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CREEP TRANSITION IN A THIN ROTATING DISC HAVING VARIABLE THICKNESS AND VARIABLE DENSITY

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Creep stresses and strain rates have been obtained for a thin rotating disc having variable thickness and variable density by using Seth's transition theory. Results have been discussed and depicted graphically. It has been seen that a rotating disc whose density and thickness ratio decreases radially is on the safer side of the design in comparison to a flat disc having variable density.

Key Words : Creep; Disc; Thickness; Density; Angular Speed

1. INTRODUCTION

Rotating discs form an essential part of the design of rotating machinery namely turbines, rotors, compressors, flywheel and computer's disc drives etc. The use of rotating disc in machine and structural applications has generated a considerable interest in many problems in domain of solid mechanics. Solutions for thin isotropic discs can be found in most of the standard elasticity, plasticity and creep text books[6,8-10]. Reddy and Srinath[11] and Chang[1] investigated the influence of material density on the stresses and displacements of a rotating discs made of orthotropic materials. it has been shown that the existence of density gradient in a rotating disc influences the stresses and displacements significantly. Wahl[15] has obtained creep stresses in a rotating disc by assuming small deformation, incompressibility condition, Tresca's yield condition, a power strain law and its associated flow rule. Seth's transition theory[12] does not require these assumptions and thus poses and solves a more general problem from which cases pertaining to the above assumption can be worked out. This theory utilizes the concept of generalized strain measure and asymptotic solution at the critical points or the turning points of the differential equations defining the deformed field and has been successfully applied to a large number of problems[2-5,12&13].

Seth[13] has defined the generalized strain measure as

$$e_{ii} = \int_0^{\overset{A}{e}_{ii}} \left[1 - 2\overset{A}{e}_{ii}\right]^{\frac{n}{2}-1} d\overset{A}{e}_{ii} = \frac{1}{n}\left[1 - 2\overset{A}{e}_{ii}\right]^{\frac{n}{2}} \quad (i = 1, 2, 3), \qquad \ldots (1.1)$$

where n is the measure and $\overset{A}{e}_{ii}$ is the principal finite strain components.

In this paper, we investigate the creep stresses in a thin rotating disc of variable thickness and variable density by using Seth's transition theory. The thickness and density are assumed to vary along the radius in the form

$$h = h_0\left(\frac{r}{b}\right)^{-k} ; \ \rho = \rho_0\left(\frac{r}{b}\right)^{-m}, \qquad \text{... (1.2)}$$

where h_0, ρ_0 are the thickness and density at $r = b$ respectively and k (> 0), m are thickness and density parameters respectively. Results have been discussed numerically and depicted graphically.

2. Governing Equations

Consider a thin rotating disc of variable thickness and variable density with a central bore of internal radius a and external radius b. The disc is rotating with an angular velocity ω of gradually increasing magnitude about an axes perpendicular to its plane and passing through the center. The disc is so thin that it is effectively in a state of plane stress ($T_{22} = 0$) and the variation of the thickness is radial and symmetric with respect to the mid plane. The displacement components in cylindrical polar co-ordinates[13] are

$$u = r(1-\beta) ; \ v = 0 \text{ and } w = dz, \qquad \text{... (2.1)}$$

where β is a function of $r = \sqrt{x^2 + y^2}$ and d is a constant.

The finite components of strain are[13]

$$\overset{A}{e}_{rr} = \frac{1}{2}[1-(r\beta' + \beta)^2],$$

$$\overset{A}{e}_{\theta\theta} = \frac{1}{2}[1-\beta^2], \qquad \text{... (2.2)}$$

$$\overset{A}{e}_{zz} = \frac{1}{2}[1-(1-d)^2]$$

and $$\overset{A}{e}_{r\theta} = \overset{A}{e}_{\theta z} = \overset{A}{e}_{zr} = 0.$$

Substituting eqs. (2.2) in (1.1), the generalized components of strain are

$$e_{rr} = \frac{1}{n}[1-(r\beta' + \beta)^n],$$

$$e_{\theta\theta} = \frac{1}{n}[1-\beta^n], \qquad \text{... (2.3)}$$

$$e_{zz} = \frac{1}{n}[1-(1-d)^n]$$

and $$e_{r\theta} = e_{\theta z} = e_{zr} = 0.$$

where n is the measure and $\beta' = \dfrac{d\beta}{dr}$

The stress-strain relations[14] are

$$T_{ij} = \lambda \delta_{ij} e_{ii} + 2\mu e_{ij} \quad (i, j = 1, 2, 3), \qquad \text{... (2.4)}$$

where T_{ij} and e_{ij} are the stress, strain tensors respectively, λ and μ are the Lame's constants and δ_{ij} is the kronecker's delta.

