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ON l^p-MULTIPLIER CONVERGENCE OF FUNCTION SERIES

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A general uniform convergence result is established for the l^p-multiplier convergence of function series. As an application, this result improves and generalizes a recent result.

Key Words : l^p-Multiplier Convergence; Countable Compactness; Uniform Convergence

Let $\lambda \subseteq \mathbb{C}^{\mathbb{N}}$. A series Σx_j on a topological vector space X is said to be λ-multiplier convergent or, simply, λ*--mc* if for every $\{t_j\} \in \lambda$, the series $\sum_{j=1}^{\infty} t_j x_j$ converges in X. The c_0-mc, l^∞-mc and $\{0, 1\}^{\mathbb{N}}$-*mc* are very important for functional analysis and measure theory, e.g., a sequentially complete locally convex space X contains no copy of $(c_0, \|\cdot\|_\infty)$ if and only if the c_0-mc, l^∞-mc and $\{0, 1\}^{\mathbb{N}}$-mc are equivalent for series on X (Li and Bu[1]); if X is a sequentially complete locally convex space, then a measure $\mu : \Sigma \to X$ is bounded if and only if for every pairwise disjoint $\{A_j\} \subseteq \Sigma$, the series $\Sigma \mu(A_j)$ is $c_0 - mc$ (Li and Kang[2]). The l^p-*mc* ($p > 0$) is also meaningful because a locally convex space X is Banach-Mackey if and only if for every $\{t_j\} \in l^1$ and weak* bounded $\{x_j'\} \subseteq X'$, the pointwise sum $\sum_{j=1}^{\infty} t_j x_j'(\cdot)$ is bounded on bounded subsets of X (Li and Swartz[3]), and X contains no copy of $(c_0, \|\cdot\|_\infty)$ if and only if for every c_0-mc series $\Sigma\, x_j$ on X, the series $\sum_{j=1}^{\infty} t_j x_j$ converges uniformly for $\sum_{j=1}^{\infty} |t_j| \leq 1$ (Li and Bu[1]).

Recently, Li et al.[4] gave a nice result as follows :

Theorem — *(Li et al.[4], Theorem 3.1) Let X be a Hausdorff locally convex space with the dual X′. For a series* $\Sigma\, x_j$ *on X, the* l^p*-mc* $(p \geq 1)$ *and the* $c_0 - mc$ *are invariants with respect to* *all (X, X')-admissible topologies. More precisely, letting $\lambda = l^p$ $(p \geq 1)$ or c_0, if for every $\{t_j\} \in l^1$, the series $\sum_{j=1}^{\infty} t_j x_j$ converges in the weak topology $\sigma(X, X')$, then for every $\{t_j\} \in \lambda$, the series $\sum_{j=1}^{\infty} t_j x_j$ also converges under the strongest (X, X')-admissible topology $\beta(X, X')$.*

This result is interesting because it is the first example of nontrivial invariants on the family of all (X, X')-admissible topologies, though there are many invariants on some proper subfamily such as the family of (X, X')-compatible topologies (see Swartz[5], Dierolf[6], Swartz[7]). In this note, we would like to establish a very general invariant result for l^p-multiplier convergence $(p > 0)$ of function series.

***Theorem* 1** — *Let X be a topological vector space, E a nonempty set and $B(E, X) = \{f \in X^E : f(E)$ is bounded$\}$. If $p > 0$ and $\{f_j\}$ is a sequence in $B(E, X)$ such that for every $\{t_j\} \in l^p$ and $a \in E$, the series $\sum_{j=1}^{\infty} t_j f_j(a)$ converges and $\left\{ \sum_{j=1}^{\infty} t_j f_j(a) : a \in E \right\}$ is bounded, then for every $\{t_j\} \in l^p$, the series $\sum_{j=1}^{\infty} t_j f_j(a)$ converges uniformly for $a \in E$.*

PROOF : Suppose that $\{t_j\} \in l^p$ and the convergence of $\sum_{j=1}^{\infty} t_j f_j(a)$ is not uniform with respect to $a \in E$, i.e., there is a neighbourhood U of $0 \in X$ such that

(*) for every $n_0 \in \mathbb{N}$, there is $n > n_0$ and $a \in E$ for which $\sum_{j=n}^{\infty} t_j f_j(a) \notin U$.

