite-dimensional vector spaces with certain continuity relations. We define vector bundles of infinite dimensional subspaces of a Hilbert space H as follows.

Definition 1.3 — Let H be an infinite dimensional Hilbert space and X be a topological space. A collection $\{E_x : x \in X\}$ of infinite dimensional subspaces of the Hilbert space H is said to be a vector bundle of infinite dimensional subspaces of H over X if the collection $\{E_x^{\perp} : x \in X\}$ where $E_x^{\perp}$ is the orthogonal complement of E_x is a vector bundle in the usual sense (cf. Proposition 1.2).

Next we give an important result on the set of all subspaces of a Hilbert space. We see that the set of all subspaces of a Hilbert space H is a partially ordered set under the set inclusion. We denote by $L(H)$ the lattice of all subspaces of H where for $A, B \in L(H)$; $A \vee B = A + B$ and $A \wedge B = A \cap B$ (cf: [4], 7.10 (1)). An important property of this lattice is the modularity relation which we state as follows:

Proposition 1.3 — *$L(H)$ is a modular lattice. That is*

$$A \subseteq C \Rightarrow A + (B \cap C) = (A + B) \cap C$$

for subspaces A, B and C of H.

In the sequel $E \in C(C')$ denote $E = \bigcup_{x \in X} E_x$ where E_x is a subspace of H for every $x \in X$. Also, $E \subseteq F$ if $E_x \subseteq F_x$ for every $x \in X$. Hence without loss of generality; the modularity relation with respect to the subspaces of H can be extended to the class of vector bundles of subspaces of H over X. We use this extended form of modularity in the construction of biordered sets and for completion we state it as follows:

Proposition 1.4 — Let A, B, C be vector bundles of subspaces of H. Then

$$A \subseteq C \Rightarrow A + (B \cap C) = (A + B) \cap C.$$

2. Biordered Set from Vector Bundles

The construction of biordered set from vector bundles is given in the following theorem.

Theorem 2.1 — *Let C be a class of vector bundles of finite dimensional subspaces of H and C' be a class of vector bundles of finite co-dimensional sub-spaces of H over X. Suppose there exists a subset $\tau \subseteq C \times C'$ such that*

(i) $(E, E') \in \tau \Rightarrow H = E_x \oplus E'_x$ for every $x \in X$.

(ii) $(E, E'); (F, F') \in \tau$ and $E' \subseteq F' \Rightarrow ((E \cap F') + F, E') \in \tau$

(iii) $(E, E'); (F, F') \in \tau$ and $F \subseteq E \Rightarrow (E, (E' + F) \cap F') \in \tau$

(iv) $(E, E'); (U, U'); (V, V') \in \tau, U' \subseteq E', V' \subseteq E', V \cap E' \subseteq U \cap E'$

$$\Rightarrow ((U \cap V') + V, U') \in \tau$$

(v) $(E, E'); (U, U'); (V, V') \in \tau, E \subseteq U, E \subseteq V, U' + E \subseteq V' + E$

$$\Rightarrow (U, (U' + V) \cap V') \in \tau$$

Define an operation $*$: $\tau \times \tau \rightarrow \{(A, B) : A = \bigcup_{x \in X} A_x; B = \bigcup_{x \in X} B_x$ where A_x, B_x are subspaces of H$\}$

Define basic product $*$ as follows :

Let

$$D_\tau = \{((E, E'), (F, F')) \in \tau \times \tau / E \subseteq F \text{ or } F \subseteq E \text{ or } E' \subseteq F'; \text{ or, } F' \subseteq E'\}.$$

Now for

$((E, E'), (F, F')) \in D_\tau$ define

$$(E, E') * (F, F') = ((E \cap F') + F, (F' + E) \cap E').$$

Then τ forms a biordered set with respect to the basic product $*$.

Proof : First we show that for $((E, E'), (F, F')) \in D_\tau$; $(E, E') * (F, F') \in \tau$.

Let $((E, E'), (F, F')) \in D_\tau$

Case (i) : $E \subseteq F$. Then $(E \cap F') + F = F$. Thus

$$(E, E') * (F, F') = ((E \cap F') + F, (F' + E) \cap E') = (F, (F' + E) \cap E')$$

Since $E \subseteq F$; by condition (iii) $(F, (F' + E) \cap E') \in \tau$, i.e. $(E, E') * (F, F') \in \tau$ in this case.

Case (ii) — $F \subseteq E$. Then

$$(E \cap F') + F = F + (F' \cap E)$$

$= (F + F') \cap E$ ... by Proposition 1.4 since $F \subseteq E$.

$= H \cap E$

$= E.$

Further, since $F \subseteq E$; $F + F' \subseteq E + F'$, i.e. $H \subseteq E + F'$. Hence $E + F' = H$.

Thus $(F' + E) \cap E' = E'$.