Eq. (2.4) for this problem become

$$T_{rr} = \frac{2\lambda\mu}{(\lambda + 2\mu)} [e_{rr} + e_{\theta\theta}] + 2\mu e_{rr},$$

$$T_{\theta\theta} = \frac{2\lambda\mu}{(\lambda + 2\mu)} [e_{rr} + e_{\theta\theta}] + 2\mu e_{\theta\theta} \qquad \text{... (2.5)}$$

and $$T_{zz} = T_{zr} = T_{r\theta} = T_{\theta z} = 0$$

Substituting eq. (2.3) in equations (2.5), the non-zero stress components are,

$$T_{rr} = \frac{2\mu}{n} [3 - 2C - \beta^n \{(1 - C) + (2 - C)(P + 1)^n\}],$$

$$T_{\theta\theta} = \frac{2\mu}{n} [3 - 2C - \beta^n \{(2 - C) + (1 - C)(P + 1)^n\}] \qquad \text{... (2.6)}$$

and $$T_{zz} = T_{zr} = T_{r\theta} = T_{\theta r} = 0$$

where $r\beta' = \beta P$ and $C = \dfrac{2\mu}{(\lambda + 2\mu)}$

The equation of equilibrium is

$$\frac{d}{dr}(hr\, T_{rr}) - hT_{\theta\theta} + \rho\omega^2 r^2 h = 0, \qquad \text{... (2.7)}$$

where ρ is the density of the material.

Using eq. (2.6) in equation (2.7), we get a non-linear differential equation in β as

$$(2 - C)\, nP\beta^{n+1} \frac{dP}{d\beta} = \left(\frac{rh'}{h}\right)[3 - 2C - \beta^n \{(1 - C) + (2 - C)(P + 1)^n\}] +$$

$$+ \beta^n [1 - (P + 1)^n - nP \;\{(1 - C) + (2 - C)(P + 1)^n\}] + \frac{n\rho\,\omega^2 r^2}{2\mu} \qquad \text{... (2.8)}$$

The critical points of β in eq. (2.8) are $P \to -1$ and $P \to \pm\infty$.

The boundary conditions are

$$T_{rr} = 0 \text{ at } r = a \text{ and at } r = b. \qquad \text{... (2.9)}$$

3. Solution Through the Principal Stress Difference

It has been shown[2-5,12&13] that the asymptotic solution through the principal stress difference at the transition point $P \to -1$ leads to the creep state. We define the transition function R as,

$$R = T_{rr} - T_{\theta\theta} = \left(\frac{2\mu\beta^n}{n}\right)[1-(P+1)^n]. \qquad \text{... (3.1)}$$

Taking the logarithmic differentiation of eq. (3.1) w.r.t.r, we get

$$\frac{d}{dr}(\log R) = \frac{nP}{r} - \frac{\left[nP\beta(P+1)^{n-1}\beta^n \frac{dP}{d\beta}\right]}{r\beta^n[1-(P+1)^n]}. \qquad \text{... (3.2)}$$

Substituting the value of $\frac{dP}{d\beta}$ from eq. (2.8) in eq. (3.2) and taking asymptotic value $P \to -1$, we get

$$\frac{d}{dr}(\log R) = -\left[\frac{n(3-2C)+1}{r(2-C)}\right] + \frac{h'}{h}\left(\frac{1-C}{2-C}\right) - \frac{r^n}{D^n(2-C)}\left[\frac{h'}{h}(3-2C) + \frac{n\rho r^2\omega^2}{2\mu}\right] \qquad \text{... (3.3)}$$

Asymptotic value of β as $P \to -1$ is D/r, D being a constant.

Integrating eq. (3.3) w.r.t. r, we get

$$R = T_{rr} - T_{\theta\theta} = Ar^d h^\nu \exp f, \qquad \text{... (3.4)}$$

where $d = -\left[\frac{n(3-2C)+1}{(2-C)}\right]$ and $\nu = \left(\frac{1-C}{2-C}\right)$ is the Poisson's ratio, A is a constant of integration

and $f = -\frac{1}{(2-C)D^n}\int\left[(3-2C)\frac{h'}{h} + \frac{n\rho r\omega^2}{2\mu}\right]r^n\,dr.$

Using eq. (3.4) in eq. (2.7), we get

$$hT_{rr} = B - A\int F dr - \omega^2\int \rho rh dr, \qquad \text{... (3.5)}$$

where B is a constant of integration and $F = r^{d-1}h^{\nu+1}\exp f$.