Pick a neighbourhood V of $0 \in X$ with $V + V \subseteq U$. By (*), there is an integer $n_1 > 1$ and an $a_1 \in E$ such that $\sum_{j=n_1}^{\infty} t_j f_j(a_1) \notin U$ and hence $\sum_{j=n_1}^{m_1} t_j f_j(a_1) \notin V$ for some $m_1 > n_1$. Similarly, there are integers $m_2 > n_2 > m_1$ and an $a_2 \in E$ such that $\sum_{j=n_2}^{m_2} t_j f_j(a_2) \notin V$.

Continuing this construction inductively, we have an integer sequence $n_1 < m_1 < n_2 < m_2 < \ldots$ and $\{a_i\} \subseteq E$ such that

(**) $$\sum_{j=n_i}^{m_i} t_j f_j(a_i) \notin V, \quad i = 1, 2, 3, \ldots .$$

Clearly, $t_j \neq 0$ for infinitely many $j \in I\!N$. Letting $\gamma_i = \sum_{j=i+1}^{\infty} |t_j|^p$ for $i = 0, 1, 2, 3, \ldots$, $\gamma_i \neq 0$ for all i and $\gamma_i \to 0$. By a classical proposition due to Dini, we have

$$\sum_{j=1}^{\infty} \left| \frac{t_j}{\sqrt[2p]{\gamma_{j-1}}} \right|^p = \sum_{j=1}^{\infty} \frac{|t_j|^p}{\sqrt{\gamma_{j-1}}} < +\infty, \text{ i.e., } \left\{ \frac{t_j}{\sqrt[2p]{\gamma_{j-1}}} \right\}_{j=1}^{\infty} \in l^p.$$

Now we consider the matrix

$$\left[\sqrt[2p]{\gamma_{ni-1}} \sum_{j=n_k}^{m_k} \frac{t_j}{\sqrt[2p]{\gamma_{n_k-1}}} f_j(a_i) \right]_{i,k}.$$

Since $\sqrt[2p]{\gamma_{n_{i-1}}} \to 0$ as $i \to +\infty$ and $f_j(E)$ is bounded for each j,

$$\lim_i \sqrt[2p]{\gamma_{n_i-1}} \sum_{j=n_k}^{m_k} \frac{t_j}{\sqrt[2p]{\gamma_{n_k-1}}} f_j(a_i) = \sum_{j=n_k}^{m_k} \frac{t_j}{\sqrt[2p]{\gamma_{n_k-1}}} \lim_i \sqrt[2p]{\gamma_{n_i-1}} f_j(a_i)$$

for each k. Let $k_1 < k_2 < \ldots$ in $I\!N$. For each $j \in I\!N$ let

$$s_j = \begin{cases} 0, & \text{if } j < n_{k_1} \text{ or } m_{k_q} < n_{k_{q+1}} \text{ for some } q \in I\!N; \\ \dfrac{t_j}{\sqrt[2p]{\gamma_{n_{k_q}-1}}}, & \text{if } n_{k_q} \le j \le m_{k_q} \text{ for some } q \in I\!N, \end{cases}$$

then $s_j = 0$ or $|s_j| = \dfrac{|t_j|}{\sqrt[2p]{\gamma_{n_{k_q}-1}}} \le \dfrac{|t_j|}{\sqrt[2p]{\gamma_{j-1}}}$ for $n_{k_q} \le j \le m_{k_q}$ and hence $\{s_j\} \in l^p$.

