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ON A STRUCTURE DEFINED BY A TENSOR FIELD OF TYPE (1, 1) SATISFYING $P^3 - P = 0$

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In this paper we have introduced a P-structure on a differentiable manifold M of dimension n that is a tensor field of type (1, 1) satisfying $P^3 - P = 0$ on M. Further if there exists a P-structure of rank $(n - 1)$, then it is shown that there exists a Riemannian metric g such that

$$g(PX, PY) = g(X, Y) + \eta^x (X)\, \eta^y (Y) \text{ and } g(PX, Y) = g(X, PY),$$

where η^x and η^y, $(x, y = r + 1, r = 2, \ldots\ldots n)$ are the linearly independent covariant vectors (that is 1-forms) corresponding to the distribution D_T of dimension $n - r$ with respeect to the operator T and r is the dimension of the distribution D_L (orthogonal to D_T) with respect to the operator L. It is also shown that CR-submanifolds of a Lorentzian Para-Sasakian manifold carries the P-structure.

Key Words : Lorentzian-para Contact Manifolds; LP-Sasakian Manifolds; f-Structure

1. INTRODUCTION

Let M be a differentiable manifold of dimension n. If I denotes the identity tensor field of type (1,1), then we must have

$$I^2 = I \qquad \ldots (1.1)$$

so that from (1.1), we have

$$I^3 - I = 0 \qquad \ldots (1.2)$$

On the other hand if φ is the tensor field of type (1,1), ξ is the characteristic vector field on M and η is the contact one form on M, then almost Lorentzian para contract structure (φ, ξ, η) satisfies the following relations [1], [3]:

$$\varphi^2 X = X + \eta(X)\xi,\ \eta(\xi) = -1,\ \varphi(\xi) = 0,\ \eta(\varphi X) = 0 \text{ and rank } (\varphi) = n - 1, \qquad \ldots (1.3)$$

where X is any vector field on M. Thus from (1.3), we must have

$$\varphi^3 - \varphi = 0. \qquad \ldots (1.4)$$

We write in brief almost LP-contact structure for Lorentzian para contact structure. It is well known that φ is symmetric, that is

$$\varphi(X, Y) = \varphi(Y, X),$$

where

$$\varphi(X, Y) = g(\varphi X, Y).$$

It is well known that CR- submanifold of a LP-Sasakian manifold carries the following structure, [1] (for CR-submanifolds see Kobayashi [2]) that is if R and Q are the projectors corresponding to the horizontal and vertical distributions, then

$$R^2 - R = 0 \text{ and } Q^2 - Q = 0.$$

Further if we choose $\Psi = \varphi R$ and $\Psi = \varphi Q$, then

$$\Psi^3 - \Psi = 0. \qquad \ldots (1.5)$$

The generalised almost contact metric hyperbolic structure (F, T^p, A_p, G) on M^{n+r}, where F, T^p, A_p, G are as given in [6], carries the structure given by

$$F^3 - F = 0. \qquad \ldots (1.5a)$$

After observing (1.2), (1.4), and (1.5)a, in this paper we have introduced the general structure in which there is a non zero tensor field P of type (1, 1) such that

$$P^3 - P = 0. \qquad \ldots (1.6)$$

If the rank of P is constant say r, then we call such a structure a *P-structure* of rank r. Further, we have shown in this paper that the P-structure of rank $(n - 1)$ is equivalent to the almost LP-contact structure (φ, ξ, η). We have shown that manifold M with P-structure admits a Riemannian metric g such that

$$g(PX, PY) = g(X, Y) + \eta^x(X)\,\eta^y(Y) \text{ and } P(X, Y) = P(Y, X)$$

and hence the rank of P must be odd where X and Y are any vector fields on M such that

$$g(PX, Y) = P(X, Y)$$

and η^x and η^y $(x, y = r + 1, r + 2, \ldots\ldots, n)$ are linearly independent covariant vectors, (that is 1-forms) in the distribution D_T of dimension $n - r$ orthogonal to the distribution D_L of dimension r (see in the next section for D_T and D_L).

2. THEOREMS

Calculation techniques used in this section are similar to those of K. Yano [4] but the philosophy is different. Suppose there exists a tensor field P of type (1, 1) satisfying (1.6). We define the operators L and T respectively by

$$L = -P^2 \text{ and } T = P^2 - I \qquad \ldots (2.1)$$

so that $L + T = -I.$

Observing (2.2), essentially, we have two types of distributions corresponding to L and T. We denote these two distributions by D_L and D_T corresponding to L and T respectively. Suppose the rank of P is constant say equal to r. Then the dimensions of D_L and D_T are r and $(n - r)$ respectively and we call such structure the P-structure of rank r satisfying $P^3 - P = 0$.

We derive the following relations for P-structure which are used later in this section.

$$PL = LP = -P, P^2L = L \qquad \text{... (2.3)}$$

and

$$PT = TP = 0, P^2T = 0. \qquad \text{... (2.4)}$$

In order to prove these relations consider,

$$PL = P(-P^2) = -P^3 = -P, LP = (-P^2)P = -P^3 = -P$$

$$\therefore PL = P = -P$$

$$P^2 L = P^2(-P^2) - P^4 = -P^2 = L.$$

This completes the proof of (2.3). For the proof of relation (2.4), consider

$$PT = P(P^2 - I) = P^3 - P = P - P = 0$$

and

$$TP = (P^2 - I)P = P^3 - P = P - P = 0, \therefore PT = TP = 0.$$

Further, $P^2 T = P^2 (P^2 - I) = P^4 - P^2 = P^2 - P^2 = 0$.

Remark : Relation (2.4) shows that P acts on D_T as a null operator.