Using boundary conditions (2.9) in eq. (3.5), we get

$$A = -\frac{\left\{\omega^2\int_a^b \rho rh dr\right\}}{\int_a^b F dr} \quad \text{and} \quad B = \omega^2[\rho rh dr]_{r=a} + A[F dr]_{r=a}$$

Now substituting value of A and B in eq. (3.5), we get

$$T_{rr} = \frac{\left\{\omega^2 \int_a^b \rho\, r dr\right\}}{h \int_a^b F dr} \int_a^r F dr - \frac{\omega^2}{h} \int_a^b \rho\, h r dr \quad \text{... (3.6)}$$

From eq. (3.4) and (3.6), we have

$$T_{\theta\theta} = T_{rr} + \frac{\left\{\omega^2 \int_a^b \rho\, r h dr\right\}}{\int_a^b F dr} \left(\frac{rF}{h}\right) \quad \text{... (3.7)}$$

Eqs. (3.6)-(3.7) give creep stresses for a thin rotating disc having variable thickness and variable density.

Now we introduce the following non-dimensional quantities

$$R = \frac{r}{b},\ R_0 = \frac{a}{b},\ \Omega^2 = \frac{\rho_0\ \omega^2 b^2}{E},\ \sigma_r = \frac{T_{rr}}{E},\ \sigma_\theta = \frac{T_{\theta\theta}}{E}.$$

Using disc profile given by eq. (1.2) in eq. (3.6) and (3.7) we get the transitional stresses in non-dimensional form as

$$\sigma_r = A_1 \int_{R_0}^{R} bF_1 dR - \Omega^2 R^k \left[\frac{R^{2-k-m} - R_0^{2-k-m}}{2-k-m}\right];\ 2-k-m \neq 0 \quad \text{... (3.8)}$$

and
$$\sigma_\theta = \sigma_r + A_1 R^{d-k(v+1)} b^d \exp f_1, \quad \text{... (3.9)}$$

where

$$A_1 = \frac{\left\{\Omega^2 \left[\frac{1 - R_0^{2-k-m}}{2-k-m}\right]\right\} R^k}{\int_{R_0}^{1} bF_1 dR};\ F_1 = R^{d-1-k(v+1)} b^{d-1} \exp f_1$$

and
$$f_1 = \frac{kb^n (3-2C) R^n}{n(2-C) D^n} - \frac{n(3-2C)\, \Omega^2 b^n R^{2+n-m}}{(2-C)^2 (2+n-m) D^n};\ 2+n-m \neq 0.$$

Eqs. (3.8)-(3.9) for $(2 - k - m = 0)$ become

$$\sigma_r = A_2 \int_{R_0}^{R} bF_2 dR - \Omega^2 R^k \log\left(\frac{R}{R_0}\right) \qquad \text{... (3.10)}$$

and

$$\sigma_\theta = \sigma_r + A_2 R^{d-k(\nu+1)} b^d \exp f_2, \qquad \text{... (3.11)}$$

where

$$A_2 = -\frac{\Omega^2 \log R_0}{\int_{R_0}^{1} bF_2 \, dR} R^k; \; F_2 = R^{d-1-k(\nu+1)} b^{d-1} \exp f_2$$

and

$$f_2 = \frac{kb^n (3-2C) R^n}{n(2-C) D^n} - \frac{n(3-2C)\Omega^2 b^n R^{2+n-m}}{(2-C)^2 D^n (2+n-m)}.$$

For a disc made of incompressible material $C - .0$ ($\nu \to 0.5$ or $\lambda \to \infty$) [2-5]; stresses (3.8)-(3.9) become

$$\sigma_r = A_3 \int_{R_0}^{R} bF_3 dR - \Omega^2 R^k \left[\frac{R^{2-k-m} - R_0^{2-k-m}}{2\ k-m}\right]; \; 2-k-m \neq 0 \qquad \text{... (3.12)}$$

and

$$\sigma_\theta = \sigma_r + A_3 R^{-0.5[3(n+k)+1]} b^{-0.5(3n+1)} \exp f_3, \qquad \text{... (3.13)}$$

where

$$A_3 = \frac{\left\{\Omega^2 \left[\frac{1-R_0^{2-k-m}}{2-k-m\cdot}\right]\right\} R^k}{\int_{R_0}^{1} bF_3 dR}; \; F_3 = R^{-15(n+k+1)} b^{-15(n+1)} \exp f_3$$

and

$$f_3 = \frac{3kb^n R^n}{2nD^n} - \frac{3n\Omega^2 b^n R^{2+n-m}}{4(2+n-m) D^n}.$$

For $(2-k-m=0)$; stresses given by eqs. (3.12) and (3.13) become

$$\sigma_r = A_4 \int_{R_0}^{R} - \Omega^2 R^k \log\left(\frac{R}{R_0}\right) \qquad \text{... (3.14)}$$

$$\sigma_\theta = \sigma_r + A_4 R^{-0.5(3(n+k)+1)} b^{-0.5(3n+1)} \exp f_4 , \quad \ldots (3.15)$$

where

$$A_4 = -\frac{\Omega^2 \log R_0}{\int_{R_0}^{1} bF_4 dR} R^k ; \; F_4 = R^{-1.5(n+k+1)} b^{-1.5(n+1)} \exp f_4$$

and

$$f_4 = \frac{3kb^n R^n}{2nD^n} - \frac{3n\Omega^2 b^n R^{2+n-m}}{4D^n (2+n-m)} .$$