By the hypothesis, for each i, the series

$$\sum_{q=1}^{\infty} \left[\sqrt[2p]{\gamma_{n_i-1}} \sum_{j=n_{k_q}}^{mk_q} \frac{t_j}{\sqrt[2p]{\gamma_{n_{k_q}-1}}} f_j(a_i) \right] = \sqrt[2p]{\gamma_{n_i-1}} \sum_{j=1}^{\infty} s_j f_j(a_i)$$

converges and $\left\{ \sum_{j=1}^{\infty} s_j f_j(a) : a \in E \right\}$ is bounded. Therefore,

$$\lim_i \sum_{q=1}^{\infty} \left[\sqrt[2p]{\gamma_{n_i-1}} \sum_{j=n_{k_q}}^{m_{k_q}} \frac{t_j}{\sqrt[2p]{\gamma_{n_{k_q}-1}}} f_j(a_i) \right] = \lim_i \sqrt[2p]{\gamma_{n_i-1}} \sum_{j=1}^{\infty} s_j f_j(a_i) = 0.$$

Now, by the Antosik-Mikusinski matrix theorem (Antosik and Swartz[8], Li and Swartz[9]),

$$\lim_i \sum_{j=n_i}^{m_i} t_j f_j(a_i) = \lim_i \sqrt[2p]{\gamma_{n_i-1}} \sum_{j=n_i}^{m_i} \frac{t_j}{\sqrt[2p]{\gamma_{n_i-1}}} f_j(a_i) = 0,$$

i.e.,

$$\sum_{j=n_i}^{m_i} t_j f_j(a_i) \in V$$

eventually. This contradicts (**).

As an application of Theorem 1, we have the following improvement of the l^p-argument in Theorem 3.1 of Li *et al.*[4]

Corollary 2 — Let X be a locally convex space with the dual X'. Then for every $p > 0$, the l^p-multiplier convergence of series on X is an invariant with respect to all (X, X')-admissible topologies, i.e., if $p > 0$ and Σx_j is a series on X such that the series $\sum_{j=1}^{\infty} t_j x_j$ converges weakly for every $\{t_j\} \in l^p$, then for every $\{t_j\} \in l^p$, the series $\sum_{j=1}^{\infty} t_j x_j$ converges in the strongest (X, X')-admissible topology $\beta(X, X')$.

PROOF : Suppose that $p > 0$ and $\sum_{j=1}^{\infty} t_j x_j$ converges weakly for every $\{t_j\} \in l^p$. Let E be a weak* bounded subset of X'. Letting $x_j(x') = x'(x_j)$ for $x' \in X'$ and $j \in \mathbb{N}, x_j \in B(E, \mathbb{C})$, i.e., $\{x_j(x') : x' \in E\} = \{x'(x_j) : x' \in E\}$ is bounded for each j. Let $\{t_j\} \in l^p$. Then, by the hypothesis, there is an $x_0 \in X$ such that $\sum_{j=1}^{\infty} t_j x_j(x') = \sum_{j=1}^{\infty} t_j x'(x_j) = x'(x_0)$ for all $x' \in X'$. Therefore, $\left\{\sum_{j=1}^{\infty} t_j x_j(x') : x' \in E\right\} = \{x'(x_0) : x' \in E\}$ is bounded and, by Theorem 1, the series $\sum_{j=1}^{\infty} t_j x'(x_j) = \sum_{j=1}^{\infty} t_j x_j(x')$ converges uniformly for $x' \in E$.

Let X be a locally convex space with the dual X'. X is said to be Banach-Mackey if $\sigma(X, X')$ and $\beta(X, X')$ have the same bounded sets. Barrelled spaces and sequentially complete locally convex spaces are Banach-Mackey spaces but the converse implications are false (Li and Swartz3, Wilansky10). A locally convex space X is Banach-Mackey if and only if for $\{x'_k\} \subseteq X'$ with $\lim_k x'_k(x) = x'(x)$ exists at each $x \in X$, the limit functional x' is bounded on each bounded subset of X (Li and Swartz3, Theorem 7).