If the rank of P is n, then the dimension of D_L is n and that of D_T is zero and hence $T = 0$ and $L = -1$, so that P satisfies

$$P^2 = I. \qquad \text{... (2.5)}$$

Thus P-structure of rank n satisfies (2.5) and n must be even.

On the other hand, if the rank of P is $(n - 1)$, the dimension of D_L is $(n - 1)$ and that of D_T is one. We can take T to be

$$TX = \omega(X)u,$$

where X is any vector field on M and ω and u are the covariant and contravariant vector fields on M respectively. Thus from relation (2.2) and (2.4), we have

$$P^2X = X + \omega(X)u, P(u) = 0, \omega(PX) = 0, \omega(u) = -1 \qquad \text{... (2.6)}$$

for any vector field X on M. Hence (2.6) is equivalent to (1.3). Thus we have the following Theorem.

Theorem 2.1 — *If, in a differentiable manifold M of dimension n, there exists a P-structure of rank (n − 1), then it is equivalent to the almost Lorentzian para contact structure* (P, u, ω).

Next we show that the P-structure satisfying (1.5) admits a Riemannian metric on M. Consider the mutually orthogonal unit vectors in D_L and D_T. Let e^i_X, for $X, Y, Z, \ldots = 1, 2, 3, \ldots, r$ be the mutually orthogonal unit vectors in D_L and e^i_x for $x, y, z, \ldots = r + 1, r = 2, \ldots, n$ be the mutually orthogonal unit vectors in D_T. Let the local co-ordinates of P and T respectively be P^i_j and T^i_j. Details of the calculations for the following results may be seen in paper by Yano[4&5]. Since $PT = 0$ or in local co-ordinates

$$P^i_j T^j_k = 0,$$

multiplying by e^k_x we find

$$P^i_j T^j_k e^k_x = 0 \Rightarrow P^i_j (T^j_k e^k_x) = 0 \Rightarrow P^i_j e^j_x = 0.$$

Thus we have

$$P^i_j e^j_x = 0.$$

We denote by $\{\eta^X_h, \eta^x_h\}$ the matrix inverse to $\{e^i_X, e^i_x\}$, then η^X_h and η^x_h are both components of linearly independent covariant vectors. Similarly, as $TP = 0$, we have

$$P^i_j \eta^x_i = 0.$$

Define a tensor field of type (0, 2) by

$$h_{ji} = \eta^X_j \eta^X_i + \eta^x_j \eta^x_i,$$

where the repeated index X or x do not represent the summation here after in this paper. Then h_{ji} is well defined and it is a Riemanniam metric on M such that

$$\eta^X_j = h_{ji} e^i_X \text{ and } \eta^x_j = h_{ji} e^i_x.$$

Next we set

$$L_{ji} = L^t_j h_{ti} \text{ and } T_{ji} = T^t_j h_{ti}.$$

It may be shown that

$$L_{ji} = e^X_j e^X_i \text{ and } T_{ji} = \eta^x_j \eta^x_i.$$

Then we can show that

$$h_{ji} = L_{ji} + T_{ji}$$

from which the following relations may be verified[4].

$$L_j^t L_i^s h_{ts} = L_{ji}, \; L_j^t T_i^s h_{ts} = 0 \text{ and } T_i^t T_i^s h_{ts} = T_{ji}.$$

Now define g by

$$g_{ji} = \frac{1}{2} [h_{ji} + P_j^t P_i^s h_{ts} + \overset{x}{\eta_j} \overset{x}{\eta_i}]$$

Then g is Riemannian metric on M such that

$$\overset{x}{\eta_j} = g_{ji} \overset{i}{e_x}, \; T_{ji} = T_j^t g_{ti}.$$

Finally it is easy to see that

$$g_{ts} P_j^t P_i^s = g_{ji} + \overset{x}{\eta_j} \overset{x}{\eta_i} . \qquad \text{... (2.7)}$$

We shall next show that P is symmetric tensor. From (2.1)

$$P^2 - T = I$$

that is

$$P_j^t P_t^i - T_j^i = \delta_j^i \text{ or } P_j^t P_{ti} - T_{ji} = g_{ji}. \qquad \text{... (2.8)}$$

But from (2.7), we have

$$P_j^t P_{it} - T_{ji} = g_{ji}. \qquad \text{... (2.9)}$$

From (2.8) and (2.9), we find

$$P_j^t (P_{ti} - P_{it}) = 0.$$

Since $\quad | P_j^t | \neq 0,$

$$P_{ji} - P_{ij} = 0$$

which proves that P_{ji} is symmetric and hence the rank of P must be odd. Thus we have the following Theorem.

Theorem 2.2 — *If, in a differentiable manifold of dimension n, there exists a P-structure of rank r, then M admits a Riemannian metric g such that (1.7) holds and hence the rank of P must be odd.*

As discussed in the introduction, it is important to note that every CR- submanifold of a Lorentzian- para Sasakian manifold carries the P-structure. In fact, if

$$\varphi X = RX + FX \text{ and } \varphi N = TN + QN, \qquad \text{... (2.10)}$$

where RX and FX are the tangential and normal parts of φX and TN and QN are the tangential and normal parts of φX respectively, and X and N are the tangent and normal vectors to M, then, multiplying both sides of (2.10) by φ and using (1.3), it follows that

$$R^3 - R = 0 \text{ and } Q^3 - Q = 0,$$

where R is an endomorphism of the tangent bundle $T(M)$. In particular, second equation above gives the P-structure on the normal bundle of M.

If M satisfies the P-structure, then we have

$$P^4 = P^2 = 0$$

which is the structure defined by Yano, Houh and Chen[7].

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