Particular Case

Case 1 — For a disc having density ($m = 0$) and variable thickness, stresses given by equations (3.12) and (3.13) become

$$\sigma_r = A_5 \int_{R_0}^{R} bF_5 dR - \Omega^2 R^k \left[R^{2-k} - \frac{R_0^{2-k}}{2-k} \right] ; \; 2-k \neq 0 \quad \ldots (3.16)$$

and $$\sigma_\theta = \sigma_r + A_5 R^{-0.5(3(n+k)+1)} b^{-0.5(3n+1)} \exp f_5 , \quad \ldots (3.17)$$

where

$$A_5 = \frac{\left\{ \Omega^2 \left[\frac{1 - R_0^{2-k}}{2-k} \right] \right\} R^k}{\int_{R_0}^{1} bF_5 dR} ; \; F_5 = R^{-1.5(n=k+1)} b^{-1.5(n+1)} \exp f_5$$

and

$$f_5 = \frac{3kb^n R^n}{2nD^n} - \frac{3n\Omega^2 b^n R^{2+n}}{4(2+n) D^n} .$$

For ($2 - k = 0$); stresses given by equations (3.14)-(3.15) become

$$\sigma_r = A_6 \int_{R_0}^{R} bF_6 dR - \Omega^2 R^k \log \left(\frac{R}{R_0} \right) \quad \ldots (3.18)$$

and $$\sigma_\theta = \sigma_r + A_6 R^{-0.5(3(n+k)+1)} b^{-0.5(3n+1)} \exp f_6 , \quad \ldots (3.19)$$

where

$$A_6 = -\frac{\Omega^2 \log R_0}{\int_{R_0}^{1} bF_6 dR} R^k; \; F_6 = R^{-1.5(n+k+1)} b^{-1.5(n+1)} \exp f_6$$

and

$$f_6 = \frac{3kb^n R^n}{2nD^n} - \frac{3n\Omega^2 b^n R^{2+n}}{4D^n(2+n)}.$$

Case 2 — For a disc having variable density and constant thickness ($k = 0$), stresses (3.12) and (3.13) become

$$\sigma_r = A_7 \int_{R_0}^{R} bF_7 dR - \Omega^2 \left[\frac{R^{2-m} - R_0^{2-m}}{2-m} \right]; \; 2 - m \neq 0 \qquad \text{... (3.20)}$$

and $$\sigma_\theta = \sigma_r + A_7 R^{-0.5(3n+1)} b^{-0.5(3n+1)} \exp f_7, \qquad \text{... (3.21)}$$

where

$$A_7 = \frac{\left\{ \Omega^2 \left[\frac{1 - R_0^{2-m}}{2-m} \right] \right\}}{\int_{R_0}^{1} bF_7 dR}; \; F_7 = R^{-1.5(n+1)} b^{-1.5(n+1)} \exp f_7$$

and

$$f_7 = -\frac{3n\,\Omega^2 b^n R^{2+n-m}}{4(2+n-m) D^n}.$$

For $(2 - m = 0)$; stresses given by equations (3.14)-(3.15) become

$$\sigma_r = A_8 \int_{R_0}^{R} bF_8 dR - \Omega^2 \log\left(\frac{R}{R_0}\right) \qquad \text{... (3.22)}$$

and $$\sigma_\theta = \sigma_r + A_8 R^{-0.5(3n+1)} b^{-0.5(3n+1)} \exp f_8, \qquad \text{... (3.23)}$$

where $$A_8 = -\frac{\Omega^2 \log R_0}{\int_{R_0}^{1} bF_8 dR}; \; F_8 = R^{-1.5(n+1)} b^{-1.5(n+1)} \exp f_8$$

and

$$f_8 = -\frac{3n\Omega^2 b^n R^{2+n-m}}{4D^n (2+n-m)}.$$

The creep strain rates are calculated as follows :

When the creep sets in, the strain should be replaced by strain-rates and the stress-strain relations are[10]

$$e_{ij} = \frac{3}{2} \lambda_1 S_{ij} \ (i, j = 1, 2, 3), \qquad \text{... (3.24)}$$

where e_{ij} is the strain rate tensor with respect to flow parameter t and S_{ij} is the stress-deviator tensor.

Differentiating eq. (3.1) w.r.t. t, we get

$$\dot{e}_{\theta\theta} = -\beta^{n-1} \beta \qquad \text{... (3.25)}$$

for SWAINGER measure (i.e. n = 1), equation (3.23) becomes

$$\dot{\varepsilon}_{\theta\theta} = -\dot{\beta}, \qquad \text{... (3.26)}$$

where $\varepsilon_{\theta\theta}$ is the SWAINGER strain measure.