Corollary 3 — Let X, Y be locally convex spaces and X Banach-Mackey. If $\{T_j\}$ is a sequence of continuous linear operators from X into Y such that for every $\{t_j\} \in l^p$ and $x \in X$, the series $\sum_{j=1}^{\infty} t_j T_j(x)$ converges, then for every $\{t_j\} \in l^p$ and bounded $B \subseteq X$, the series $\sum_{j=1}^{\infty} t_j T_j(x)$ converges uniformly for $x \in B$ and hence the restriction $\sum_{j=1}^{\infty} t_j T_j(\cdot)|_B$ is continuous.

PROOF : Let $\{t_j\} \in l^p$ and B a bounded subset of X. For every $y' \in Y'$,

$$\left\{\sum_{j=1}^{k} t_j (y' \circ T_j) : k \in \mathbb{N}\right\} \subseteq X' \text{ and } \lim_k \sum_{j=1}^{k} t_j (y' \circ T_j)(x) = y'\left(\sum_{j=1}^{\infty} t_j T_j(x)\right)$$

at each $x \in X$ and $\left\{ y'\left(\sum_{j=1}^{\infty} t_j T_j(x)\right) : x \in B\right\}$ is bounded by Theorem 7 of Li and Swartz[3]. By the Mackey Theorem, $\left\{\sum_{j=1}^{\infty} t_j T_j(x) : x \in B\right\}$ is bounded. Now Theorem 1 shows that $\sum_{j=1}^{\infty} t_j T_j(x)$ converges uniformly for $x \in B$.

If X is sequentially complete, then we can improve this result. A linear operator T from a topological vector space X into a topological vector space is said to be bounded if $T(B)$ is bounded for every bounded $B \subseteq X$, and T is sequentially continuous if $T(x_n) \to 0$ whenever $x_n \to 0$ in X. Continuous operators are sequentially continuous and sequentially continuous operators are bounded but, in general, the converse implications are not true.

Corollary 4 — Let X be a sequentially complete locally convex space and Y a topological vector space. If $\{T_j\}$ is a sequence of sequentially continuous Y-valued linear operators on X such that $\sum_{j=1}^{\infty} t_j T_j(x)$ converges for every $\{t_j\} \in l^p$ and $x \in X$, then for every $\{t_j\} \in l^p$, the series $\sum_{j=1}^{\infty} t_j T_j(x)$ converges uniformly on every bounded subset of X and $\sum_{j=1}^{\infty} t_j T_j(\cdot)$ is sequentially continuous.

PROOF : Let B be a bounded subset of X and $\{t_j\} \in l^p$. Since X is a sequentially complete locally convex space, B is $\mathcal{K}$-bounded (Li and Swartz[9], Swartz[11]). Clearly, for each $x \in X$, $\left\{\sum_{j=1}^{n} t_j T_j(x) : n \in \mathbb{N}\right\}$ is bounded because $\lim_n \sum_{j=1}^{n} t_j T_j(x) = \sum_{j=1}^{\infty} t_j T_j(x)$ exists and hence

$\left\{\sum_{j=1}^{n} t_j T_j : n \in I\!N\right\}$ is uniformly bounded on B, i.e., $\left\{\sum_{j=1}^{n} t_j T_j(x) : n \in I\!N, x \in B\right\}$ is bounded (Li and Swartz[9], Swartz[11]). Therefore, for every $\{t_j\} \in l^p$, $\left\{\sum_{j=1}^{\infty} t_j T_j(x) : x \in B\right\}$ is bounded and, by Theorem 1, $\sum_{j=1}^{\infty} t_j T_j(x)$ converge uniformly for $x \in B$.

Now let $\{t_j\} \in l^p$ and $x_n \to 0$ in X. Then $\{x_n\}$ is bounded and

$$\lim_n \sum_{j=1}^{\infty} t_j T_j(x_n) = \lim_k \lim_n \sum_{j=1}^{k} t_j T_j(x_n) = 0,$$

i.e., $\sum_{j=1}^{\infty} t_j T_j(\cdot)$ is sequentially continuous.