Hence $(E, E') * (F, F') = ((E \cap F') + F, (F' + E) \cap E') = (E, E') \in \tau$.

Case (iii) — $E' \subseteq F'$. Since $E' \subseteq F'$; we have $(F' + E) \cap E' = E'$. Further, since $E' \subseteq F'$; by condition (ii) $((E \cap F') + F, E') \in \tau$. Hence

$$(E, E') * (F, F') = ((E \cap F') + F, (F' + E) \cap E') = ((E \cap F') + F, E') \in \tau.$$

Case (iv) — $F' \subseteq E'$. Since $F' \subseteq E'$, by Proposition 1.4, we have

$$(F' + E) \cap E' = F' + (E \cap E') = F'. \text{ Since } F' \subseteq E'; E \cap F' \subseteq E \cap E' = \{0\}.$$

Hence $E \cap F' = \{0\}$. Thus $(E \cap F') + F = F$.

Hence $(E, E') * (F, F') = ((E \cap F') + F, (F' + E) \cap E') = (F, F') \in \tau$. Hence for $((E, E'), (F, F')) \in D_\tau$; $(E, E') * (F, F') \in \tau$.

Next assume that $(F, F') * (E, E') = (E, E')$.

Then $((F \cap E') + E, (E' + F) \cap F') = (E, E')$.

Hence $E' = (E' + F) \cap F' \subseteq F'$.

Conversely assume that $E' \subseteq F'$.

Then $(F, F') * (E, E') = ((F \cap E') + E, (E' + F) \cap F')$.

Now $(E' + F) \cap F' = E' + (F \cap F') = E'$.

Also, since $E' \subseteq F'$; $F \cap E' \subseteq F \cap F' = \{0\}$. Hence $F \cap E' = \{0\}$. Thus

$$(F \cap E') + E = E. \text{ So, } (F, F') * (E, E') = (E, E').$$

Next assume that $(E, E') * (F, F') = (E, E')$. Then $((E \cap F') + F, (F' + E) \cap E') = (E, E')$

Hence $F \subseteq (E \cap F') + F = E$.

Conversely, assume that $F \subseteq E$. Then

$$(E \cap F') + F = F + (F' \cap E) = (F + F') \cap E, \text{ since } F \subseteq E$$

$$= H \cap E = E.$$

Further, since $F \subseteq E$; $F + F' \subseteq F' + E$, i.e. $H \subseteq F' + E$. Hence $F' + E = H$. So, $(F' + E) \cap E' = E'$.

Thus;

$$(E, E') * (F, F') = ((E \cap F') + F, (F' + E) \cap E') = (E, E').$$

Axiom B_1 — We define ω^l and ω^r relations in τ as follows:
For $(E, E'), (F, F') \in \tau$;

$$(E, E')\, \omega^l\, (F, F') \Leftrightarrow F \subseteq E$$

and

$$(E, E')\, \omega^r\, (F, F') \Leftrightarrow E' \subseteq F'.$$

Then we observe that ω^l and ω^r are quasiorders and

$$D_\tau = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}.$$

Hence axiom B_1.

Axiom B_{21} : Assume that $(F, F') \in \omega^r (E, E')$. Then $F' \subseteq E'$. Also,

$$(F, F') * (E, E') = ((F \cap E') + E, (E' + F) \cap F')$$

$$= ((F \cap E') + E, F') \text{ since } F' \subseteq E' + F$$

Thus $(F, F') * (E, E') = ((F \cap E') + E, F')$. Hence

$(F, F')\, \mathcal{R} (F, F') * (E, E')$. Further, since $F' \subseteq E'$;

$(F, F') * (E, E') = ((F \cap E') + E, F')\, \omega^r\, (E, E')$ and since

$E \subseteq (F \cap E') + E$; $(F, F') * (E, E')\, \omega^l\, (E, E')$. Thus $(F, F') * (E, E')\, \omega\, (E, E')$.

Hence axiom B_{21}.

Axiom B_{22} — Assume that $(U, U')\, \omega^l\, (F, F')$ and $(F, F'), (U, U') \in \omega^r (E, E')$. Hence $F \subseteq U$, $F' \subseteq E'$ and $U' \subseteq E'$.

Now $(U, U') * (E, E') = ((U \cap E') + E, (U' + E) \cap E')$

$$= ((U \cap E') + E, U') \text{ since } (U' + E) \cap E' = U' + (E \cap E')$$

and

$$(F, F') * (E, E') = ((F \cap E') + E, (F' + E) \cap E')$$

$$= ((F \cap E') + E, F') \text{ since } (F' + E) \cap E' = F' + (E \cap E')$$

We have to prove that $(F \cap E') + E \subseteq (U \cap E') + E$ which is true, since

$$F \subseteq U \Rightarrow F \cap E' \subseteq U \cap E'$$

$$\Rightarrow (F \cap E') + E \subseteq (U \cap E') + E$$

Hence $(U, U') * (E, E') \; \omega^l \; (F, F') * (E, E')$. Thus axiom B_{22}.