From eq. (3.1) the transitional value of β is

$$\beta = \left(\frac{n}{2\mu}\right)^{\frac{1}{n}} [T_{rr} - T_{\theta\theta}]^{\frac{1}{n}}. \qquad \text{... (3.27)}$$

Using eqs. (3.25), (3.26) and (3.27) in equation (3.24), we get

$$\dot{\varepsilon}_{\theta\theta} = \kappa \left[\frac{3n(\sigma_r - \sigma_\theta)}{2}\right]^{\frac{1}{n}-1} \left[\frac{S_{ij}}{E}\right], \qquad \text{... (3.28)}$$

where $\kappa = \frac{3}{2} \lambda_1 E$ is a constant and λ_1 has the same dimensions as that of $\frac{1}{E}$.

Since the form in eq. (3.28) is to be valid, we must have

$$\dot{\varepsilon}_{ij} = \kappa \left[\frac{3n(\sigma_r - \sigma_\theta)}{2}\right]^{\frac{1}{n}-1} \left[\frac{S_{ij}}{E}\right]. \qquad \text{... (3.29)}$$

These are the constitutive equations used by Odquist[13] for finding the creep stresses, provided we put $n = \frac{1}{N}$.

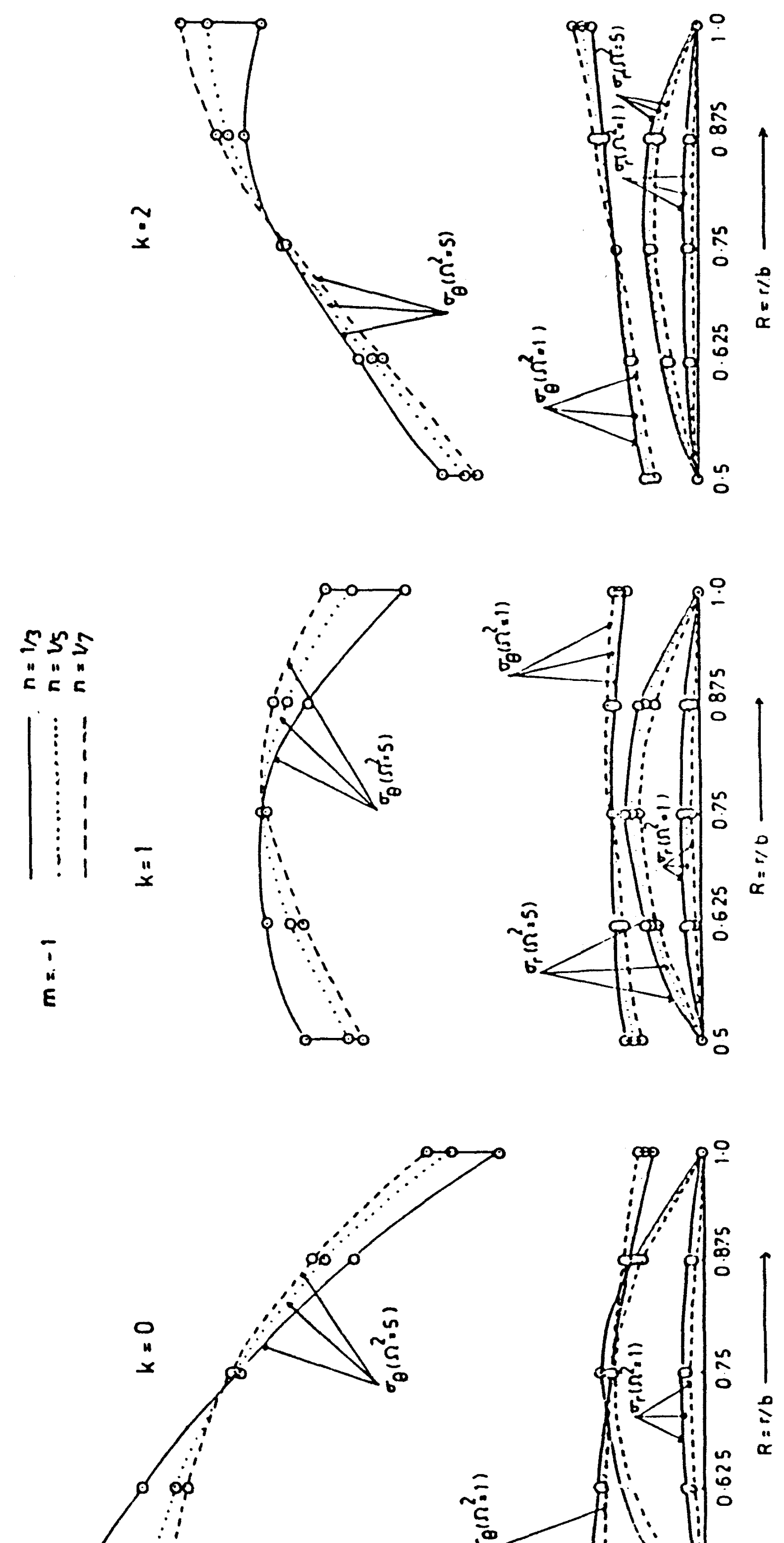

FIG. 1. Creep stresses in a thin rotating disc having variable thickness and variable density ($m = -1$) for various radii ratios