The Thomas theorem says that if Ω is a compact space and X is a Banach space, then for $\{f_j\} \subseteq C(\Omega, X)$, the following (A) and (B) are equivalent (see Thomas[12], Li and Swartz[13]).

(A) Σf_j is $\{0, 1\}^{I\!N}$-multiplier convergent in the topology of pointwise convergence on Ω, i.e., for every increasing sequence $\{j_k\} \subseteq I\!N$, and $\omega \in \Omega$, the series $\sum_{k=1}^{\infty} f_{jk}(\omega)$ converges and the pointwise sum $\sum_{k=1}^{\infty} f_{jk}(\cdot)$ is continuous.

(B) Σf_j is $\{0, 1\}^{I\!N}$-multiplier convergent in the topology of uniform convergence on Ω, i.e., for every increasing sequence $\{j_k\} \subseteq I\!N$, the series $\sum_{k=1}^{\infty} f_{jk}(\omega)$ converges uniformly on Ω and hence $\sum_{k=1}^{\infty} f_{jk}(\cdot)$ is continuous on Ω.

Recently, Li[14] has improved this result by weakening the compactness of Ω to the countable compactness of Ω (see also Cui[15]).

It is easy to show that the continuity of $\sum_{k=1}^{\infty} f_{jk}(\cdot)$ in (A) can not be substituted by the boundedness of $\sum_{k=1}^{\infty} f_{jk}(\cdot)$. However, we have the following.

Corollary 5 — Let Ω be a countably compact space and X a topological vector space. Then for $\{f_j\} \subseteq C(\Omega, X)$ and $p > 0$, the following (I) and (II) are equivalent.

(I) For every $\{t_j\} \in l^p$ and $\omega \in \Omega$, the series $\sum_{j=1}^{\infty} t_j f_j(\omega)$ converges and $\left\{\sum_{j=1}^{\infty} t_j f_j(\omega) : \omega \in \Omega\right\}$ is bounded.

(II) For every $\{t_j\} \in l^p$, the series $\sum_{j=1}^{\infty} t_j f_j(\omega)$ converges uniformly for $\omega \in \Omega$ and hence $\sum_{j=1}^{\infty} t_j f_j(\cdot) : \Omega \to X$ is continuous.

PROOF : If $f(\Omega)$ is bounded for each $f \in C(\Omega, X)$, then the desired follows from Theorem 1 immediately. We need to show that $C(\Omega, X) \subseteq B(\Omega, X)$.

Let $f \in C(\Omega, X)$. Then $f(\Omega)$ is countably compact. Suppose that the vector topology on X is generated by a family P of paranorms on X (see Wilansky[10], p. 55, Prob. 111). Then $f(\Omega)$ is countably compact in $(X, \sigma p)$ for each $p \in P$. Since $(X, \sigma p)$ is first countable (Wilansky[10], p.52), $f(\Omega)$ is also sequentially compact in $(X, \sigma p)$ (Wilansky[16], p. 124) and hence $f(\Omega)$ is bounded in $(X, \sigma p)$ for each $p \in P$. Therefore, $f(\Omega)$ is bounded in X with the original topology $V\{\sigma p : p \in P\}$ (Wilansky[10], p. 48).

We would like to restate Corollary 5 in the following different way.

Corollary 6 — Let Ω be a countably compact space and X a topological vector space. Suppose that $p > 0$ and $\{f_j\} \subseteq C(\Omega, X)$ such that for every $\{t_j\} \in l^p$ and $\omega \in \Omega$, the series $\sum_{j=1}^{\infty} t_j f_j(\omega)$ converges. Then for every $\{t_j\} \in l^p$, the function $\sum_{j=1}^{\infty} t_j f_j(\cdot)$ is continuous if and only if for every $\{t_j\} \in l^p$, the function $\sum_{j=1}^{\infty} t_j f_j(\cdot)$ is bounded on Ω.

Note that in the case of $\{0, 1\}^{\mathbb{N}}$-multiplier convergence, as was stated above, the similar equivalance of continuity and boundedness is not true.

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