Axion B_{31} — Assume that $(U, U') \; \omega^r \; (F, F') \; \omega^r \; (E, E')$. Then $U' \subseteq F' \subseteq E'$. Also,

$$(U, U') * (F, F') = ((U \cap F') + F, (U' + F) \cap F')$$

$$= ((U \cap F') + F, U') \text{ since } (U' + F) \cap F' = U' + (F \cap F')$$

and

$$(U, U') * (E, E') = ((U \cap E') + E, (U' + E) \cap E')$$

$$= ((U \cap E') + E, U') \text{ since } (U' + E) \cap E' = U' + (E \cap E')$$

$$= (G, U') \text{ where } G = (U \cap E') + E$$

Now

$$[(U, U') * (E, E')] * (F, F') = (G, U') * (F, F')$$

$$= ((G \cap F') + F, (F' + G) \cap U')$$

$$= ((G \cap F') + F, U') \text{ since } U' \subseteq F' + G.$$

Next, to prove that $(U, U') * (F, F') = [(U, U') * (E, E')] * (F, F')$ it is enough to prove that $(U \cap F') + F = (G \cap F') + F$, i.e. it is enough to prove that $U \cap F' = G \cap F'$. For, we see that

$$((U \cap E') + E) \cap E' = (U \cap E') + (E \cap E') = U \cap E'.$$

Hence $\quad ((U \cap E') + E) \cap E' \cap F' = U \cap E' \cap F'$

i.e. $\quad ((U \cap E') + E) \cap F' = U \cap F'$ since $F' \subseteq E'$

Thus $G \cap F' = U \cap F'$ since $G = (U \cap E') + E$

Hence the axiom B_{31}.

Axiom B_{32} — Assume that $(U, U') \; \omega^l \; (F, F')$ and $(U, U'), (F, F') \in \omega^r (E, E')$. Then $F \subseteq U$, $U' \subseteq E'$ and $F' \subseteq E'$.

Hence $(F, F') * (U, U') = ((F \cap U') + U, (U' + F) \cap F')$

$$= (U, (U' + F) \cap F') \text{ since } F \subseteq U, F \cap U' \subseteq U$$

and

$$[(F, F') * (U, U')] * (E, E') = (U, (U' + F) \cap F') * (E, E')$$

$$= ((U \cap E') + E, (E' + U) \cap ((U' + F) \cap F'))$$

$$= ((U \cap E') + E, (U' + F) \cap F') \quad \text{... (1)}$$

since

$$((U' + F) \cap F') \subseteq F' \subseteq E' + U$$

Also,

$$[(F, F') * (E, E')] * [(U, U') * (E, E')]$$

$$= ((F \cap E') + E, (E' + F) \cap F') * ((U \cap E') + E, (E' + U) \cap U')$$

$$= ((((F \cap E') + E) \cap U') + ((U \cap E') + E), (U' + ((F \cap E') + E)) \cap F')$$

$$= ((U \cap E') + E, (U' + ((F \cap E') + E)) \cap F') \quad \text{... (2)}$$

Therefore, to prove that (1) and (2) are equal we have to prove that

$$(U' + F) \cap F' = (U' + ((F \cap E') E)) \cap F'$$

For,

$$[U' + ((F \cap E') + E)] \cap E') = [(U' + (F \cap E')) + E] \cap E'$$

$$= [((U' + F) \cap E') + E] \cap E' \text{ by Proposition 1.4 since } U' \subseteq E'$$

$$= ((U' + F) \cap E') + (E \cap E') \text{ by Proposition 1.4 since}$$

$$(U' + F) \cap E' \subseteq E'. = (U' + F) \cap E'$$

Therefore,

$$(U' + ((F \cap E') + E)) \cap E' \cap F' = (U' + F) \cap E' \cap F'$$

i.e. $(U' + ((F \cap E') + E)) \cap F' = (U' + F) \cap F'$ since $F' \subseteq E'$.

Hence the axiom B_{32}.

Axiom B_4' — Assume that $(U, U'), (V, V') \in \omega^r(E, E')$ and $(U, U') * (E, E')\ \omega^l\ (V, V') * (E, E')$. Since $(U, U'), (V, V') \in \omega^r(E, E')$; $U' \subseteq E', V' \subseteq E'$. Also, since

$$(U, U') * (E, E') = ((U \cap E') + E, (E' + U) \cap U') = ((U \cap E') + E, U')$$

and

$$(V, V') * (E, E') = ((V \cap E') + E, (E' + V) \cap V') = ((V \cap E') + E, V')$$

and $(U, U') * (E, E')\ \omega^l\ (V, V') * (E, E')$; we have $(V \cap E') + E \subseteq (U \cap E') + E$. Since $U' \subseteq E'$, we have $V \cap U' \subseteq V \cap E'$. Also, we have $(V \cap E') + E \subseteq (U \cap E') + E$ which implies that $((V \cap E') + E) \cap E' \subseteq ((U \cap E') + E) \cap E'$ ... (a).