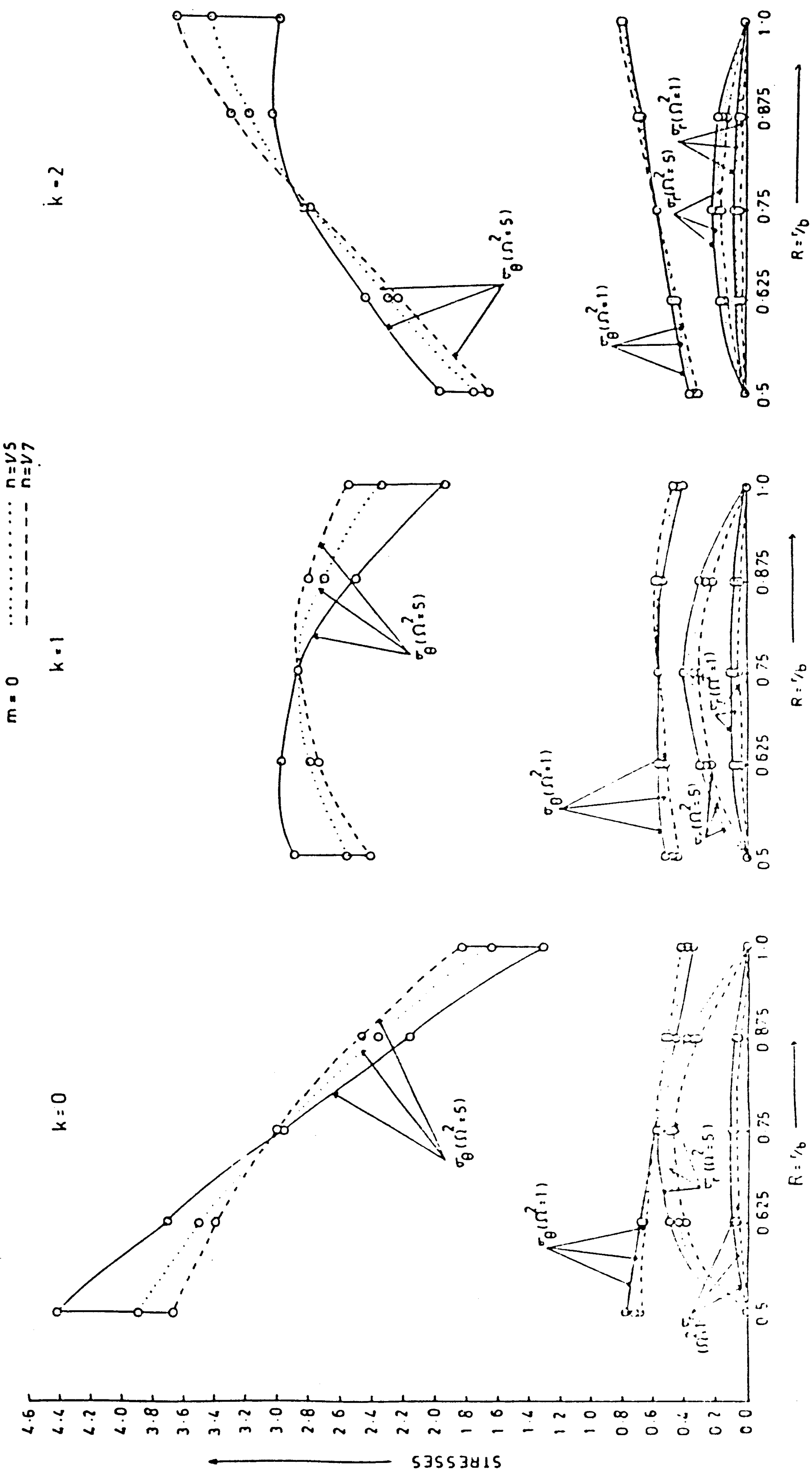

FIG. 2. Creep stresses in a thin rotating disc having variable thickness and variable density (m = 0) for various radii ratios

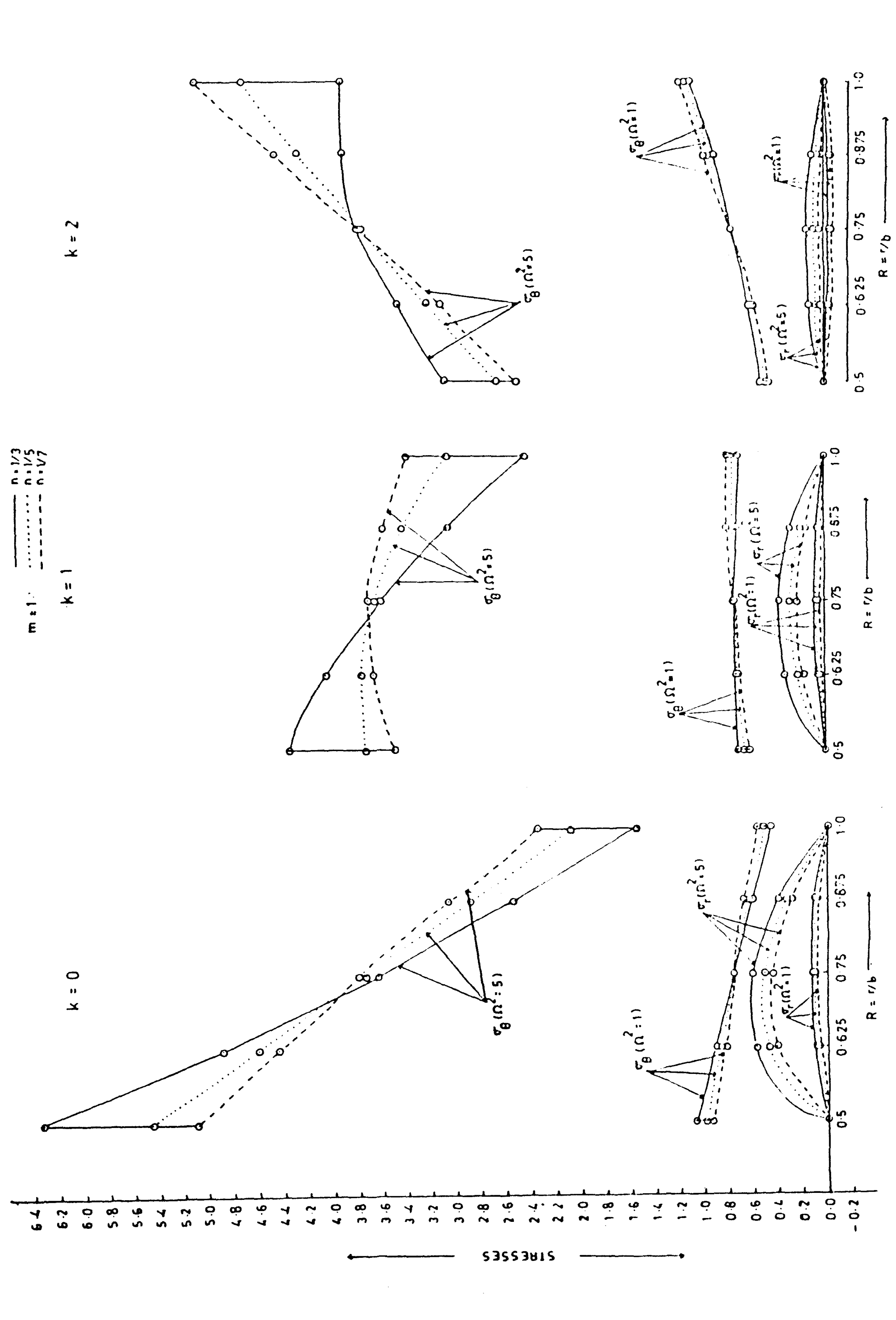

FIG. 3. Creep stresses in a thin rotating disc having variable thickness and variable density (m = 1) for various radii ratios