Now $((V \cap E') + E) \cap E' = (V \cap E') + (E \cap E')$ since $V \cap E' \subseteq E'$

$$= V \cap E'$$

and

$$((U \cap E') + E) \cap E' = (U \cap E') + (E \cap E') \text{ since } U \cap E' \subseteq E'$$

$$= U \cap E'.$$

Hence, (a) implies that $V \cap E' \subseteq U \cap E'$. Now by condition (iv),

$((U \cap V') + V, U') \in \tau$. Let $x = ((U \cap V') + V, U')$. Since $v \subseteq (U \cap V') + V$;

$x \omega^l (V, V')$ and since $U' \subseteq E'$; $x\ \omega^r (E, E')$. Further

$$x * (E, E') = (U \cap V') + V, U') * (E, E')$$

$$= ((((U \cap V') + V) \cap E') + E, (E' + ((U \cap V') + V)) \cap U')$$

$$= ((((U \cap V') + V) \cap E') + E, U')$$

since $U' \subseteq E'$

Now $(U, U') * (E, E') = ((U \cap E') + E, (E' + U) \cap U')$

$$= ((U \cap E') + E, U') \text{ since } U' \subseteq E'$$

Also, $((U \cap V' + V) \cap E' = (U \cap V') + (V \cap E')$ since $U \cap V' \subseteq V' \subseteq E'$

Again, $U \cap E' = H \cap (U \cap E')$

$$= (V' + V) \cap (U \cap E')$$

$$= ((V' + V) \cap E') \cap U$$

$$= (V' + (V \cap E')) \cap U \text{ since } V' \subseteq E'$$

$$= ((V \cap E') + V') \cap U$$

$$= (V \cap E') + (V' \cap U) \text{ since } V \cap E' \subseteq U \cap E' \subseteq U$$

$$= ((U \cap V') + V) \cap E'$$

Hence, $x * (E, E') = (U, U') * (E, E')$. Thus axiom B_4' holds.

Dual of Axiom B_4' — Assume that $(U, U'), (V, V') \in \omega^J (E, E')$ and $(E, E') * (U, U')\ \omega^r\ (E, E') * (V, V')$.
Hence $E \subseteq U, E \subseteq V$
and $(U' + E) \cap E' \subseteq (V' + E) \cap E'$.
Then

$$U' + E = (U' + E) \cap H$$

$$= (U' + E) \cap (E + E')$$

$$= (E + E') \cap (U' + E)$$

$$= E + (E' \cap (U' + E)) \text{ since } E \subseteq U' + E$$

$$\subseteq E + (E' \cap (V' + E)) \text{ since } (U' + E) \cap E' \subseteq (V' + E) \cap E'$$

$$= (E + E') \cap (V' + E) \text{ since } E \subseteq V' + E$$

$$= V' + E$$

Thus $U' + E \subseteq V' + E$. Hence by condition (v)

$(U, (U' + V) \cap V') \in \tau$. Let $x = (U, (U' + V) \cap V')$

Then $x\ \omega^r (V, V')$ and $x\ \omega^J (E, E')$ since $(U' + V) \cap V' \subseteq V'$ and $E \subseteq U$.

Further

$$(E, E') * x = (E, E') * (U, (U' + V) \cap V')$$

$$= ((E \cap (U' + V) \cap V') + U, (((U' + V) \cap V') + E) \cap E')$$

$$= (U, (((U' + V) \cap V') + E) \cap E') \text{ since } E \cap (U' + V) \cap V' \subseteq E \subseteq$$

and

$$(E, E') * (U, U') = ((E \cap U') + U, (U' + E) \cap E') = (U, (U' + E) \cap E')$$

Now

$$((U' + V) \cap V') + E = E + (V' \cap (U' + V))$$

$$= (E + V') \cap (U' + V) \text{ since } E \subseteq U' + V$$

$$= (V' + V) \cap (V' + E)$$

$$= U' + (V \cap (V' + E)) \text{ since } U' \subseteq U' + E \subseteq V' + E$$

$$= U' + ((E + V') \cap V)$$

$$= U' + (E + (V' \cap V)) \text{ since } E \subseteq V$$

$$= U' + E$$

Therefore, $((U' + V) \cap V' + E) \cap E' = (U' + E) \cap E'$. Hence $(E, E') * x = (E, E') * (U, U')$. Thus the dual of axiom B_4' holds. This completes the proof.

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