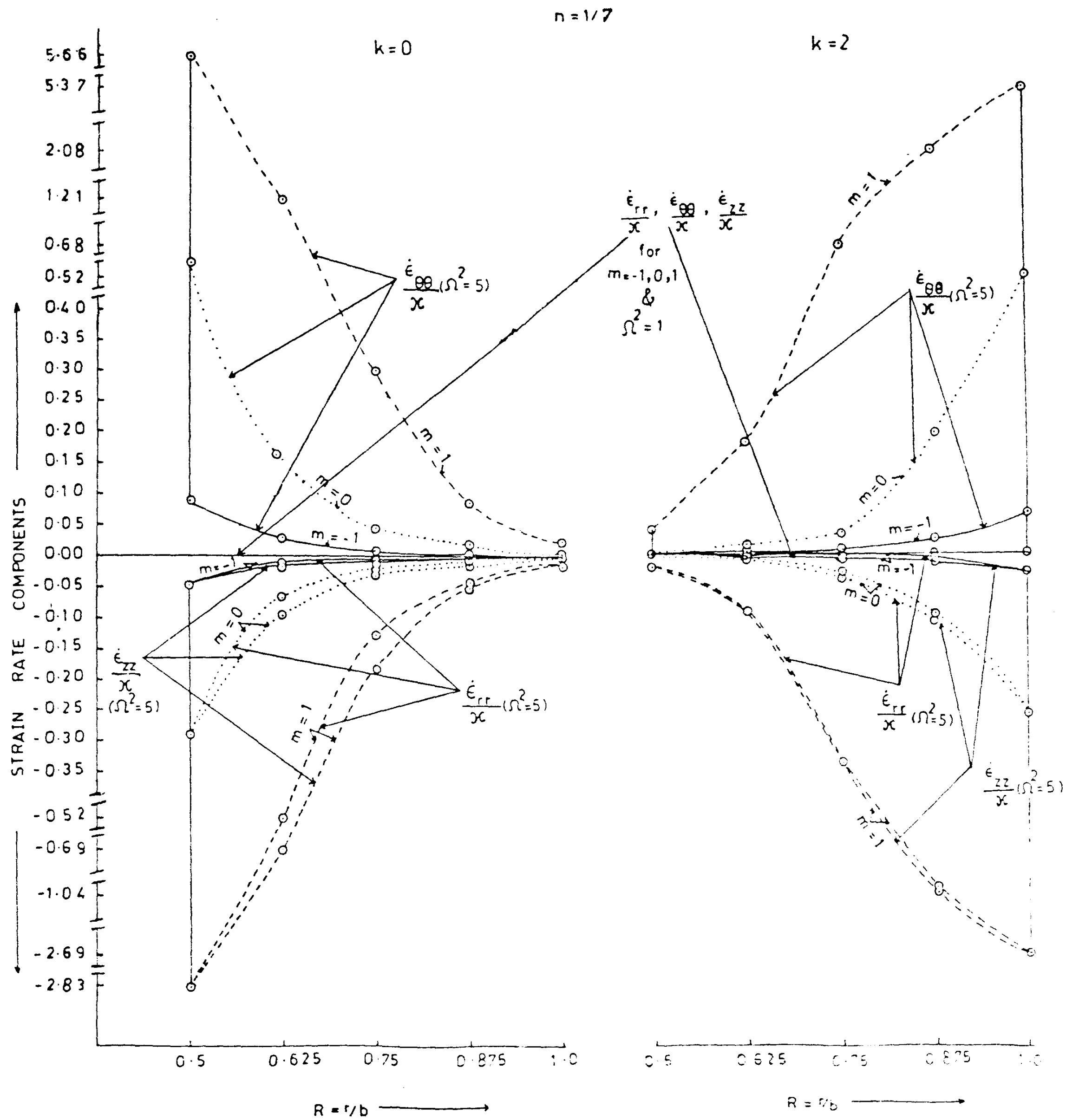

FIG. 1. Strain rates for a thin rotating disc having variable thickness and variable density for various radii ratios

4. Numerical Illustration and Discussion

Curves have been drawn in figures 1, 2 and 3 between stresses and radii ratio $R = (r/b)$ for a disc rotating with angular speed $\Omega^2 = 1, 5$ having variable thickness ($k = 0, 1, 2$), variable density ($m = -1, 0, 1$) and $n = 1/3, 1/5, 1/7$ (*i.e.* N = 3, 5, 7).

For a flat disc ($k = 0$) rotating with angular speed $\Omega^2 = 5$ and whose density decreases radially ($m = 1$), it is seen from figure 3 that the circumferential stress is maximum at the internal surface of the disc having constant density ($m = 0$) or density increases radially ($m = -1$) (see figure 1 and 2). This means that a disc rotating with higher angular speed and whose density decreases radially ($m = 1$) increases the possibility of fracture at the bore whereas a rotating disc whose density increases radially recedes the possibility of a fracture at the bore.

For a disc rotating with higher angular speed ($\Omega^2 = 5$) whose thickness decreases radially ($k > 1$), $n = 1/7$ (or $N = 7$) and density decreases radially ($m = 1$), the circumferential stress is maximum at the external surface. For $m = 0, -1$ and $k > 1$ though the circumferential stress is maximum at the external surface yet it has smaller value than those for $m = 1$. Therefore, it can be concluded that a rotating disc whose density and thickness decreases radially and $n = 1/7$ ($N = 7$) is on the safer side of the design in comparison to a flat disc having variable density.

In figure 4. curves have been drawn between the strain rates and radii ratio R. It has been observed that a rotating disc whose thickness ratio and density decreases radially experiences a significant deformation for angular speed $\Omega^2 = 5$ and $n = 1/7$.

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